Turkish Journal of Mathematics

Volume 46 | Number 1 Article 10

1-1-2022

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 130 – 144
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doi:10.3906/mat-2107-99

Geodesics of fiberwise cigar soliton deformation of the Sasaki metric

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Received: 28.07.2021 • **Accepted/Published Online:** 19.12.2021 • **Final Version:** 19.01.2022

Abstract: Ricci solitons arose in proof the Poincare conjecture by R. Hamilton and G. Perelman. The first example of a noncompact steady Ricci soliton on a plane was found by R. Hamilton. This two-dimensional manifold is conformally equivalent to the plane and it is called by R. Hamilton's cigar soliton. The cigar soliton metric can be considered as a fiber-wise conformal deformation of the Euclidean metric on a fiber of the tangent bundle. In the paper we propose a deformation of the classical Sasaki metric on the tangent bundle of an n-dimensional Riemannian manifold that induces the cigar soliton type metric on the fibers. The purpose of the research is to study geodesics of the cigar soliton deformation of the Sasaki metric on the tangent bundle of the Riemannian manifold with focus on the locally symmetric/constant curvature base manifold.

Key words: Sasaki metric, tangent bundle, geodesics

1. Introduction

A Riemannian manifold (M, g) is called a *Ricci soliton* if and only if there exist a smooth vector field V and constant λ such that the metric tensor g satisfies the equation

$$\frac{1}{2}\mathcal{L}_V g + Ric = \lambda g, \tag{1.1}$$

where $\mathcal{L}_V g$ is he Lie derivative of g along the vector field V, and Ric is the Ricci curvature tensor of the metric tensor g. The Ricci solitons are divided into three classes according to the sign of the constant λ. Namely,

- if $\lambda < 0$, then a Ricci soliton is called *expanding*;
- if $\lambda > 0$, then a Ricci soliton is called *shrinking*;
- if $\lambda = 0$, then a Ricci soliton is called *steady*.

If V is the gradient of some function F (potential function), then a Ricci soliton is called *gradient*. For a gradient Ricci soliton the equation (1.1) can be expressed as

$$Hess(F) + Ric = \lambda g, \tag{1.2}$$

where $Hess(F)_{ij} = \nabla_i \nabla_j F$ denotes the Hessian of the potential function.

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2010 *AMS Mathematics Subject Classification:* 53B20, 53B21, 53B99

Interest in Ricci solitons is associated with their appearance in the process of proving the famous Poincare conjecture by R. Hamilton and G. Perelman. There are a large number of publications on the geometry of Ricci solitons (see, e.g. [2–4] and references therein).

The first example of a noncompact steady Ricci soliton on a plane was found by R. Hamilton [6]. This two-dimensional manifold is conformally equivalent to a plane and it is called by R. Hamilton's *cigar soliton*. Its first fundamental form is as follows

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

Hamilton's cigar soliton is gradient with potential function $F = -\ln(1 + x^2 + y^2)$.

The standard metric on the tangent bundle of Riemannian manifold (M, g) is the Sasaki metric [10]. It can be completely defined by scalar products of various combinations of *vertical* and *horizontal* lifts of vector fields from the base to the tangent bundle by

$$G(X^h, Y^h) = g(X, Y), \quad G(X^h, Y^v) = 0, \quad G(X^v, Y^v) = g(X, Y).$$

(see Section 2 for details). Note that a vertical lift of a vector field is tangent to the fiber. Therefore, the Sasaki metric being descended to the fibers coincides with the metric of the base manifold. One can generalize the Sasaki metric definition allowing the fiber metric to be different from the base one. The latter idea brings another (nonflat in general) geometry to the fibers and the whole tangent bundle. A general question can be posed as follows: *to what extent the new fiber-wise metric changes the geometry of the tangent bundle?* In the present paper we analyze the case when the fiber-wise metric is the R. Hamilton's cigar soliton one and focus on geodesics of generalized Sasaki metric.

The general idea comes from the two-dimensional case. Namely, if E^2 is Euclidean plane with the Cartesian coordinates (x, y), then TE^2 with the Sasaki metric is Euclidean with the Cartesian coordinates $(x, y; \xi_1, \xi_2)$. The line element is of the form

$$d\sigma^2 = dx^2 + dy^2 + d\xi_1^2 + d\xi_2^2.$$

The "fiber part" of the latter line element can be deformed in the following way

$$d\sigma^2 = dx^2 + dy^2 + \frac{1}{1 + \xi_1^2 + \xi_2^2} \left(d\xi_1^2 + d\xi_2^2\right)$$

in order to get the cigar soliton metric on each fiber. In what follows we refer to the following Definition.

Definition 2.1 *Fiberwise Hamiltonian cigar soliton deformation of the Sasaki metric on the tangent bundle of the Riemannian manifold (M, g) is defined by*

$$G_Q(X^h, Y^h) = g_q(X, Y), \quad G_Q(X^h, Y^v) = 0, \quad G_Q(X^v, Y^v) = \frac{1}{1+t} \cdot g_q(X, Y),$$

where $Q = (q, \xi) \in TM$, $t = |\xi|^2$.

It is natural to consider a bit more general case involving the properties of R. Hamilton's deformating function according to the following definition.

Definition 2.2 *The fiberwise cigar soliton deformation of the Sasaki metric on the tangent bundle of the Riemannian manifold* (M,g) *of the form*

$$G_Q(X^h,Y^h)=g_q(X,Y),\quad G_Q(X^h,Y^v)=0,\quad G_Q(X^v,Y^v)=f(t)g_q(X,Y),$$

where $Q=(q,\xi)\in TM$, $t=|\xi|^2$, *and* f *is a smooth function with the properties* $f>0$, $f(0)=1$, $\lim_{t\to\infty}f=0$, $f'<0$, $\lim_{t\to\infty}f'=0$ *is called fiberwise cigar soliton deformation of the Sasaki metric.*

Remark that the fiberwise cigar soliton deformation is similar but is not the same as the metric on the tangent bundle introduced by M.T.K. Abbassi and M. Sarih [1], Cheeger-Gromoll metric [8] or fiber-wise deformed metric introduced by A. Yampolsky [14], but is a specific particular case of metric introduced by A. Zagane and M. Djaa in [15].

A regular parameterized curve $\Gamma(\sigma)\subset TM$ can be considered, in general, as a pair $\{x(\sigma),\xi(\sigma)\}$, where $x(\sigma)\subset M$ is a curve and $\xi(\sigma)$ is a vector field along $x(\sigma)$. As a result,

- we obtain the differential equations of naturally parameterized geodesics in terms of $x(\sigma)$ and $\xi(\sigma)$ with respect to Definitions 2.1 and 2.2 (Theorem 3.1);
- we prove that $f(t)|\xi'_\sigma|=c\ (=const)$, $\quad 0\le c\le 1$ and classify geodesics on TM with the fiberwise cigar soliton deformed Sasaki metric with respect to the parameter c namely: if $c=0$, then the geodesic is called *horizontal*; if $c=1$, then the geodesic is called *vertical*; if $0<c<1$, then the geodesic is called *oblique* (Definition 3.2);
- we prove (cf. [9]) that in case of locally symmetric base the projection of oblique geodesic $\Gamma(\sigma)$ to the base has all geodesic curvatures constant and in case of the base constant $k_i=0$ for all $k_i\ge 3$ (Theorem 4.2);
- we obtain (cf. [11]) the equations of geodesics on $T(E^n)$, $T(S^n)$ and $T(H^n)$ with the fiberwise (Hamiltonian) cigar soliton deformed Sasaki metric (Theorems 4.3, 4.4, 4.5, 4.6, 4.7).

2. Basic properties of the cigar soliton deformed Sasaki metric

Let (M,g) be n-dimensional Riemannian manifold with metric g. Denote by $g(\cdot,\cdot)$ a scalar product with respect to g. Denote by TM tangent bundle of (M,g). It is well known that at each point $Q=(q,\xi)\in TM$ the tangent space T_QTM splits into vertical and horizontal parts:

$$T_QTM=\mathcal{H}_QTM\oplus\mathcal{V}_QTM.$$

The vertical part $\mathcal{V}_Q$ is tangent to the fiber, while the horizontal part $\mathcal{H}_Q$ is transversal to it. Denote by $(x^1,\dots,x^n;\xi^1,\dots,\xi^n)$ the natural induced local coordinate system on TM. Denote $\partial_i=\frac{\partial}{\partial x^i}$, $\partial_{n+i}=\frac{\partial}{\partial\xi^i}$. Then for $\tilde X\in T_QTM$ we have $\tilde X=\tilde X^i\partial_i+\tilde X^{n+i}\partial_{n+i}$.

Denote by $\pi:TM\to M$ the tangent bundle projection. The mapping $\pi_*:T_QTM\to TM$ defines a point-wise linear isomorphism between $\mathcal{H}_Q(TM^n)$ and T_qM^n. Remark that $\ker\pi_*|_Q=\mathcal{V}_Q$.

The so-called connection mapping $\mathcal{K}:T_QTM\to T_qM$ acts on $\tilde X$ by $\mathcal{K}\tilde X=(\tilde X^{n+i}+\Gamma^i_{jk}\xi^j\tilde X^k)\partial_i$. Here Γ^i_{jk} are the Christoffel symbols of g. The connection mapping $\mathcal{K}$ defines a point-wise linear isomorphism between $\mathcal{V}_QTM$ and T_qM. Remark that $\ker\mathcal{K}|_Q=\mathcal{H}_Q$.

The images $\pi_* \tilde{X}$ and $\mathcal{K}\tilde{X}$ are called horizontal and vertical projections of $\tilde{X}$, respectively. The operations inverse to projections are called lifts. Namely, if $X \in T_qM$, then $X^h = X^i \partial_i - \Gamma^i_{jk}\xi^j X^k \partial_{n+i}$ is in $\mathcal{H}_Q TM$ and is called the horizontal lift of X, and $X^v = X^i \partial_{n+i}$ is in $\mathcal{V}_Q TM$ and is called the vertical lift of X.

Let $\tilde{X}, \tilde{Y} \in T_Q TM$. The standard Sasaki metric on TM is defined at each point $Q = (q, \xi) \in TM$ by the following scalar product

$$G(\tilde{X}, \tilde{Y})\big|_Q = g(\pi_* \tilde{X}, \pi_* \tilde{Y})\big|_q + g(K\tilde{X}, K\tilde{Y})\big|_q.$$

Horizontal and vertical subspaces are mutually orthogonal with respect to Sasaki metric.

The Sasaki metric can be completely defined by scalar product of various combinations of lifts of vector fields from M to TM by

$$G_Q(X^h, Y^h) = g_q(X, Y), \quad G_Q(X^h, Y^v) = 0, \quad G_Q(X^v, Y^v) = g_q(X, Y).$$

Define the fiberwise cigar soliton deformation of the Sasaki metric as follows.

Definition 2.1 *Fiberwise Hamiltonian cigar soliton deformation of the Sasaki metric on the tangent bundle of the Riemannian manifold* (M, g) *is defined by*

$$G_Q(X^h, Y^h) = g_q(X, Y), \quad G_Q(X^h, Y^v) = 0, \quad G_Q(X^v, Y^v) = \frac{1}{1+t} \cdot g_q(X, Y),$$

where $Q = (q, \xi) \in TM$, $t = |\xi|^2$.

Definition 2.2 *The fiberwise cigar soliton deformation of the Sasaki metric on the tangent bundle of the Riemannian manifold* (M, g) *of the form*

$$G_Q(X^h, Y^h) = g_q(X, Y), \quad G_Q(X^h, Y^v) = 0, \quad G_Q(X^v, Y^v) = f(t) g_q(X, Y),$$

where $Q = (q, \xi) \in TM$, $t = |\xi|^2$, *and* f *is a smooth function with the properties* $f > 0$, $f(0) = 1$, $\lim_{t\to\infty} f = 0$, $f' < 0$, $\lim_{t\to\infty} f' = 0$ *is called fiberwise cigar soliton deformation of the Sasaki metric.*

Let R be the curvature tensor of ∇. Denote by $\tilde{\nabla}$ the Levi-Civita connection of the cigar soliton deformed Sasaki metric G. The following lemma contains Kowalski-type formulas [7] and is the main tool for the further considerations (see also [15]).

Lemma 2.3 *Let* (M, g) *be the Riemannian manifold. The Levi-Civita connection* $\tilde{\nabla}$ *of the fiberwise cigar soliton deformed Sasaki metric* G *on the tangent bundle* TM *is completely defined by*

$$\tilde{\nabla}_{X^h} Y^h = \Big(\nabla_X Y\Big)^h - \frac{1}{2}\Big(R(X, Y)\xi\Big)^v, \tag{2.1}$$

$$\tilde{\nabla}_{X^h} Y^v = \frac{1}{2} f(t) \Big(R(\xi, Y)X\Big)^h + \Big(\nabla_X Y\Big)^v, \tag{2.2}$$

$$\tilde{\nabla}_{X^v} Y^h = \frac{1}{2} f(t) \Big(R(\xi, X)Y\Big)^h, \tag{2.3}$$

$$\tilde{\nabla}_{X^v} Y^v = \big(\ln f(t)\big)'\Big(g(X,\xi)Y + g(Y,\xi)X - g(X,Y)\xi\Big)^v, \tag{2.4}$$

where ∇ *is the Levi-Civita connection on* (M,g), R *is the curvature tensor of* ∇.

Proof Remark, first, that the following formulas are independent on the choice of tangent bundle metric and are known as Dombrowski formulas [5]: at each point $Q=(q,\xi)\in TM$ the brackets of lifts of vector fields from M to TM are

$$\big[X^h, Y^h\big] = \big[X,Y\big]^h - \big(R(X,Y)\xi\big)^v, \quad \big[X^h, Y^v\big] = \big(\nabla_X Y\big)^v, \quad \big[X^v, Y^v\big] = 0.$$

Prove, now, that the derivative of the function $f(t)$ along the lifts of vector fields from M to TM are (cf. [13])

$$X^h(f) = 0, \quad X^v(f) = 2f'(t)g(X,\xi).$$

Indeed, keeping in mind $X^h = X^i\frac{\partial}{\partial u^i} - \Gamma^i_{jk}X^j\xi^k\frac{\partial}{\partial \xi^i}$ and $X^v = X^i\frac{\partial}{\partial \xi^i}$, we have

$$\begin{aligned}
X^h(f) &= X^i\frac{\partial}{\partial u^i}(f) - \Gamma^i_{jk}X^j\xi^k\frac{\partial}{\partial \xi^i}(f)\\
&= f'_t\Big(X^i\frac{\partial}{\partial u^i}(g_{sp}\xi^s\xi^p) - \Gamma^i_{jk}X^j\xi^k\frac{\partial}{\partial \xi^i}(g_{sp}\xi^s\xi^p)\Big)\\
&= f'_t\Big(X^i(\Gamma^m_{pi}g_{ms} + \Gamma^m_{si}g_{mp})\xi^s\xi^p - 2\Gamma^i_{jk}X^j\xi^k g_{ip}\xi^p)\Big)\\
&= f'_t\Big(\Gamma_{ip,s}X^i\xi^s\xi^p + \Gamma_{is,p}X^i\xi^s\xi^p - 2\Gamma_{jk,p}X^j\xi^k\xi^p\Big)\\
&= f'_t\Big(2\Gamma_{ip,s}X^i\xi^s\xi^p - 2\Gamma_{jk,p}X^j\xi^k\xi^p\Big) = 0.
\end{aligned}$$

$$X^v(f) = X^i\frac{\partial}{\partial \xi^i}(f) = f'_t X^i\frac{\partial}{\partial \xi^i}(g_{sp}\xi^s\xi^p) = 2f'_t g_{ip}X^i\xi^p = 2f'_t\, g(X,\xi).$$

Finally, the derivative of cigar soliton deformed Sasaki metric G along the lifts of vector fields from M to TM are

$$X^hG(Y^h,Z^h) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{2.5}$$

$$X^hG(Y^v,Z^v) = f(t)\Big(g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\Big), \tag{2.6}$$

$$X^vG(Y^h,Z^h) = 0, \tag{2.7}$$

$$X^vG(Y^v,Z^v) = 2f'(t)g(X,\xi)g(Y,Z). \tag{2.8}$$

Now we can use the Koszul formula. We have

$$\begin{aligned}
2G(\tilde{\nabla}_{X^h}Y^h, Z^h) &= X^hG(Y^h, Z^h) + Y^hG(X^h, Z^h) - Z^hG(X^h, Y^h) \\
&+ G([X^h, Y^h], Z^h) - G([X^h, Z^h], Y^h) - G([Y^h, Z^h], X^h) \\
&= Xg(Y^h, Z^h) + Yg(X^h, Z^h) - Zg(X^h, Y^h) \\
&+ G([X, Y]^h, Z^h) - G([X, Z]^h, Y^h) - G([Y, Z]^h, X^h) \\
&= Xg(Y^h, Z^h) + Yg(X^h, Z^h) - Zg(X^h, Y^h) \\
&+ g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) = 2g(\nabla_X Y, Z) = 2G((\nabla_X Y)^h, Z^h);
\end{aligned}$$

$$\begin{aligned}
2G(\tilde{\nabla}_{X^h}Y^h, Z^v) &= X^hG(Y^h, Z^v) + Y^hG(X^h, Z^v) - Z^vG(X^h, Y^h) \\
&+ G([X^h, Y^h], Z^v) - G([X^h, Z^v], Y^h) - G([Y^h, Z^v], X^h) = -G((R(X, Y)\xi)^v, Z^v)
\end{aligned}$$

and get (2.1). Then

$$\begin{aligned}
2G(\tilde{\nabla}_{X^h}Y^v, Z^h) &= X^hG(Y^v, Z^h) + Y^vG(X^h, Z^h) - Z^hG(X^h, Y^v) \\
&+ G([X^h, Y^v], Z^h) - G([X^h, Z^h], Y^v) - G([Y^v, Z^h], X^h) \\
&= G((R(X, Z)\xi)^v, Y^v) = fg(R(X, Z)\xi, Y) = fg(R(\xi, Y)X, Z) \\
&= g(fR(\xi, Y)X, Z) = G(f(R(\xi, Y)X)^h, Z^h);
\end{aligned}$$

$$\begin{aligned}
2G(\tilde{\nabla}_{X^h}Y^v, Z^v) &= X^hG(Y^v, Z^v) + Y^vG(X^h, Z^v) - Z^hG(X^h, Y^v) \\
&+ G([X^h, Y^v], Z^v) - G([X^h, Z^v], Y^v) - G([Y^v, Z^v], X^h) \\
&= f(g(\nabla_X Y, Z) + g(Y, \nabla_X Z)) + G((\nabla_X Y)^v, Z^v) - G((\nabla_X Z)^v, Y^v) \\
&= f(g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_X Y, Z) - g(\nabla_X Z, Y)) \\
&= 2fg(\nabla_X Y, Z) = 2G((\nabla_X Y)^v, Z^v)
\end{aligned}$$

and get (2.2). In a similar way

$$\begin{aligned}
2G(\tilde{\nabla}_{X^v}Y^h, Z^h) &= X^vG(Y^h, Z^h) + Y^hG(X^v, Z^h) - Z^hG(X^v, Y^h) \\
&+ G([X^v, Y^h], Z^h) - G([X^v, Z^h], Y^h) - G([Y^h, Z^h], X^v) \\
&= G((R(Y, Z)\xi)^v, X^v) = fg(R(Y, Z)\xi, X) = fg(R(\xi, X)Y, Z) \\
&= g(fR(\xi, X)Y, Z) = G(f(R(\xi, X)Y)^h, Z^h);
\end{aligned}$$

$$
\begin{aligned}
2G(\tilde{\nabla}_{X^v}Y^h, Z^v) &= X^v G(Y^h, Z^v) + Y^h G(X^v, Z^v) - Z^v G(X^v, Y^h) \\
&\quad + G([X^v, Y^h], Z^v) - G([X^v, Z^v], Y^h) - G([Y^h, Z^v], X^v) \\
&= f(g(\nabla_Y X, Z) + g(X, \nabla_Y Z)) - G((\nabla_Y X)^v, Z^v) - G((\nabla_Y Z)^v, X^v) \\
&= f(g(\nabla_Y X, Z) + g(X, \nabla_Y Z) - g(\nabla_Y X, Z) - g(\nabla_Y Z, X)) = 0
\end{aligned}
$$

and get (2.3). Finally,

$$
\begin{aligned}
2G(\tilde{\nabla}_{X^v}Y^v, Z^h) &= X^v G(Y^v, Z^h) + Y^v G(X^v, Z^h) - Z^h G(X^v, Y^v) \\
&\quad + G([X^v, Y^v], Z^h) - G([X^v, Z^h], Y^v) - G([Y^v, Z^h], X^v) \\
&= -f(g(\nabla_Z X, Y) + g(X, \nabla_Z Y)) + G((\nabla_Z X)^v, Y^v) + G((\nabla_Z Y)^v, X^v) \\
&= f(-g(\nabla_Z X, Y) - g(X, \nabla_Z Y) + g(\nabla_Z X, Y) + g(\nabla_Z Y, X)) = 0;
\end{aligned}
$$

$$
\begin{aligned}
2G(\tilde{\nabla}_{X^v}Y^v, Z^v) &= X^v G(Y^v, Z^v) + Y^v G(X^v, Z^v) - Z^v G(X^v, Y^v) \\
&\quad + G([X^v, Y^v], Z^v) - G([X^v, Z^v], Y^v) - G([Y^v, Z^v], X^v) \\
&= 2f'_t(g(X,\xi)g(Y,Z) + g(Y,\xi)g(X,Z) - g(Z,\xi)g(X,Y)) \\
&= 2f'_t(g(g(X,\xi)Y, Z) + g(g(Y,\xi)X, Z) - g(g(X,Y)\xi, Z)) \\
&= 2f'_t g(g(X,\xi)Y + g(Y,\xi)X - g(X,Y)\xi, Z) \\
&= 2fg(\frac{f'_t}{f}(g(X,\xi)Y + g(Y,\xi)X - g(X,Y)\xi), Z) \\
&= 2G(\frac{f'_t}{f}(g(X,\xi)Y + g(Y,\xi)X - g(X,Y)\xi)^v, Z^v)
\end{aligned}
$$

and get (2.4). □

3. Geodesics

Let $\Gamma = \{x(\sigma), \xi(\sigma)\}$ be a naturally parameterized curve on the tangent bundle TM with the fiberwise cigar soliton deformed Sasaki metric G. Denote $x'_\sigma = \frac{dx}{d\sigma}$, $x''_\sigma = \nabla_{\frac{dx}{d\sigma}} x'_\sigma$, $\xi'_\sigma = \nabla_{\frac{dx}{d\sigma}} \xi$, $\xi''_\sigma = \nabla_{\frac{dx}{d\sigma}} \xi'_\sigma$. Then

$$
\Gamma'_\sigma = (x'_\sigma)^h + (\xi'_\sigma)^v, \quad |\Gamma'_\sigma|^2 = |x'_\sigma|^2 + f(t)|\xi'_\sigma|^2 = 1. \tag{3.1}
$$

Now we derive the differential equations of geodesics (see also [16]).

Theorem 3.1 *Let (M, g) be Riemannian manifold and TM its tangent bundle with the fiberwise cigar soliton deformed Sasaki metric, R is the curvature operator of the base manifold M. A naturally parameterized curve $\Gamma = \{x(\sigma), \xi(\sigma)\}$ is geodesic on TM if and only if $x(\sigma)$ and $\xi(\sigma)$ satisfy the equations*

$$
x''_\sigma + f(t)R(\xi, \xi'_\sigma)x'_\sigma = 0, \tag{3.2}
$$

$$
\xi''_\sigma + \big(\ln f(t)\big)'\big((|\xi|^2)'_\sigma \xi'_\sigma - |\xi'_\sigma|^2 \xi\big) = 0. \tag{3.3}
$$

Respectively the equations of geodesic lines on the tangent bundle with the fiberwise Hamiltonian cigar soliton deformed Sasaki metric are

$$x''_\sigma + \frac{1}{1+|\xi|^2} R(\xi, \xi'_\sigma) x'_\sigma = 0, \tag{3.4}$$

$$\xi''_\sigma - \frac{2g(\xi'_\sigma, \xi)}{1+|\xi|^2} \xi'_\sigma + \frac{|\xi'_\sigma|^2}{1+|\xi|^2} \xi = 0. \tag{3.5}$$

Proof The curve Γ is geodesic if and only if $\tilde{\nabla}_{\Gamma'_\sigma} \Gamma'_\sigma = 0$. Since we have

$$\begin{aligned}
\tilde{\nabla}_{\Gamma'_\sigma} \Gamma'_\sigma &= \tilde{\nabla}_{((x'_\sigma)^h + (\xi'_\sigma)^v)} ((x'_\sigma)^h + (\xi'_\sigma)^v) \\
&= \tilde{\nabla}_{(x'_\sigma)^h} (x'_\sigma)^h + \tilde{\nabla}_{(x'_\sigma)^h} (\xi'_\sigma)^v + \tilde{\nabla}_{(\xi'_\sigma)^v} (x'_\sigma)^h + \tilde{\nabla}_{(\xi'_\sigma)^v} (\xi'_\sigma)^v \\
&= (\nabla_{x'_\sigma} x'_\sigma)^h - \frac{1}{2}(R(x'_\sigma, x'_\sigma)\xi)^v + \frac{1}{2} f (R(\xi, \xi'_\sigma) x'_\sigma)^h + (\nabla_{x'_\sigma} \xi'_\sigma)^v \\
&+ \frac{1}{2} f (R(\xi, \xi'_\sigma) x'_\sigma)^h + \big(\ln f\big)' (g(\xi'_\sigma, \xi)\xi'_\sigma + g(\xi'_\sigma, \xi)\xi'_\sigma - g(\xi'_\sigma, \xi'_\sigma)\xi)^v \\
&= (x''_\sigma + f R(\xi, \xi'_\sigma) x'_\sigma)^h + (\xi''_\sigma + \big(\ln f\big)' (2g(\xi'_\sigma, \xi)\xi'_\sigma - |\xi'_\sigma|^2 \xi))^v = 0,
\end{aligned}$$

so we have (3.2) and (3.3). Put $f = \frac{1}{1+t}$ in (3.2) and (3.3). Since $f'_t = -\frac{1}{(1+t)^2}$, $\big(\ln f(t)\big)' = \frac{f'_t}{f} = -\frac{1}{1+t}$ and $t = |\xi|^2$, it follows that $x''_\sigma + \frac{1}{1+|\xi|^2} R(\xi, \xi'_\sigma) x'_\sigma = 0$, $\xi''_\sigma - \frac{1}{1+|\xi|^2}\big(2g(\xi'_\sigma, \xi)\sigma' - |\xi'_\sigma|^2 \xi\big) = 0$ and then we get (3.4) and (3.5). □

Lemma 3.2 *Let (M, g) be Riemannian manifold and TM its tangent bundle with the fiberwise cigar soliton deformed Sasaki metric, $\Gamma = \{x(\sigma), \xi(\sigma)\}$ be a geodesic curve on TM. Then*

$$f(t)|\xi'_\sigma|^2 = c^2, \tag{3.6}$$

where $c = const$, $0 \leq c \leq 1$. As a consequence,

$$|\xi'_\sigma|^2 = c^2(1+|\xi|^2), \quad \textit{where } 0 \leq c \leq 1,\ c = const \tag{3.7}$$

in Hamiltonian case.

Proof Since s is an arc length parameter on $x(s)$, using (3.1) we have

$$\frac{ds}{d\sigma} = |x'_\sigma| = \sqrt{1 - f|\xi'_\sigma|^2}. \tag{3.8}$$

Using (3.3), we get

$$\xi''_\sigma = -\frac{f'_t}{f}(2g(\xi'_\sigma, \xi)\xi'_\sigma - |\xi'_\sigma|^2 \xi). \tag{3.9}$$

On the one hand,

$$g(\xi'_\sigma, \xi''_\sigma) = \frac{1}{2}\frac{d}{d\sigma}(|\xi'_\sigma|^2). \tag{3.10}$$

On the other hand, using (3.9) we obtain

$$g(\xi'_\sigma, \xi''_\sigma) = -\frac{f'_t}{f} g(\xi'_\sigma, \xi)|\xi'_\sigma|^2. \tag{3.11}$$

Now if we recall $t = |\xi|^2$, we get $f'_\sigma = f'_t \frac{dt}{d\sigma} = 2f'_t g(\xi'_\sigma, \xi)$. Therefore, $g(\xi'_\sigma, \xi) = \frac{1}{2}\frac{f'_\sigma}{f'_t}$. Substituting it in (3.11) and using (3.10), we obtain

$$\frac{1}{2}\frac{d}{d\sigma}(|\xi'_\sigma|^2) = -\frac{1}{2}\frac{f'_\sigma}{f}|\xi'_\sigma|^2 \Rightarrow \int \frac{d(|\xi'_\sigma|^2)}{|\xi'_\sigma|^2} = -\int \frac{f'_\sigma}{f} d\sigma \Rightarrow$$

$$\Rightarrow |\xi'_\sigma|^2 = \frac{c^2}{f}. \tag{3.12}$$

Combining (3.8) and (3.12), we complete the proof of the lemma. □

Definition 3.3 (Classification of geodesics) *According to the Lemma 3.2, the set of geodesics of TM with the cigar soliton deformed Sasaki metric can be splitted naturally into 3 classes, namely*

- *horizontal geodesics $(c = 0)$ generated by parallel vector fields along the geodesics on the base manifold;*
- *vertical geodesics $(c = 1)$ represented by geodesics on a fixed fiber, their equations are*

$$\xi''_\sigma + \big(\ln f(t)\big)'\big((|\xi|^2)'_\sigma \xi'_\sigma - \frac{1}{f(t)}\xi\big) = 0; \tag{3.13}$$

- *oblique geodesics $(0 < c < 1)$ satisfy the equations*

$$x''_\sigma + f(t)R(\xi, \xi'_\sigma)x'_\sigma = 0, \tag{3.14}$$

$$\xi''_\sigma + \big(\ln f(t)\big)'\big((|\xi|^2)'_\sigma \xi'_\sigma - \frac{c^2}{f(t)}\xi\big) = 0. \tag{3.15}$$

Respectively if we put $f = \frac{1}{1+t}$ and $t = |\xi|^2$ in (3.13), (3.14), and (3.15), then the vertical and oblique geodesic on TM with the fiberwise *Hamiltonian* cigar soliton deformed Sasaki metric can be expressed as follows

- vertical geodesics $(c = 1)$ represented by geodesics on a fixed fiber, their equations are

$$\xi''_\sigma - \frac{2g(\xi'_\sigma, \xi)}{1 + |\xi|^2}\xi'_\sigma + \xi = 0; \tag{3.16}$$

- oblique geodesics $(0 < c < 1)$ satisfy the equations

$$x''_\sigma + \frac{1}{1 + |\xi|^2}R(\xi, \xi'_\sigma)x'_\sigma = 0, \tag{3.17}$$

$$\xi''_\sigma - \frac{2g(\xi'_\sigma, \xi)}{1 + |\xi|^2}\xi'_\sigma + c^2\xi = 0. \tag{3.18}$$

In what follows, we study oblique geodesics in more details.

4. Oblique geodesics on the tangent bundle of manifolds of constant sectional curvature

Consider some properties of the curvature operator of Riemannian manifold of constant curvature. Define a power of curvature operator $R^p(X,Y)$ recurrently in the following way: $R^p(X,Y)Z = R^{p-1}(X,Y)R(X,Y)Z$ for $p \geq 2$. Note that the following statements for the curvature tensor of Riemannian manifold of constant curvature ε hold:

- $R'_\sigma = 0$, namely, every space of constant curvature is locally symmetric;
- $R(X,Y)Z = \varepsilon(g(Y,Z)X - g(X,Z)Y)$.

The proof of the following lemma is in [12].

Lemma 4.1 *Let (M,g) be n-dimensional Riemannian manifold of constant curvature ε. Then for any X and Y*

$$R^p(X,Y) = \begin{cases} (-B^2\varepsilon^2)^{k-1}R(X,Y), & \text{for } p = 2k-1 \\ (-B^2\varepsilon^2)^{k-1}R^2(X,Y), & \text{for } p = 2k \end{cases},$$

where $k \geq 1$ and $B^2 = |X|^2|Y|^2 - g(X,Y)^2$ is the square of norm of bivector $X \wedge Y$.

Theorem 4.2 *Let (M,g) be a locally symmetric n-dimensional Riemannian manifold, a naturally parameterized curve $\Gamma = \{x(\sigma), \xi(\sigma)\}$ be an oblique geodesic on TM^n with the fiberwise cigar soliton deformed Sasaki metric, $\gamma = \pi \circ \Gamma = x(\sigma)$ be projection of the oblique geodesic Γ on the base manifold. Let $k_1, k_2, k_3, ..., k_{n-1}$ be geodesic curvatures of $x(\sigma)$. Then $k_i \equiv const$ for $i \geq 1$. In a particular case, if (M,g) is a Riemannian manifold of constant curvature, then $k_i \equiv 0$ for $i \geq 3$.*

Proof Using (3.2), we have: $x''_\sigma = fR(\xi'_\sigma, \xi)x'_\sigma$. It is easy to see that $g(x''_\sigma, x'_\sigma) = fg(R(\xi'_\sigma,\xi)x'_\sigma, x'_\sigma) = 0$, hence $|x'_\sigma| \equiv const$. Calculate the third derivative.

$$\begin{aligned} x'''_\sigma &= f'_t \cdot 2g(\xi'_\sigma,\xi)R(\xi'_\sigma,\xi)x'_\sigma + fR(\xi''_\sigma,\xi)x'_\sigma + fR(\xi'_\sigma,\xi)x''_\sigma \\ &= 2f'_t g(\xi'_\sigma,\xi)R(\xi'_\sigma,\xi)x'_\sigma - 2f'_t g(\xi'_\sigma,\xi)R(\xi'_\sigma,\xi)x'_\sigma + fR(\xi'_\sigma,\xi)x''_\sigma = fR(\xi'_\sigma,\xi)x''_\sigma. \end{aligned}$$

On the one hand, $x'''_\sigma = fR(\xi'_\sigma,\xi)x''_\sigma$. Since $g(x'''_\sigma, x''_\sigma) = fg(R(\xi'_\sigma,\xi)x''_\sigma, x''_\sigma) = 0$, it follows that $|x''_\sigma| \equiv const$. Continuing the process we obtain

$$x^{(p)}_\sigma = fR(\xi'_\sigma,\xi)x^{(p-1)}_\sigma, \qquad |x^{(p)}_\sigma| \equiv const, \qquad p \geq 2. \tag{4.1}$$

On the other hand, $x'''_\sigma = fR(\xi'_\sigma,\xi)x''_\sigma = f^2R^2(\xi'_\sigma,\xi)x'_\sigma$. Therefore continuing the process we obtain

$$x^{(p)}_\sigma = f^{p-1}R^{p-1}(\xi'_\sigma,\xi)x'_\sigma. \tag{4.2}$$

Denote by $\nu_1, ..., \nu_n$ the Frenet frame along $\gamma = x(s)$, where s is an arc length parameter. Then the Frenet formulas hold

$$\begin{cases} (\nu_1)'_s = k_1\nu_2, \\ (\nu_i)'_s = -k_{i-1}\nu_{i-1} + k_i\nu_{i+1}, & \text{for } i = 2, ..., n-1 \\ (\nu_n)'_s = -k_{n-1}\nu_{n-1}. \end{cases}$$

Now if we recall $\frac{ds}{d\sigma} = (1-c^2)^{1/2}$, we get $\nu_1 = x'_s = x'_\sigma \frac{d\sigma}{ds} = x'_\sigma(1-c^2)^{-1/2}$. Therefore, $x'_\sigma = (1-c^2)^{1/2}\nu_1$. Using the Frenet formulas, we obtain

$$x''_\sigma = (1-c^2)^{1/2}(\nu_1)'_\sigma = (1-c^2)^{1/2}(\nu_1)'_s\frac{ds}{d\sigma} = (1-c^2)k_1\nu_2.$$

Since $|x''_\sigma| \equiv const$, it follows that $k_1 \equiv const$.

In a similar way, we have

$$x'''_\sigma = (1-c^2)k_1(\nu_2)'_\sigma = (1-c^2)k_1(\nu_2)'_s\frac{ds}{d\sigma} = -(1-c^2)^{3/2}k_1^2\nu_1 + (1-c^2)^{3/2}k_1k_2\nu_3,$$

and since $|x'''_\sigma| \equiv const$, then $k_2 \equiv const$.

On the one hand, $x^{(4)}_\sigma = -(1-c^2)^2k_1^3\nu_2 - (1-c^2)^2k_1k_2^2\nu_2 + (1-c^2)^2k_1k_2k_3\nu_4 = -(1-c^2)^2k_1(k_1^2+k_2^2)\nu_2 + (1-c^2)^2k_1k_2k_3\nu_4$.

On the other hand, $x^{(4)}_\sigma = f^3R^3(\xi'_\sigma,\xi)x'_\sigma = -B^2f^3R(\xi'_\sigma,\xi)x'_\sigma = -B^2f^2x''_\sigma$, where $B^2 = \frac{c^2}{f(t)}|\xi|^2 - g(\xi',\xi)^2$.

Let $k_1, k_2 \neq 0$. Then

$$-(1-c^2)^2k_1(k_1^2+k_2^2)\nu_2 + (1-c^2)^2k_1k_2k_3\nu_4 = -B^2f^2(1-c^2)k_1\nu_2,$$

$$(B^2f^2 - (1-c^2)(k_1^2+k_2^2))\nu_2 + (1-c^2)k_2k_3\nu_4 = 0.$$

Therefore, we have $k_3 = 0$, and $B^2f^2 = (1-c^2)(k_1^2+k_2^2)$. Finally, we obtain $k_i = 0$ for $i \geq 3$ and $B^2f^2 \equiv const$, where we denote $\omega^2 = B^2f^2$. □

Consider the oblique geodesics on TM of manifolds of constant sectional curvature ε with the cigar soliton deformed Sasaki metric. Consider the following cases: $\varepsilon = 0$, 1 or -1 according as M is E^n, S^n or H^n.

The following theorem holds for the oblique geodesics on TM of Euclidean space with the cigar soliton deformed Sasaki metric.

Theorem 4.3 *Any oblique geodesics on $T(E^n)$ with the fiberwise cigar soliton deformed Sasaki metric is a vector field ξ which moves along a straight line by*

$$\xi''_\sigma + \big(\ln f(t)\big)'\big((|\xi|^2)'_\sigma\xi'_\sigma - \frac{c^2}{f(t)}\xi\big) = 0. \tag{4.3}$$

Respectively the oblique geodesic on $T(E^n)$ with the fiberwise Hamiltonian cigar soliton deformed Sasaki metric is a vector field ξ which moves along a straight line by

$$\xi''_\sigma - \frac{2g(\xi'_\sigma,\xi)}{1+|\xi|^2}\xi'_\sigma + c^2\xi = 0. \tag{4.4}$$

Proof In the case of Euclidean space we have $R = 0$, then, using (3.14), we get $x''_\sigma = 0$. Since $\frac{ds}{d\sigma} = \sqrt{1-c^2} \equiv const$, then the curve $x(\sigma)$ is geodesic in Euclidean space, namely, $x(\sigma)$ is a straight line. □

Now we consider oblique geodesics on the tangent bundle of sphere ($\varepsilon = 1$) and hyperbolic space ($\varepsilon = -1$) with the cigar soliton deformed Sasaki metric. In these cases the geodesic equation are

$$x''_\sigma = \varepsilon f(t)(a\xi'_\sigma - b\xi), \tag{4.5}$$

$$\xi''_\sigma + \big(\ln f(t)\big)'\big((|\xi|^2)'_\sigma \xi'_\sigma - |\xi'_\sigma|^2\xi\big) = 0, \tag{4.6}$$

where $a = g(\xi, x'_\sigma)$, $b = g(\xi'_\sigma, x'_\sigma)$. Since the theorem 4.2 holds for geodesic curvatures k_1, k_2, $k_3, ..., k_{n-1}$ of projection of the geodesic on TM, then we can classify projection of the geodesic as follow:

- $k_1 = 0$, namely, the projection of the geodesic on TM is geodesic on the base manifold M;
- $k_1 > 0$, $k_2 \geq 0$.

Therefore, we have the following theorem.

Theorem 4.4 *Let M be S^n or H^n, TM be the tangent bound with the fiberwise cigar soliton deformed Sasaki metric and $k_1 = 0$, namely, let the projection $\gamma = \pi \circ \Gamma = x(\sigma)$ of the geodesic on TM be a geodesic on sphere S^n or hyperbolic space H^n. If along $x(\sigma)$ choose orthonormal frame $e_1(\sigma) = \frac{x'(\sigma)}{|x'(\sigma)|}, e_2(\sigma), ..., e_n(\sigma)$, consisting of parallel vector fields along $x(\sigma)$, and expand the vector field $\xi(\sigma) = y_1(\sigma)e_1(\sigma) + y_2(\sigma)e_2(\sigma) + ... + y_n(\sigma)e_n(\sigma)$, then the coordinate functions $y_i(\sigma)$ can be found from the following system*

$$\begin{cases} y_i = \alpha_i y_1, \quad i = 2, ..., n \\ y_1'' + \frac{c^2}{f(t)}(\ln f(t))'_t y_1 = 0, \\ (1 + \alpha_2^2 + ... + \alpha_n^2) f(t)(y_1')^2 = c^2, \quad \alpha_i \equiv const. \end{cases}$$

Proof If we recall $x' = (1 - c^2)^{1/2} e_1$, we get

$$a = g(\xi, x') = g(y_1 e_1 + y_2 e_2 + ... + y_n e_n, (1 - c^2)^{1/2} e_1) = (1 - c^2)^{1/2} y_1,$$

$$b = g(\xi', x') = g(y_1' e_1 + y_2' e_2 + ... + y_n' e_n, (1 - c^2)^{1/2} e_1) = (1 - c^2)^{1/2} y_1'.$$

Using $x'' = \varepsilon f(a\xi' - b\xi)$, we have

$$(1 - c^2)^{1/2} y_1 (y_1' e_1 + y_2' e_2 + ... + y_n' e_n) - (1 - c^2)^{1/2} y_1' (y_1 e_1 + y_2 e_2 + ... + y_n e_n) = 0,$$

$$y_1 y_i' - y_1' y_i = 0, \quad i = 2, ..., n,$$

$$\frac{y_i'}{y_i} = \frac{y_1'}{y_1}, \quad i = 2, ..., n,$$

$$\int (\ln y_i)'_\sigma d\sigma = \int (\ln y_1)'_\sigma d\sigma, \quad i = 2, ..., n,$$

$$y_i = \alpha_i y_1, \quad i = 2, ..., n.$$

Using $f|\xi'|^2 = c^2$, we have $(y_1')^2 + (y_2')^2 + ... + (y_n')^2 = \frac{c^2}{f}$, and since $y_i = \alpha_i y_1$, we obtain

$$(y_1')^2 + (y_2')^2 + ... + (y_n')^2 = (y_1')^2 + (\alpha_2^2 + ... + \alpha_n^2)(y_1')^2 = (1 + \alpha_2^2 + ... + \alpha_n^2)(y_1')^2 = \frac{c^2}{f}$$

If we denote $A_n^2 = 1 + \alpha_2^2 + ... + \alpha_n^2$, then $(y_1')^2 = \frac{c^2}{A_n^2 f}$. Therefore, using $\xi'' + \big(\ln f\big)'\big((|\xi|^2)'\xi' - \frac{c^2}{f}\xi\big) = 0$, we get

$$y_1'' + (\ln f)'(2A_n^2 y_1 (y_1')^2 - \frac{c^2}{f} y_1) = 0 \Rightarrow y_1'' + \frac{c^2}{f}(\ln f)_t' y_1 = 0.$$

□

Corollary 4.5 *Under the conditions of the Theorem 4.4 in the case of the fiberwise Hamiltonian cigar soliton deformed Sasaki metric the solution of the system can be expressed as follows*

$$\begin{cases} y_1 = \beta^+ e^{c\sigma} + \beta^- e^{-c\sigma} \\ y_i = \alpha_i y_1, \quad i = 2, ..., n \\ (1 + \alpha_2^2 + ... + \alpha_n^2)\beta^+\beta^- = -\frac{1}{4} \end{cases},$$

where $\alpha_i, \beta^+, \beta^- \equiv const$, $\beta^+ > 0$, $\beta^- < 0$.

Proof Let $k_1 = 0$. If $f(t) = \frac{1}{1+t}$, then $y_1'' + \frac{c^2}{f(t)}(\ln f(t))' y_1 = 0$ can be expressed as $y_1'' - c^2 y_1 = 0$ with the solution $y_1 = \beta^+ e^{c\sigma} + \beta^- e^{-c\sigma}$. Substituting it in $A_n^2 f(t)(y_1')^2 = c^2$, where $A_n^2 = 1 + \alpha_2^2 + ... + \alpha_n^2$, we get

$$\frac{A_n^2 c^2 (\beta^+ e^{c\sigma} - \beta^- e^{-c\sigma})^2}{1 + A_n^2(\beta^+ e^{c\sigma} + \beta^- e^{-c\sigma})^2} = c^2, \quad A_n^2(\beta^+ e^{c\sigma} - \beta^- e^{-c\sigma})^2 = 1 + A_n^2(\beta^+ e^{c\sigma} + \beta^- e^{-c\sigma})^2,$$

$$-4A_n^2 \beta^+\beta^- = 1, \qquad A_n^2 \beta^+\beta^- = -\frac{1}{4}.$$

□

Theorem 4.6 *Let* M *be* S^n *or* H^n, TM *be the tangent bound with the fiberwise cigar soliton deformed Sasaki metric and* $k_1 > 0$, $k_2 \geq 0$. *Let* $\nu_1, ..., \nu_n$ *be the Frenet frame along* $x(\sigma)$. *Then the oblique geodesics on* $T(S^n)$ *and* $T(H^n)$ *with* $k_1 > 0$, $k_2 \geq 0$ *can be expressed as*

$$\begin{cases} x'' = \varepsilon f(t)(a\xi' - b\xi), \\ \xi = a(1-c^2)^{-1/2}\nu_1 + (a' - b)(1-c^2)^{-1}k_1^{-1}\nu_2 - a(1-c^2)^{-1/2}k_1^{-1}k_2\nu_3, \end{cases}$$

where a, b *can be found from the following differential equations*

$$a' = \frac{\varepsilon}{2} f(t)(|\xi|^2)' a + (1 - \varepsilon f(t)|\xi|^2) b,$$

$$b' = \frac{c^2}{f(t)}((\ln f(t))_t' + \varepsilon f(t)) a - (|\xi|^2)'((\ln f(t))_t' + \frac{\varepsilon}{2} f(t)) b.$$

Proof Let $k_1 > 0$, $k_2 \geq 0$. Using $x'' = \varepsilon f(a\xi' - b\xi)$ and $\xi'' = (\ln f)'(\frac{c^2}{f}\xi - (|\xi|^2)'\xi')$, we get

$$a' = g(\xi', x') + g(\xi, x'') = b + \varepsilon f(\frac{1}{2}a(|\xi|^2)' - b|\xi|^2) = \frac{\varepsilon}{2} f(|\xi|^2)' a + (1 - \varepsilon f|\xi|^2) b,$$

$$b' = g(\xi'', x') + g(\xi', x'') = (\ln f)'(\frac{c^2}{f} a - (|\xi|^2)' b) + \varepsilon f(a\frac{c^2}{f} - \frac{1}{2} b(|\xi|^2)')$$

$$= \frac{c^2}{f(t)}((\ln f(t))' + \varepsilon f(t))a - (|\xi|^2)'((\ln f(t))' + \frac{\varepsilon}{2} f(t))b.$$

Using $x'' = \varepsilon f(a\xi' - b\xi)$, we get $\xi' = \frac{1}{\varepsilon f a}(x'' + \varepsilon f b\xi)$. Hence

$$x''' = fR(\xi', \xi)x'' = \varepsilon f(g(\xi, x'')\xi' - g(\xi', x'')\xi)$$

$$= f^2(g(\xi', \xi)a - |\xi|^2 b)\xi' + f^2(g(\xi', \xi)b - \frac{c^2}{f}a)\xi$$

$$= \frac{f}{\varepsilon a}(g(\xi', \xi)a - |\xi|^2 b)x'' + f^2(2g(\xi', \xi)b - |\xi|^2\frac{b^2}{a} - \frac{c^2}{f}a)\xi.$$

Therefore, ξ can be expressed as

$$\xi = \frac{\varepsilon a x''' - f(g(\xi'\xi)a - |\xi|^2 b)x''}{\varepsilon f^2(2g(\xi', \xi)ab - |\xi|^2 b^2 - \frac{c^2}{f}a^2)}.$$

Note that $f(g(\xi', \xi)a - |\xi|^2 b) = \varepsilon(a' - b)$ and $|x''|^2 = f^2(\frac{c^2}{f}a^2 + |\xi|^2 b^2 - 2g(\xi', \xi)ab)$. Then $\xi = \frac{(a'-b)x'' - ax'''}{|x''|^2}$. Note that $x'' = (1-c^2)k_1\nu_2$, and $x''' = -(1-c^2)^{3/2}k_1^2\nu_1 + (1-c^2)^{3/2}k_1k_2\nu_3$. Hence $|x''|^2 = (1-c^2)^2k_1^2$ and

$$\xi = a(1-c^2)^{-1/2}\nu_1 + (a'-b)(1-c^2)^{-1}k_1^{-1}\nu_2 - a(1-c^2)^{-1/2}k_1^{-1}k_2\nu_3.$$

□

Corollary 4.7 *Under the conditions of the theorem 4.6 in the case of the fiberwise Hamiltonian cigar soliton deformed Sasaki metric, we have that geodesics can be expressed as:*

$$\begin{cases} x'' = \frac{\varepsilon}{1+|\xi|^2}(a\xi' - b\xi), \\ \xi = a(1-c^2)^{-1/2}\nu_1 + (a'-b)(1-c^2)^{-1}k_1^{-1}\nu_2 - a(1-c^2)^{-1/2}k_1^{-1}k_2\nu_3, \end{cases}$$

where a, b can be found from the following differential equations

$$a' = \frac{\varepsilon g(\xi', \xi)}{1+|\xi|^2}a + \left(1 - \frac{\varepsilon|\xi|^2}{1+|\xi|^2}\right)b, \tag{4.7}$$

$$b' = c^2(\varepsilon - 1)a + \frac{g(\xi', \xi)}{1+|\xi|^2}(2-\varepsilon)b. \tag{4.8}$$

Acknowledgment

The author thanks Alexander YAMPOLSKY for posing a problem and helpful discussions concerning the research. The author thanks the anonymous reviewers for their careful reading of the manuscript and constructive comments. The research was supported by the National Research Foundation of Ukraine funded by the Ukrainian State budget in frames of project 2020.02/0096 “Operators in infinite-dimensional spaces: the interplay between geometry, algebra and topology”.

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