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Sober spaces

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Abstract: The goal of this paper is to introduce various forms of sober objects in a topological category and investigate the relationships among these various forms. Moreover, we characterize quasi-sober objects, T_0 objects and each of various forms of sober objects in some topological categories and give some invariance properties of them. Finally, we compare our results with some well-known results in the category of topological spaces.

Key words: Topological category, T_0 objects, T_1 objects, sober spaces, irreducible spaces.

1. Introduction

The sober spaces, which are one of the most important classes in the theory of non-Hausdorff topological spaces, were introduced in [12]. The usual notion of closedness and T_0 -axiom of topology were generalized the topological categories [1, 17, 20] and they are being used in defining separation axioms $T_i, i = 2, 3, 4$, compactness and perfectness [1, 5].

In this paper, we introduce various forms of sober objects and quasi sober objects in topological categories. Moreover, we try to:

- (i) find the relationships as well as interrelationships among these notions,
- (ii) give the characterization of these notions in the category of **PBorn** (resp. **CP** and **RRel**) of all prebornological (resp. pair and reflexive relation) spaces and show whether the subcategories $\mathbf{T}'_0\mathbf{Sob}\mathcal{E}$ (resp. $\mathbf{T}_0\mathbf{Sob}\mathcal{E}$, $\overline{\mathbf{T}}_0\mathbf{Sob}\mathcal{E}$) of \mathcal{E} consisting of all T'_0 sober (resp. T_0 sober, \overline{T}_0 sober) objects of \mathcal{E} are isomorphic or not, where $\mathcal{E} = \mathbf{PBorn}, \mathbf{CP}$ or \mathbf{RRel} ,
- (iii) examine some invariance properties of these various forms of sober objects in these topological categories,
- (iv) compare our results with some results in the category of topological spaces.

2. Preliminaries

The category of prebornological spaces, **PBorn**, has as objects the pair (X, \mathcal{F}) , where \mathcal{F} is a family of subsets of X that is closed under finite union and contains all finite subsets of X and has morphisms $(X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$ those functions $f : X \rightarrow Y$ such that $f(C) \in \mathcal{G}$ if $C \in \mathcal{F}$ [16]. **PBorn** is a topological category over **Set**, the category of all sets and functions [16].

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The category of pair, **CP**, has as objects the pair (X, Y) , where $Y \subset X$ and has as morphisms $(X, Y) \rightarrow (X_1, Y_1)$ those functions $f : X \rightarrow X_1$ such that $f(Y) \subset Y_1$. This is a topological category [2].

The category of reflexive relation spaces (spatial graphs), **RRel**, has as objects the pair (B, R) , where B is a set and R is reflexive relation on B , and as morphisms $(B, R) \rightarrow (B_1, R_1)$ those functions $f : B \rightarrow B_1$ such that if sRt , then $f(s)R_1f(t)$ for all $s, t \in B$ [9, 18].

A source $f_i : (B, R) \rightarrow (B_i, R_i), i \in I$ is initial in **RRel** iff for all $s, t \in B$, sRt iff $f_i(s)R_i f_i(t)$ for all $i \in I$ [9, 18].

An epimorphism $f : (B, R) \rightarrow (B_1, R_1)$ is final in **RRel** iff for all $s, t \in B_1$, sR_1t holds in B_1 precisely when there exist $c, d \in B$, such that cRd and $f(c) = s$ and $f(d) = t$ [18].

RRel is a topological category.

Let B be a set, $p \in B$, and the wedge $B^2 \vee_{\Delta} B^2$ (resp. the infinity wedge $\vee_p^{\infty} B$) be two distinct copies of B^2 identified along the diagonal Δ (resp. taking countably many disjoint copies of B and identifying them at the point p) [1]. Define $A : B^2 \vee_{\Delta} B^2 \rightarrow B^3$ by

$$A((a, b)_i) = \begin{cases} (a, b, a), & i = 1 \\ (a, a, b), & i = 2 \end{cases}$$

$\nabla : B^2 \vee_{\Delta} B^2 \rightarrow B^2$ by

$$\nabla((a, b)_i) = (a, b)$$

for $i = 1, 2$

$A_p^{\infty} : \vee_p^{\infty} B \rightarrow B^{\infty}$ by

$$A_p^{\infty}(a_i) = (p, \dots, p, a, p, p, \dots)$$

where a_i is in the i -th component of $\vee_p^{\infty} B$ and $B^{\infty} = B \times B \times \dots$ is the countable cartesian product of B , and $\nabla_p^{\infty} : \vee_p^{\infty} B \rightarrow B$ by

$$\nabla_p^{\infty}(a_i) = a$$

for all $i \in I$, where I is the index set $\{i : a_i \text{ is in the } i\text{-th component of } \vee_p^{\infty} B\}$ [1].

Definition 2.1 (cf. [1, 17]) Let $U : \mathcal{E} \rightarrow \mathbf{Set}$ be topological and X be an object in \mathcal{E} with $p \in U(X) = B$.

(1) If the initial lift of the U -source

$$\{A : B^2 \vee_{\Delta} B^2 \rightarrow U(X^3) = B^3 \quad \text{and} \quad \nabla : B^2 \vee_{\Delta} B^2 \rightarrow U(D(B^2)) = B^2\}$$

is discrete, then X is called a \overline{T}_0 object, where D is the discrete functor. (2) If the initial lift of the U -source

$$\{id : B^2 \vee_{\Delta} B^2 \rightarrow U(B^2 \vee_{\Delta} B^2)' = B^2 \vee_{\Delta} B^2 \quad \text{and} \quad \nabla : B^2 \vee_{\Delta} B^2 \rightarrow U(D(B^2)) = B^2\}$$

is discrete, then X is called T'_0 object, where $(B^2 \vee_{\Delta} B^2)'$ is the final lift of the U -sink

$$\{q \circ i_1, q \circ i_2 : U(X^2) = B^2 \rightarrow B^2 \vee_{\Delta} B^2\},$$

where $i_k : B^2 \rightarrow B^2 \coprod B^2, k = 1, 2$ are the canonical injection maps and $q : B^2 \coprod B^2 \rightarrow B^2 \bigvee_{\Delta} B^2$ is the quotient map. **(3)** If X does not contain an indiscrete subspace with (at least) two points, then X is called a T_0 object. **(4)** $\{p\}$ is closed iff the initial lift of the U -source

$$\{A_p^\infty : \bigvee_p^\infty B \rightarrow B^\infty = U(X^\infty) \quad \text{and} \quad \nabla_p^\infty : \bigvee_p^\infty B \rightarrow UD(B) = B\}$$

is discrete.

(5) $\emptyset \neq Z \subset B$ is closed iff $\{*\}$, the image of Z , is closed in X/Z , where X/Z is the final lift of the epi U -sink

$$Q : U(X) = B \rightarrow B/Z = (B \setminus Z) \cup \{*\},$$

identifying Z with a point $*$. **(6)** If $Z = B = \emptyset$, then we define Z to be closed.

Remark 2.2 For the category **Top** of topological spaces and continuous functions, all of $\overline{T_0}$, T'_0 , and T_0 are equivalent and they reduce to the usual T_0 separation axiom and the notion of closedness coincides with the usual closedness [1, 17, 20].

3. Sober spaces

Let \mathcal{E} be a topological category over **Set** and $X \in Ob(\mathcal{E})$ [18].

Definition 3.1

$$cl(Z) = \bigcap \{U \subset X : Z \subset U \quad \text{and} \quad U \text{ is closed}\}$$

is called the closure of a subobject Z of X .

It is shown that the notion of closedness forms closure operator [8] in some topological categories [5–7, 10, 11, 14, 19].

In Exercise 2.D of [9], let \mathcal{M} be the class of all closed subobjects of an object of \mathcal{E} . If \mathcal{M} is stable under pullback, then cl is an idempotent closure operator of \mathcal{E} .

Definition 3.2 ([7]) X is said to be irreducible if A, B are closed subobjects of X and $X = A \cup B$, then $A = X$ or $B = X$.

Definition 3.3 (1) X is called quasi-sober if every nonempty irreducible closed subset of X is the closure of a point.

(2) X is called $\overline{T_0}$ sober if X is $\overline{T_0}$ and quasi-sober.

(3) X is called T'_0 sober if X is T'_0 and quasi-sober.

(4) X is called T_0 sober if X is T_0 and quasi-sober.

Remark 3.4 For **Top** all of $\overline{T_0}$ sober, T'_0 sober, and T_0 sober are equivalent and they reduce to the usual sober [15] and quasi-sober reduces to the usual quasi-sober [13].

Theorem 3.5 If X is $\overline{T_0}$ sober, then X is T'_0 sober.

Proof If X is $\overline{T_0}$ sober, then in particular, X is $\overline{T_0}$ and by Theorem 3.2, of [3], X is T'_0 . Hence, by Definition 3.3, X is T'_0 sober. \square

Recall that if the initial lift of the U -source

$$\{S : B^2 \vee_{\Delta} B^2 \rightarrow U(X^3) = B^3 \quad \text{and} \quad \nabla : B^2 \vee_{\Delta} B^2 \rightarrow U(D(B^2)) = B^2\}$$

is discrete, then X is called a T_1 object and if the initial lift of the U -source

$$S : B^2 \vee_{\Delta} B^2 \rightarrow U(X^3) = B^3$$

and the initial lift of the U -source

$$A : B^2 \vee_{\Delta} B^2 \rightarrow U(X^3) = B^3$$

coincide, then X is called a $Pre\overline{T}_2$ object, where S is the skewed axis map defined as

$$S((a, b)_i) = \begin{cases} (a, b, b), & i=1 \\ (a, a, b), & i=2 \end{cases}$$

[1]. If X is T'_0 and $Pre\overline{T}_2$, then X is called KT_2 [4].

Theorem 3.6 *Let (X, \mathcal{F}) be a prebornological space.*

(A) (1) (X, \mathcal{F}) is T'_0 .

(2) (X, \mathcal{F}) is \overline{T}_0 .

(3) (X, \mathcal{F}) is T_1 .

(B) *The following are equivalent:*

(1) (X, \mathcal{F}) is quasi-sober.

(2) (X, \mathcal{F}) is \overline{T}_0 sober.

(3) (X, \mathcal{F}) is T'_0 sober.

(4) (X, \mathcal{F}) is irreducible.

(C) *The following are equivalent*

(1) (X, \mathcal{F}) is T_0 .

(2) (X, \mathcal{F}) is T_0 sober.

(3) $X = \emptyset$ or X is a one-point set.

Proof (A) (1) Let (X, \mathcal{F}) be a prebornological space, \mathcal{G} be the final prebornological structure on $X^2 \coprod X^2$ induced by the injection maps

$$i_1, i_2 : (X^2, \mathcal{F}^2) \longrightarrow X^2 \coprod X^2,$$

\mathcal{H} be the final prebornological structure on $X^2 \vee_{\Delta} X^2$ induced by the quotient map

$$q : (X^2 \coprod X^2, \mathcal{G}) \longrightarrow X^2 \vee_{\Delta} X^2,$$

and $\overline{\mathcal{F}}$ be the initial structure on $X^2 \vee_{\Delta} X^2$ induced by

$$id : X^2 \vee_{\Delta} X^2 \longrightarrow (X^2 \vee_{\Delta} X^2, \mathcal{H}) \quad \text{and} \quad \nabla : X^2 \vee_{\Delta} X^2 \longrightarrow (X^2, \mathcal{D}\mathcal{F}),$$

where \mathcal{DF} is discrete structure on X^2 i.e. \mathcal{DF} is the set of all finite subsets of X^2 [16] and \mathcal{F}^2 is the product structure on X^2 .

Let U be any element in $\overline{\mathcal{F}}$. In particular, $\nabla(U) \in \mathcal{DF}$,

If $\nabla(U) = \emptyset$, then it follows that $U = \emptyset$. Suppose that $\nabla(U) \neq \emptyset$. Since \mathcal{DF} is discrete structure on X^2 and $\nabla(U) \in \mathcal{DF}$, $\nabla(U)$ is finite and consequently, U is finite. Hence, $\overline{\mathcal{F}}$ is discrete and by Definition 2.1, (X, \mathcal{F}) is T'_0 .

(2) Let $\overline{\mathcal{F}}$ be the initial prebornological structure on $X^2 \vee_{\Delta} X^2$ induced by

$$A : X^2 \vee_{\Delta} X^2 \rightarrow (X^3, \mathcal{F}^3) \quad \text{and} \quad \nabla : X^2 \vee_{\Delta} X^2 \rightarrow (X^2, \mathcal{DF}),$$

where \mathcal{F}^3 is the product structure on X^3 .

Let U be any element in $\overline{\mathcal{F}}$. Since $\overline{\mathcal{F}}$ is initial by [16], $\nabla(U) \in \mathcal{DF}$ and $\pi_i A(U) \in \mathcal{F}$, where $\pi_i : X^3 \rightarrow X$, $i = 1, 2, 3$ are the projection maps.

If $\nabla(U) = \emptyset$, then it follows that $U = \emptyset$. Suppose that $\nabla(U) \neq \emptyset$. Since \mathcal{DF} is discrete structure on X^2 and $\nabla(U) \in \mathcal{DF}$, $\nabla(U)$ is finite and consequently, U is finite. Hence, $\overline{\mathcal{F}}$ is discrete and by Definition 2.1, (X, \mathcal{F}) is \overline{T}_0 .

The proof for T_1 is similar.

(B) By Definition 3.3 and part (A), (1)-(3) are equivalent.

Suppose (X, \mathcal{F}) is quasi-sober and $X = M \cup N$, where M and N are closed subsets of X . By Theorem 3.9 of [2], $M = X$ or $M = \emptyset$ and $N = X$ or $N = \emptyset$. Hence, by Definition 3.2, (X, \mathcal{F}) is irreducible.

Suppose (X, \mathcal{F}) is irreducible and $\emptyset \neq M \subset X$ is irreducible closed. Since M is closed, by Theorem 3.9 of [2], $M = X$ and by Definition 3.1, $X = M = \mathbf{cl}\{a\}$ for some $a \in X$. Hence, by Definition 3.3, (X, \mathcal{F}) is quasi-sober. Hence, (1) and (4) are equivalent.

(C) (1) \Rightarrow (2): Suppose (X, \mathcal{F}) is T_0 and M is $\emptyset \neq M \subset X$ is irreducible closed. Since M is closed, by Theorem 3.9 of [2], $M = X$ and by Definition 3.1, $M = X = \mathbf{cl}\{a\}$ for some $a \in X$. Hence, by Definition 3.2, (X, \mathcal{F}) is quasi-sober and by Definition 3.3, (X, \mathcal{F}) is T_0 sober.

(2) \Rightarrow (3): Suppose (X, \mathcal{F}) is T_0 sober and $X \neq \emptyset$ and X is not a one-point set. Then, there exist distinct points c and d of X and by [16] $(\{c, d\}, P(\{c, d\}))$ is the indiscrete subspace of (X, \mathcal{F}) , contradicting to (X, \mathcal{F}) is being T_0 sober. Hence, $X = \emptyset$ or X is a one-point set.

(3) \Rightarrow (1): If $X = \emptyset$ or X is a one-point set, then by Definition 2.1, (X, \mathcal{F}) is T_0 . □

Theorem 3.7 *Let (X, Y) be a pair space.*

(A) (1) (X, Y) is \overline{T}_0 .

(2) (X, Y) is T'_0 .

(3) (X, Y) is T_1 .

(B) *The following are equivalent:*

(1) (X, Y) is \overline{T}_0 sober.

(2) (X, Y) is T'_0 sober.

(3) (X, Y) is quasi-sober.

(4) *The nonempty proper irreducible closed subsets of X are exactly the one-point subsets.*

(C) The following are equivalent:

- (1) (X, Y) is T_0 .
- (2) (X, Y) is T_0 sober.
- (3) $X = \emptyset$ or X is a one point set.
- (4) (X, Y) is irreducible.

Proof (A) (1) Note that (X, \emptyset) is the discrete pair space [2]. Let (X, Y) be a pair space and V be the initial structure on $X^2 \vee_{\Delta} X^2$ induced by

$$A : X^2 \vee_{\Delta} X^2 \rightarrow (X^3, Y^3) \quad \text{and} \quad \nabla : X^2 \vee_{\Delta} X^2 \rightarrow (X^2, \emptyset),$$

where Y^3 is the product structure on X^3 [2]. It follows that $\nabla(V) \subset \emptyset$. Hence, $V = \emptyset$ and by Definition 2.1, (X, Y) is $\overline{T_0}$.

(2) Let (X, Y) be a pair space, Z be the final structure on $X^2 \vee_{\Delta} X^2$ induced by

$$q \circ i_1, q \circ i_2 : (X^2, Y^2) \rightarrow X^2 \vee_{\Delta} X^2,$$

where the maps i_1, i_2 , and q are defined in Section 2 and V be the initial structure on $X^2 \vee_{\Delta} X^2$ induced by

$$id : X^2 \vee_{\Delta} X^2 \rightarrow (X^2 \vee_{\Delta} X^2, Z) \quad \text{and} \quad \nabla : X^2 \vee_{\Delta} X^2 \rightarrow (X^2, \emptyset).$$

In particular, $\nabla(V) \subset \emptyset$ and consequently, $V = \emptyset$ and by Definition 2.1, (X, Y) is T'_0 .

(3) The proof of Part (3) is similar to Part (1).

(B) By Part (A) and by Definition 3.3, (1)-(3) are equivalent.

(3) \Rightarrow (4): (X, Y) is quasi-sober and $\emptyset \neq A \subset X$ is proper irreducible closed. Since (X, Y) is quasi-sober, then $A = \mathbf{cl}\{a\}$ for some $a \in X$. By Theorem 3.8 of [2], $\{a\}$ is closed. Thus, $\mathbf{cl}\{a\} = \{a\}$ and $A = \{a\}$.

(4) \Rightarrow (3): Suppose the nonempty proper irreducible closed subsets of X are exactly the one-point subsets and $\emptyset \neq A \subset X$ is proper irreducible closed. Then by assumption, $A = \{a\}$ for some $a \in X$. By Theorem 3.8 of [2], $\{a\}$ is closed, we get $\mathbf{cl}(\{a\}) = \{a\} = A$. Hence, by Definition 3.3, (X, Y) is quasi-sober.

(C) Suppose (X, Y) is T_0 sober and $X \neq \emptyset$ and X is not a one point set. Then there exist distinct points s and t of X and by [2], $(\{s, t\}, \{s, t\})$ is the indiscrete subspace of (X, Y) contradicting to (X, Y) is being T_0 sober. Hence, $X = \emptyset$ or X is a one-point set.

If $X = \emptyset$ or X is a one-point set, then by Definitions 2.1 and 3.3, (X, Y) is T_0 and quasi-sober. This shows Parts (1)-(3) are equivalent.

Suppose (X, Y) is irreducible and $X \neq \emptyset$ and X is not a one-point set. Then there exist distinct points s and t of X . By Theorem 3.8 of [2] both $\{t\}$ and $\{t\}^C$ are closed subsets of X and $X = \{t\} \cup \{t\}^C$ contradicting to (X, Y) is being irreducible.

If $X = \emptyset$ or X is a one-point set, then by Definition 3.2, (X, Y) is irreducible.

This shows Parts (3) and (4) are equivalent. □

Theorem 3.8 Let (B, R) be a reflexive space and $p \in B$.

- (1) (B, R) is T'_0 .

(2) The following are equivalent:

- (a) (B, R) is $\overline{T_0}$.
- (b) R is antisymmetric.
- (c) (B, R) is T_0 .
- (3) (a) $\{p\}$ is closed iff for each $a \in B$, if aRp and pRa , then $a = p$.
- (b) (B, R) is $\overline{T_0}$ iff $\{x\}$ is closed for each $x \in B$.
- (4) The following are equivalent:
 - (a) $A \subset B$ is closed.

(b) For each $x \in B$ if there exist $c, d \in A$ such that xRc and dRx , then $x \in A$.

(c) A is upward-closed or downward-closed.

(5) (B, R) is quasi-sober if and only if (B, R) is T'_0 sober.

(6) The following are equivalent:

- (a) (B, R) is $\overline{T_0}$ sober.
- (b) (B, R) is T_0 sober.
- (c) $\{x\}$ is closed for each $x \in B$ and the nonempty proper irreducible closed subsets of B are exactly the one-point subsets.

Proof (1) Let (B, R) be a reflexive space, $(B^2 \vee_{\Delta} B^2)'$ be the final lift of

$$q \circ i_1, q \circ i_2 : (B^2, R^2) \rightarrow B^2 \vee_{\Delta} B^2,$$

where R^2 is the product relation on B^2 , and R_1 be the initial structure on the wedge $B^2 \vee_{\Delta} B^2$ induced by

$$id : B^2 \vee_{\Delta} B^2 \rightarrow (B^2 \vee_{\Delta} B^2)' \quad \text{and} \quad \nabla : B^2 \vee_{\Delta} B^2 \rightarrow (B^2, N)$$

where N is discrete relation on B^2 . We have to show that R_1 is discrete, i.e. for any points s and t of the wedge $B^2 \vee_{\Delta} B^2$ if sR_1t , then $s = t$. If sR_1t , then, in particular, $\nabla(s) = (a, b) = \nabla(t)$ for some $a, b \in B$, and consequently, s and t have the form. $(a, b)_1$ and $(a, b)_2$. If $s = (a, b)_1$ (resp. $(a, b)_2$) and $t = (a, b)_2$ (resp. $(a, b)_1$), then $i_k((a, b)) = (a, b)_1 = s, i_k((a, b)) = (a, b)_2 = t$ for $k=1$ or 2 , which implies s and t must lie in the same component of the wedge $B^2 \vee_{\Delta} B^2$ which means $s = t$ for all s and t . Thus, R_1 is discrete and by 2.1, (B, R) is T'_0 .

(2) $(a) \Rightarrow (b)$: Suppose (B, R) is $\overline{T_0}$ and for any $x, y \in B$, xRy and yRx . We show that $x = y$. Assume $x \neq y$. Let $s = (x, y)_1$ and $t = (x, y)_2$. Note that

$$\pi_1 A(s)R\pi_1 A(t) = xRx,$$

$$\pi_2 A(s)R\pi_2 A(t) = yRx,$$

$$\pi_3 A(s)R\pi_3 A(t) = xRy,$$

and

$$\nabla(s) = (x, y) = \nabla(t)$$

. Since (B, R) is \overline{T}_0 , by Definition 2.1, $s = t$. Thus, $x = y$, i.e. R is antisymmetric.

(b) \Rightarrow (c): Suppose that R is antisymmetric and A is any subset of B with at least two points. Let $c, d \in A$ with $c \neq d$, $D = \{c, d\}$, and R_D be the initial relation on D induced by the inclusion map $i : D \subset B$. If $R_D = D^2$, the indiscrete reflexive relation on D^2 , then, in particular, $cR_D d = cRd$ and $dR_D c = dRc$. Since R is antisymmetric, $c = d$, a contradiction. Hence, a reflexive space (B, R) cannot contain an indiscrete subspace with at least two points. Thus, by Definition 2.1, (B, R) is T_0 .

(c) \Rightarrow (a): Suppose (B, R) is T_0 . Let R^3 is the product relation on B^3 and R_1 be the initial structure on the wedge $B^2 \vee_{\Delta} B^2$ induced by

$$A : B^2 \vee_{\Delta} B^2 \rightarrow (B^3, R^3) \quad \text{and} \quad \nabla : B^2 \vee_{\Delta} B^2 \rightarrow (B^2, B^2 \times B^2).$$

We need to show that R_1 is discrete. If $sR_1 t$ for any points s and t of the wedge $B^2 \vee_{\Delta} B^2$, then $\pi_1 A(s)R\pi_1 A(t)$, $\pi_2 A(s)R\pi_2 A(t)$, $\pi_3 A(s)R\pi_3 A(t)$, and $\nabla(s) = \nabla(t)$. Since $\nabla(s) = \nabla(t)$, s and t have the form: $(x, y)_1$ and $(x, y)_2$ for some x and y in B .

If $s = (x, y)_1$ (resp. $(x, y)_2$) and $t = (x, y)_2$ ((resp. $(x, y)_1$)), then

$$\begin{aligned} \pi_1 A(s)R\pi_1 A(t) &= xRx, \\ \pi_2 A(s)R\pi_2 A(t) &= yRx \quad (\text{resp. } xRy), \\ \pi_3 A(s)R\pi_3 A(t) &= xRy \quad (\text{resp. } yRx), \end{aligned}$$

and

$$\nabla(s) = (x, y) = \nabla(t).$$

Since xRy and yRx , $x = y$. Otherwise, $(\{x, y\}, \{x, y\} \times \{x, y\})$ is the indiscrete subspace of (B, R) , which contradicts to (B, R) is being T_0 . Hence, $s = t$ and R_1 is discrete, i.e. (B, R) is \overline{T}_0 . This completes the proof of Part (2).

(3) (a): Suppose $\{p\}$ is closed and for each $a \in B$ aRp and pRa . Note that

$$\begin{aligned} \nabla_p^\infty(a, p, p, \dots) &= a = \nabla_p^\infty(p, a, p, \dots), \\ \pi_1 A_p^\infty(a, p, p, \dots)R(p, a, p, \dots) &= aRp, \\ \pi_2 A_p^\infty(a, p, p, \dots)R(p, a, p, \dots) &= pRa, \end{aligned}$$

and

$$\pi_i A_p^\infty(a, p, p, \dots)R(p, a, p, \dots) = pRp$$

for all $i \geq 3$, where $\pi_i : B^\infty \rightarrow B$ are the projection maps, for all $i \in I$. Since $\{p\}$ is closed, by Definition 2.1, $(a, p, p, \dots) = (p, a, p, \dots)$ which implies $a = p$.

Conversely, suppose for each $a \in B$, if aRp and pRa , then $a = p$. Let s and t be any points in $\vee_p^\infty B$ with $\pi_i A_p^\infty(s)R\pi_i A_p^\infty(t)$ for all i , and $\nabla_p^\infty(s) = \nabla_p^\infty(t)$. It follows $s = a_k$ and $t = a_n$ for some k and n . Note that

$$\pi_k A_p^\infty(s)R\pi_k A_p^\infty(t) = aRp,$$

$$\pi_n A_p^\infty(s)R\pi_n A_p^\infty(t) = pRa,$$

and

$$\pi_i A_p^\infty(s)R\pi_i A_p^\infty(t) = pRp$$

for all $i \neq k$. By the assumption, $a = p$, and consequently, $s = a_k = a_n = t$. Thus, by Definition 2.1, $\{p\}$ is closed.

(b): Combine Part (2) and Part 3(a).

(4) (a) \Rightarrow (b): Suppose A is closed and for each $x \in B \exists c, d \in A$ such that xRc and dRx . Note that $Q(x)R_1(Q(c) = *)$ and $(* = Q(d))R_1Q(x)$, where R_1 is the final structure on B/A that is induced by $Q : B \rightarrow B/A$. Since $\{*\}$ is closed, by (3)(a), $Q(x) = *$ which means $x \in A$.

(b) \Rightarrow (c): Assume (b) holds. If for each $x \in B$ there exists $c \in A$ such that xRc (resp. cRx), then, by assumption, $x \in A$ and so, A is downward-closed (resp. upward-closed) [9, 15].

(c) \Rightarrow (a): Suppose A is upward-closed or downward-closed. Assume yR_1* and $*R_1y$ for some $y \in B/A$. It follows that there exist $x \in B$ and $c, d \in A$ such that xRc and cRx with $Q(x) = y$ and $Q(c) = * = Q(d)$.

If xRc , then $x \in A$ since A is downward-closed. Thus, $y = Q(x) = *$.

If dRx , then $x \in A$ since A is upward-closed. Thus, $y = Q(x) = *$ and by Part (3)(a), $\{*\}$ is closed. Hence, by Definition 2.1, A is closed.

(5) We get the proof from Part (1) and Definition 3.3.

(6) By Part (2), (a) and (b) are equivalent.

(b) \Rightarrow (c): Suppose (B, R) is T_0 sober. In particular, by Parts (2) and (3), $\{x\}$ is closed for each $x \in B$. If $\emptyset \neq A \subset B$ is proper irreducible closed, then $A = \mathbf{cl}\{a\}$ for some $a \in B$ since (B, R) is quasi-sober. Since $\{a\}$ is closed, we have $\{a\} = \mathbf{cl}\{a\} = A$.

(c) \Rightarrow (b): Suppose that $\{x\}$ is closed for each $x \in B$ and every nonempty proper irreducible closed subset of B is a one-point set. By Parts (2) and (3), (B, R) is T_0 .

If $\emptyset \neq A \subset B$ is proper irreducible closed subset of B , then by assumption, $A = \{a\}$ for some $a \in B$. But since $\{a\}$ is closed, $\mathbf{cl}\{a\} = \{a\} = A$. Hence, by Definition 3.3, (B, R) is quasi-sober and consequently, (B, R) is T_0 sober. \square

Let $\mathbf{T}'_0\mathcal{E}$ (resp. $\mathbf{T}_0\mathcal{E}$, $\overline{\mathbf{T}}_0\mathcal{E}$, $\mathbf{T}_1\mathcal{E}$) be the full subcategory of \mathcal{E} consisting of all T'_0 (resp. T_0 , \overline{T}_0 , T_1) objects of \mathcal{E} . Let $\mathbf{T}'_0\mathbf{Sob}\mathcal{E}$ (resp. $\mathbf{T}_0\mathbf{Sob}\mathcal{E}$, $\overline{\mathbf{T}}_0\mathbf{Sob}\mathcal{E}$) be the full subcategory of \mathcal{E} consisting of all T'_0 sober (resp. T_0 sober, \overline{T}_0 sober) objects of \mathcal{E} .

By Theorem 3.6, we get:

Theorem 3.9 (1) *The categories $\mathbf{T}'_0\mathbf{PBorn}$, $\overline{\mathbf{T}}_0\mathbf{PBorn}$, and $\mathbf{T}_1\mathbf{PBorn}$ are pairwise isomorphic.*

(2) *The categories $\mathbf{T}'_0\mathbf{SobPBorn}$, $\overline{\mathbf{T}}_0\mathbf{SobPBorn}$ and $\mathbf{QSobPBorn}$ are pairwise isomorphic, where $\mathbf{QSobPBorn}$ is the full subcategory of \mathbf{PBorn} consisting of all quasi-sober prebornological spaces.*

(3) The categories $\mathbf{T}_0\mathbf{PBorn}$ and $\mathbf{T}_0\mathbf{SobPBorn}$ are isomorphic.

By Theorem 3.7, we get:

Theorem 3.10 (1) The categories $\mathbf{T}'_0\mathbf{CP}$, $\overline{\mathbf{T}}_0\mathbf{CP}$, and $\mathbf{T}_1\mathbf{CP}$ are pairwise isomorphic.

(2) The categories $\mathbf{T}'_0\mathbf{SobCP}$, $\overline{\mathbf{T}}_0\mathbf{SobCP}$, and \mathbf{QSobCP} are pairwise isomorphic, where \mathbf{QSobCP} is the full subcategory of \mathbf{CP} consisting of all quasi-sober pair spaces.

(3) The categories $\mathbf{T}_0\mathbf{CP}$ and $\mathbf{T}_0\mathbf{SobCP}$ are isomorphic.

Theorem 3.11 (1) The categories \mathbf{RRel} and $\mathbf{T}'_0\mathbf{RRel}$ are isomorphic.

(2) The categories $\mathbf{T}_0\mathbf{RRel}$ and $\overline{\mathbf{T}}_0\mathbf{RRel}$ are isomorphic.

(3) The categories $\mathbf{T}'_0\mathbf{SobRRel}$ and $\mathbf{QSobRRel}$ are isomorphic, where $\mathbf{QSobRRel}$ is the full subcategory of \mathbf{RRel} consisting of all quasi-sober reflexive spaces.

(4) The categories $\mathbf{T}_0\mathbf{SobRRel}$ and $\overline{\mathbf{T}}_0\mathbf{SobRRel}$ are isomorphic.

Proof It follows from Theorem 3.8. □

We now state some well-known results [15] in \mathbf{Top} .

Theorem 3.12 (1) Every T_2 topological space is sober.

(2) Sobriety is between T_0 and T_2 .

(3) Being T_1 and being sober are incomparable properties.

(4) Being T_0 and being quasi-sober are incomparable properties.

(5) Every irreducible and T_2 is a one-point space.

(6) X is T_1 iff $\{x\}$ is closed for each $x \in X$.

(7) Let X be a topological space, $S(X)$ be the set of all irreducible closed subsets of X , and the map $f : X \rightarrow S(X)$ be defined as $f(x) = \mathbf{cl}\{x\}$.

(i) X is sober iff f is bijective.

(ii) X is T_0 iff f is injective.

(iii) X is quasi-sober iff f is surjective.

Theorem 3.13 Let (X, d) be an extended pseudo-quasi-semi metric space.

(1) (X, d) is quasi-sober iff (X, d) is T'_0 sober,

(2) (X, d) is \overline{T}_0 sober iff $\{x\}$ is closed for each $x \in X$ and the nonempty proper irreducible closed subsets of X are exactly the one-point subsets,

(3) (X, d) is T_0 sober iff (X, d) is a quasi-sober and an extended quasi-semi metric space.

Proof (1) By Theorem 3.5 of [14], (X, d) is T'_0 and by Definition 3.3, the result follows.

(2) Suppose that (X, d) is \overline{T}_0 sober. By Theorem 3.5 of [14] and Theorem 3.2 of [7], $\{x\}$ is closed for each $x \in X$. If A is any nonempty proper irreducible closed subset of X , then by Definition 3.3, $A = \mathbf{cl}\{a\}$ for some $a \in X$ and Theorem 3.5 of [14], $\mathbf{cl}\{a\} = \{a\}$. Hence, $A = \{a\}$.

Suppose that $\{x\}$ is closed for each $x \in X$ and every nonempty proper irreducible closed subset of X is a one-point set. By Theorem 3.5 of [14] and Theorem 3.2 of [7], (X, d) is $\overline{T_0}$. If A is any nonempty proper irreducible closed subset of X , then by the assumption, $A = \{a\} = cl\{a\}$ for some $a \in X$. Hence, by Definition 3.3, (X, d) is quasi-sober and $\overline{T_0}$ sober.

(3) By Theorem 3.3 of [14], an extended pseudo-quasi-semi metric space (X, d) is T_0 iff d is pseudo. The result follows from this fact and Definition 3.3. \square

We can infer the following results:

(1) By Theorems 3.9-3.11, the categories $\mathbf{T'_0PBorn}$, $\overline{\mathbf{T_0PBorn}}$, $\mathbf{T_1PBorn}$, $\mathbf{T'_0CP}$, $\overline{\mathbf{T_0CP}}$, $\mathbf{T_1CP}$, $\mathbf{T'_0RRel}$, $\mathbf{T'_0SobPBorn}$, $\overline{\mathbf{T_0SobPBorn}}$, and $\mathbf{QSobPBorn}$ have all limits and colimits.

(2) By Theorems 3.3 and 3.25 of [14], if the extended pseudo-quasi-semi metric space (X, d) is T_1 , then each subset of X is closed and if (X, d) is a nonempty irreducible T_1 , then (X, d) must be a one-point space. Every irreducible and KT_2 may not be a one-point space. For example, let $X = \{a, b\}$ and d be defined as $d(a, a) = 0 = d(b, b), d(a, b) = 2 = d(b, a)$. By Theorem 3.13 of [14], (X, d) is KT_2 and by Theorem 3.19 of [14], (X, d) is irreducible but (X, d) is not a one-point space. This shows Theorem 3.12(5) does not hold in the category \mathbf{pqsMet} of all extended pseudo-quasi-semi metric spaces and nonexpansive maps. Let $X = \{a, b\}$ and d be defined as $d(a, a) = 0 = d(b, b), d(a, b) = \infty$ and $d(b, a) = 1$. By Theorem 3.4 of [7], both $\{a\}$ and $\{b\}$ are closed but by Theorem 3.3 of [14], (X, d) is not T_1 . This shows Theorem 3.12(6) does not hold in \mathbf{pqsMet} . If (R, d) is the indiscrete extended pseudo-quasi-semi metric space, i.e. $d(a, b) = 0$ for all $a, b \in R$, where R is the set of reel numbers, then by Theorem 3.13 of [14] and Theorem 3.13, (R, d) is both T_0 sober and KT_2 but by Theorem 3.4 of [14], (R, d) is neither $\overline{T_0}$ sober nor $\overline{T_0}$. This shows Theorem 3.12(2) is not valid in \mathbf{pqsMet} .

(3) By Theorem 3.6, a T_0 sober prebornological space is a quasi-sober, T'_0 sober, and $\overline{T_0}$ sober. The prebornological space $(R, P(R))$ is quasi-sober, T'_0 sober, and $\overline{T_0}$ sober but it is not T_0 sober, where R is the set of reel numbers. Moreover, By Theorem 3.6, all of quasi-sober prebornological spaces, T'_0 sober prebornological spaces, and $\overline{T_0}$ sober prebornological spaces are equivalent. By Theorem 3.6, every T_0 is $\overline{T_0}$ sober but the reverse implication may not be true. This shows Theorem 3.12(2) does not hold in \mathbf{PBorn} .

(4) By Theorem 3.9 of [2] and Definition 3.1, $cl = \iota$, the trivial closure operator [9] of \mathbf{PBorn} .

(5) By Theorem 3.7, each of a quasi-sober pair space, T'_0 sober pair space, T_0 sober pair space, and $\overline{T_0}$ sober pair space is T_1 . The pair space (R, N) is T_1 but it none of quasi-sober, T'_0 sober, T_0 sober, and $\overline{T_0}$ sober, where N is the set of natural numbers. Moreover, by Theorem 3.7, T_0 implies T'_0 and $\overline{T_0}$ and the pair space (R, N) is T'_0 and $\overline{T_0}$ but it is not T_0 .

(6) By Theorem 3.8 of [2] and Definition 3.1, $cl = \delta$, the discrete closure operator [9] of \mathbf{CP} .

(7) By Theorems 3.3 and 3.4 of [14], the indiscrete extended pseudo-quasi-semi metric space (R, d) is both T_0 sober and quasi-sober, (R, d) is neither T_1 nor $\overline{T_0}$. By Theorem 3.7, the pair space (R, N) is both $\overline{T_0}$ and T_1 but it is neither T_0 sober nor quasi-sober. By Theorem 3.6, being T_1 (resp. $\overline{T_0}$) and being $\overline{T_0}$ sober (resp. quasi-sober) are the same. This shows Theorem 3.12(3) and (4) do not hold in \mathbf{PBorn} .

(8) By Theorem 3.6, a nonempty prebornological space (X, \mathcal{F}) is $\overline{T_0}$ but $f : X \rightarrow S(X)$ is not injective, where $S(X)$ is the set of all irreducible closed subsets of X . This shows Theorem 3.12(7) does not hold in an arbitrary topological category.

- (9) By Theorem 3.8, \overline{T}_0 sober= T_0 sober (resp. $\overline{T}_0=T_0$) implies T'_0 sober (resp. T'_0). Let B be an infinite set. Then by Theorem 3.8, the indiscrete reflexive space (B, B^2) is T'_0 sober and quasi sober but it is neither T_0 sober nor T_0 . Moreover, a reflexive space (B, R) is T_0 iff $\{x\}$ is closed for each $x \in B$.
- (10) In arbitrary topological category, by Theorem 3.5, every \overline{T}_0 sober object is T'_0 sober. By Theorem 3.13 and Parts (2) and (3), there is no implication between T_0 sober and \overline{T}_0 sober.

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