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NAJMA MOSADEGH

ESMAIEL ABEDI

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Variational geometry for surfaces in conformally flat space

Najma MOSADEGH*, **Esmaiel ABEDI**
Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran

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Abstract: In this paper, it is shown that a closed surface in 3-dimensional harmonic conformally flat space is minimal if the sign of the mean curvature does not change. Also, it is determined that the critical point of mean curvature functional of the surface is homeomorphic to the sphere.

Key words: Conformally flat spaces, variational aspect

1. Introduction

The research in the theory of surfaces in the Euclidean space focuses on the basic properties and the mean tools. The elementary theory of surfaces are presented, for example see [1]. Historically, as an interesting material, surfaces are the local solutions of one of the oldest problems in geometry, called the isoperimetric problem. Because they are compatible as mathematical models in outline in which the physical system seeks a situation of least energy, the surfaces are of great interest for both mathematicians and physicists. Also, we refer to [3] for the constant mean curvature surfaces with boundary, [5, 6] for minimal surfaces.

In recent years, several papers have presented the results which studied on surfaces with constant either extrinsic curvature or intrinsic curvature. For example the studies on surfaces with constant Gaussian curvature and with constant extrinsic curvature have been worked by the authors in studies in [2, 4], respectively.

Indeed, expanding studies on the surfaces renewed interest to deal with the closed surfaces with the induced metric which are immersed in the 3-dimensional conformally flat space in which the ambient manifold is equipped with two Riemannian metrics $\widetilde{g} = e^{\sigma} g_0$, where g_0 denotes the Euclidean one. Also, if in this space the conformal map satisfies $\Delta_{\widetilde{g}}\sigma = \mathsf{trace}_{\widetilde{g}}\widetilde{\nabla}^2\sigma = 0$ then it is called harmonic conformally flat manifold. Both the intrinsic and extrinsic geometry of a surface from variational point of view is the aim of this paper, where they inherit the metric of ambient 3-dimensional harmonic conformally flat space. Indeed, we make use of some topological characterizations such as the Gauss-Bonnet theory and the Euler number to achieve the new results concerning surfaces in sense of the variation in our fundamental object of studying. Briefly, we come to the corresponding variation to the curvature K and the mean curvature H of surfaces, where surfaces are tangent to the conformal vector field, then up to now the following results obtain:

- If the sign of mean curvature H of a closed surface with the induced metric does not change in the 3-dimensional harmonic conformally flat space, then the surface is minimal.

*Correspondence: n.mosadegh@azaruniv.ac.ir
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- Under an isometric immersion a closed complete oriented surface with the induced metric in the 3-dimensional harmonic conformally flat space is homomorphic to a sphere, if the immersion is as a critical point of the mean curvature functional.

- The Euler-Lagrange equation associated to the critical points of the Willmore functional holds

$$\Delta_g H + H\big(\|\omega^\sharp\|^2_{\tilde{g}} - \lambda_1\lambda_2 + 2H^2 - K\big) = 0,$$

  where, λ_i is an eigenvalue of the Weingarten operator and the vector field $\omega^\sharp$ is correlated to 1-form $\omega = \mathsf{d}\sigma$, metrically.

The paper is organized as follows. First of all, by means of the preparation for our investigation in this object, we need to have an overview of the geometry of the ambient 3-dimensional harmonic conformally flat space $(\widetilde{M}^3, \tilde{g})$ and determine some results concerning isometrically immersed closed surfaces with the induced metric there. Then, the principal results achive in the rest of the section 3 for the closed surfaces in the 3-dimensional harmonic conformally flat space, where surfaces with the induced metric are tangent to the conformal vector field. Indeed, by applying the Gauss-Bonnet theory and integrating of the mean curvature in sense of variation most of the results are obtained.

2. Preliminaries

Let $\widetilde{M}^n$ be a smooth Riemannian manifold which is equipped with the two conformal Riemannian metrics g_0 and $\tilde{g}$ in which $\tilde{g} = exp(\sigma)g_0$ for a smooth map $\sigma \in C^\infty(\widetilde{M}^n)$. Then, by the Koszul formula for X and Y tangent to $\widetilde{M}^n$ the Levi-Civita connections $\tilde{\nabla}$ and ∇^0 associated to $\tilde{g}$ and g_0 are related.

$$\tilde{\nabla}_X Y = \nabla^0_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - \tilde{g}(X,Y)\omega^\sharp\}, \tag{2.1}$$

where $\omega^\sharp$ is conformal vector field corresponding to 1-form $\omega = d\sigma$ metrically, which in a coordinate (x^i) is written $\omega^\sharp = \mathsf{grad}_{\tilde{g}}\sigma = \tilde{g}^{ij}\frac{\partial\sigma}{\partial x^i}\frac{\partial}{\partial x^j}$ at any point $x \in \widetilde{M}^n$. Next, we consider a $(0,2)$ tensor field

$$B(X,Y) = (\nabla^0_X\omega)(Y) - \frac{1}{2}\omega(X)\omega(Y), \tag{2.2}$$

in such a way that, the curvature tensor fields $\tilde{R}$ and R^0 corresponding to $\tilde{g}$ and g_0 are connected.

$$\begin{aligned}\tilde{R}(X,Y)Z =& R^0(X,Y)Z + \frac{1}{2}\big\{B(X,Z)Y - B(Y,Z)X \\ &+ g(X,Z)\tilde{\nabla}_Y\omega^\sharp - g(Y,Z)\tilde{\nabla}_X\omega^\sharp\big\} \\ &+ \frac{1}{4}\big(g(Y,Z)\omega(X) - g(X,Z)\omega(Y)\big)\omega^\sharp,\end{aligned} \tag{2.3}$$

where X, Y and Z are tangent to $\widetilde{M}^n$.

A Riemannian manifold $(\widetilde{M}^n, \tilde{g})$ is called locally conformally flat if for any $x \in \widetilde{M}^n$, there existes a neighborhood U of x and a smooth function $\sigma \in C^\infty(\widetilde{M}^n)$, so that $\tilde{g} = e^\sigma g_0$ and the metric g_0 is flat.

Under the conformal change $\tilde{g} = e^{\sigma} g_0$, and take (2.1) with a strightforward computation the Laplace-Beltrami operator transforms like

$$\Delta_{\tilde{g}} = e^{-\sigma}\Delta_{g_0} + (n-2)e^{-\sigma} g_0^{ij} \frac{\partial \sigma}{\partial x^j}\frac{\partial}{\partial x^i}. \tag{2.4}$$

We concentrate on the conformally flat spaces $\widetilde{M}^3$ where $\Delta_{\tilde{g}}\sigma = \mathsf{trace}_{\tilde{g}}\tilde{\nabla}\omega^\sharp = 0$, and these spaces are called harmonic conformally flat spaces. In approach of harmonic conformally flat space the equation (2.4) holds, $\Delta_{g_0}\sigma = -\|\mathsf{grad}_{g_0}\sigma\|^2_{g_0}$.

Example 2.1 *In the conformally flat manifold $R^3_+ = \{(x,y,z) \in R^3; z > 0\}$ with the conformally flat metric $\tilde{g} = e^{\sigma} g_0$, where g_0 is flat, we consider a smooth map $\sigma : \tilde{R^3_+} \longrightarrow R$ where $\sigma(x,y,z) = \mathsf{ln} z$. Then $\Delta_{g_0}\mathsf{ln} z = -\frac{1}{z^2} = -\|\mathsf{grad}_{\mathsf{g}_0}(\mathsf{ln} z)\|^2_{g_0}$. So, it yields that $(R^3, \tilde{g} = e^{\mathsf{ln} z} g_0)$, for $z > 0$ is a harmonic conformally flat space. Moreovere, in this case the curvature tensore $\tilde{K} = -\frac{1}{z^2}$.*

Proposition 2.2 *Let (M,g) be an isometrically immersed surface with the induced metric g in the 3-dimensional conformally flat space $(\widetilde{M}^3, \widetilde{g})$. Let $\{e_1, e_2\}$ be a locall orthonormal frame field on M, where M is tangent to the $\omega^\sharp$. Then,*

- $\omega(e_1)^2 = \omega(e_2)^2$ *and* $\omega^\sharp = \|\omega^\sharp\|_{\tilde{g}} \frac{\sqrt{2}}{2}(e_1 + e_2)$.
- $\widetilde{K} = \left(\frac{1}{4}\|\omega^\sharp\|^2_{\tilde{g}} - \frac{1}{2}\mathsf{div}\omega^\sharp\right)$.

where, $\mathsf{div}\omega^\sharp = \mathsf{trace}_g \nabla\omega^\sharp$ *is considered on* M.

Proof From (2.3), the sectional curvature of plane, which is spanned by $\{e_1, e_2\}$, holds

$$\widetilde{g}(\widetilde{R}(e_1,e_2)e_2,e_1) = \frac{1}{2}\big(\|\omega^\sharp\|^2_{\tilde{g}} - \mathsf{div}\omega^\sharp - \omega(e_2)^2\big), \tag{2.5}$$

analogously,

$$\widetilde{g}(\widetilde{R}(e_2,e_1)e_1,e_2) = \frac{1}{2}\big(\|\omega^\sharp\|^2_{\tilde{g}} - \mathsf{div}\omega^\sharp - \omega(e_1)^2\big), \tag{2.6}$$

Then, immediately these equations follow $\omega(e_1)^2 = \omega(e_2)^2$. From here we can have $\theta = \frac{\pi}{4}$, where θ is angle between $\omega^\sharp$ and the vector field e_i. Then

$$\omega^\sharp = \widetilde{g}(\omega^\sharp, e_1)e_1 + \widetilde{g}(\omega^\sharp, e_2)e_2 = \frac{\sqrt{2}}{2}\|\omega^\sharp\|_{\tilde{g}}(e_1 + e_2). \tag{2.7}$$

Next, summing the equations (2.5) and (2.6) yield

$$\widetilde{K} = \frac{1}{2}\big(\frac{1}{2}\|\omega^\sharp\|^2_{\tilde{g}} - \mathsf{div}\omega^\sharp\big). \tag{2.8}$$

Hence, we reach the result. □

Now, we restrict our attention to a Riemannian closed surface (M,g) in the 3-dimensional conformally flat space $(\widetilde{M},\tilde{g})$, where M with induced metric g is tangent to the $\omega^\sharp$. We consider the Levi-Civta connections of conformal metrics $\tilde{g}$ and g by $\widetilde{\nabla}$ and ∇. Let e_i, N, and H be the principal direction associated to the eigenvalues λ_i for $i=1,2$, the local unit normal vector field, and the mean curvature on M, respectively. We denote the sectional curvature spanned by $\{e_1,e_2\}$ on $\widetilde{M}$ by $\widetilde{K}$ and the Gaussian curvature on M by K. Also, $\Gamma(TM)$ means the set of all the vector fields on M. Furthermore, the Codazzi equation

$$\tilde{g}\big((\widetilde{\nabla}_X A)Y-(\widetilde{\nabla}_Y A)X,Z\big)=\frac{1}{2}\big(\omega(AY)\tilde{g}(X,Z)-\omega(AX)\widetilde{g}(Y,Z)\big), \tag{2.9}$$

holds for any $X,Y,Z\in\Gamma(TM)$, and A is the Weingarten operator on M.
By regarding the idea in the Verpoort's thesis [7], we used it in our investigating to proof the following propositions.

Proposition 2.3 *Let M be an isometrically immersed closed surface with the induced metric g in the 3-dimensional conformally flat space $(\widetilde{M}^3,\widetilde{g})$. Then for $X\in\Gamma(TM)$*

$$\int_M \lambda_2\widetilde{g}(\nabla_{e_1}X,e_1)+\lambda_1\widetilde{g}(\nabla_{e_2}X,e_2)-\frac{1}{2}\omega(AX)d\Omega=0. \tag{2.10}$$

Proof Let e_i for $i=1,2$ be the eigenvectors of the Weingarten operator A corresponding to the eigenvalues λ_i. By taking (2.9), we calculate

$$\begin{aligned}
\lambda_2\widetilde{g}(\nabla_{e_1}X,e_1)+\lambda_1\widetilde{g}(\nabla_{e_2}X,e_2) &= 2H\mathsf{div}X-\lambda_2\widetilde{g}(\nabla_{e_2}X,e_2)+\lambda_1\widetilde{g}(\nabla_{e_1}X,e_1)\\
&= 2H\mathsf{div}X-\widetilde{g}(\nabla_{e_2}X,Ae_2)+\widetilde{g}(\nabla_{e_1}X,Ae_1)\\
&= 2H\mathsf{div}X-\widetilde{g}(\nabla_{e_2}AX-(\nabla_{e_2}A)X,e_2)\\
&\quad +\widetilde{g}(\nabla_{e_1}AX-(\nabla_{e_1}A)X,e_1)\\
&= 2H\mathsf{div}X-\mathsf{div}AX\\
&+ \widetilde{g}((\nabla_{e_2}A)X,e_2)+\widetilde{g}((\nabla_{e_1}A)X,e_1)\\
&= 2H\mathsf{div}X-\mathsf{div}AX\\
&+ \widetilde{g}((\nabla_X A)e_1,e_1)+\frac{1}{2}\big(\widetilde{g}(\omega(AX)e_1,e_1)-\widetilde{g}(\omega(Ae_1)X,e_1)\big)\\
&+ \widetilde{g}((\nabla_X A)e_2,e_2)+\frac{1}{2}\big(\widetilde{g}(\omega(AX)e_2,e_2)-\widetilde{g}(\omega(Ae_2)X,e_2)\big)\\
&= 2H\mathsf{div}X-\mathsf{div}AX+\frac{1}{2}\omega(AX)\\
&+ \widetilde{g}((\nabla_X A)e_2,e_2)+\widetilde{g}((\nabla_X A)e_1,e_1)\\
&= 2H\mathsf{div}X-\mathsf{div}AX+(X\lambda_1)+(X\lambda_2)+\frac{1}{2}\omega(AX)\\
&= 2H\mathsf{div}X-\mathsf{div}AX+2\widetilde{g}(\nabla H,X)+\frac{1}{2}\omega(AX)\\
&= \mathsf{div}(2HX)-\mathsf{div}AX+\frac{1}{2}\omega(AX),
\end{aligned}$$

for all $X \in \Gamma(TM)$. Since, the surface is closed by applying the divergence theorem we conclude the proof. $\square$

Proposition 2.4 *Let (M,g) be an isometrically immersed closed surface with the induced metric g in the 3-dimensional conformally flat space $(\widetilde{M}^3, \widetilde{g})$. Then*

$$\int_M \lambda_2 f \mathsf{Hess}_h(e_1,e_1) + \lambda_1 f \mathsf{Hess}_h(e_2,e_2) - \frac{1}{2} f\omega(A\nabla h) d\Omega$$
$$= \int_M \lambda_2 h \mathsf{Hess}_f(e_1,e_1) + \lambda_1 h \mathsf{Hess}_f(e_2,e_2) - \frac{1}{2} h\omega(A\nabla f) d\Omega, \tag{2.11}$$

where $f, h \in C^\infty(M)$.

Proof We are in the hypotheses of the last proposition. Let $X = (f\nabla h - h\nabla f) \in \Gamma(TM)$, where $f, h \in C^\infty(M)$, then the divergence theorem states $\int_M f\Delta h d\Omega = \int_M h\Delta f d\Omega$. Next, Proposition 2.3 follows

$$\begin{aligned} 0 \;\; = \;\; & \int_M \big\{\lambda_2 \widetilde{g}(\widetilde{\nabla}_{e_1}(f\nabla h - h\nabla f), e_1) + \lambda_1 \widetilde{g}(\widetilde{\nabla}_{e_2}(f\nabla h - h\nabla f), e_2) \\ & -\frac{1}{2}\omega(A(f\nabla h - h\nabla f))\big\} d\Omega, \end{aligned} \tag{2.12}$$

then

$$\int_M \big\{f\lambda_2 \widetilde{g}\big(\widetilde{\nabla}_{e_1}\nabla h, e_1\big) + f\lambda_1 \widetilde{g}\big(\widetilde{\nabla}_{e_2}\nabla h, e_2\big) - \frac{1}{2} f\omega(A\nabla h)\big\} d\Omega,$$
$$= \int_M \big\{h\lambda_2 \widetilde{g}(\widetilde{\nabla}_{e_1}\nabla f, e_1) + h\lambda_1 \widetilde{g}(\widetilde{\nabla}_{e_2}\nabla f, e_2) - \frac{1}{2} h\omega(A\nabla f))\big\} d\Omega, \tag{2.13}$$

since $\widetilde{g}(\widetilde{\nabla}_{e_i}\nabla f, e_i) = \mathsf{Hess}_f(e_i, e_i)$ and analogously for h, then the proof is completed. $\square$

A variation of an isometric immersion $\varphi : M \to \widetilde{M}$ is considered a differentiable map $\varphi : M \times (-\epsilon, \epsilon) \to \widetilde{M} : (p,t) \mapsto \varphi_t(p) = \varphi(p,t)$, where $\varphi_t : M \to \widetilde{M}$ is an immersion such that for $t = 0$, $\varphi_0 = \varphi$. Furthermore, the variational vector field ξ, associated to $\{\varphi_t\}$ attaches $\xi_p = \frac{\partial \varphi(p,t)}{\partial t}|_{t=0}$ for $p \in M$. The variation $\{\varphi_t\}$ is said to be a tangent variation where the vector field ξ is tangent to the surface in any points. It is convenient to set convention the variation is normal where the variational vector field $\xi = fN$, where N is a local unit normal vector field on M and $f \in C^\infty(M)$ is an arbitrary smooth function.

In the rest of this section, we come up to establish the variation of the eigenvalues of the Weingarten operator inspired by the variation of the Weingarten operator that was given:

Theorem 2.5 *([7] Variation of the Weingarten operator) Let (M,g) be a semi-Riemannian hypersurface of a semi-Riemannian manifold $\widetilde{M}$ under a variation of M, with variational vector field $\xi = fN + \xi^t$, then for every $X \in \chi(M)$ holds :*

$$(\delta A)(X) = f\big(\widetilde{R}(N,X)N + A^2(X)\big) + \mathsf{Hs_f}(X) + (\pounds_{\xi^t} A)(X), \tag{2.14}$$

where $\mathsf{Hs_f}(X)$ is an operator associated to tensor $\mathsf{Hess}_f(X,Y) = \widetilde{g}(\mathsf{Hs_f}(X), Y) = \widetilde{g}(\widetilde{\nabla}_X \nabla f, Y)$ and $(\pounds_{\xi^t} A)(X)$ is the Lei derivative of the Weingarten operator along ξ^t.

Let $\varphi_t : M \to \widetilde{M}^3$ be an immersed normal variation of a closed surface M with the induced metric g in the 3-dimensional conformally flat space $(\widetilde{M}^3, \tilde{g})$ where $\xi = fN$. Let e_i be the principal direction on M, corresponding to the eigenvalue λ_i of the Weingarten operator A. Now, taking variation of both sides of the equation $A(\varphi_t)e_i(\varphi_t) = \lambda_i(\varphi_t)e_i(\varphi_t)$ leads to

$$(\delta A)(e_i) + A(\delta e_i) = (\delta\lambda_i)e_i + \lambda_i(\delta e_i).$$

We have $g(\delta e_i, e_i) = 0$ which yields $\widetilde{g}(\delta e_i, e_i) = 0$ then $\delta e_i = \widetilde{g}(\delta e_i, e_j)e_j$ for $i, j = 1, 2$ and $j \neq i$, it follows

$$(\delta A)(e_i) + \lambda_j(\delta e_i) = (\delta\lambda_i)e_i + \lambda_i(\delta e_i), \qquad i \neq j$$

then $\delta\lambda_i = \widetilde{g}((\delta A)e_i, e_i)$. From here, by applying (2.14) we get

$$\delta\lambda_i = f\big(\widetilde{g}(\widetilde{R}(N, e_i)N, e_i) + \lambda_i^2\big) + \mathsf{Hess}_f(e_i, e_i). \tag{2.15}$$

3. Variational aspects of surfaces in the 3-dimensional harmonic conformally flat space

From now we concentrate on the main results.

Theorem 3.1 *Let $\varphi_t : M \to \widetilde{M}^3$ be an immersed normal variation of a closed surface M with the induced metric g in the 3-dimensional harmonic conformally flat space $(\widetilde{M}, \tilde{g})$. Then, the variation of the Guassian curvature on M holds*

$$\begin{aligned}\delta K \quad &= \quad \frac{f}{2}\big((\lambda_1 - \lambda_2)\widetilde{g}(\widetilde{\nabla}_{e_1}\omega^\sharp, e_1) + (\lambda_2 - \lambda_1)\widetilde{g}(\widetilde{\nabla}_{e_2}\omega^\sharp, e_2)\big)\\ &\quad + \lambda_1\mathsf{Hess}_f(e_2, e_2) + \lambda_2\mathsf{Hess}_f(e_1, e_1) - \frac{1}{2}\mathsf{div}(\delta\omega^\sharp)\\ &\quad + f\big(2H(\lambda_1\lambda_2) - H\omega(\frac{\nabla f}{f}) + \frac{1}{8}H\|\omega^\sharp\|^2_{\tilde{g}} - \omega(\frac{A\nabla f}{f})\big).\end{aligned} \tag{3.1}$$

Proof Let $\varphi : M \to \widetilde{M}^3$ be an isometric immersion of a closed surface (M, g) into the 3-dimensional harmonic conformally flat space $(\widetilde{M}, \tilde{g})$. Let e_i and N be a principal direction corresponding to the eigenvalues λ_i for $i = 1, 2$ and local unit normal vector field on M, respectively. We consider the Guass equation that denotes $K = \widetilde{K} + \lambda_1\lambda_2$. Then

$$\frac{\partial}{\partial t}\Big|_{t=0}K = \delta K = \delta\widetilde{K} + \delta(\lambda_1\lambda_2). \tag{3.2}$$

We consider equations (2.2) and (2.8) implying

$$\delta\widetilde{K} = -\frac{1}{2}\delta\big(B(e_1, e_1) + B(e_2, e_2)\big). \tag{3.3}$$

such that

$$\frac{\partial}{\partial t}\Big|_{t=0}B(e_i, e_i) = \delta B(e_i, e_i) = \delta\big(\widetilde{g}(\widetilde{\nabla}_{e_i}\omega^\sharp, e_i) - \frac{1}{4}\|\omega^\sharp\|^2_{\tilde{g}}\big).$$

The next step, variation of the first and second terms of the last equation is calculated as

$$\begin{aligned}
\delta\widetilde{g}(\widetilde{\nabla}_{e_i}\omega^\sharp, e_i) &= (\delta\widetilde{g})(\widetilde{\nabla}_{e_i}\omega^\sharp, e_i) + f\widetilde{g}(\widetilde{\nabla}_N\widetilde{\nabla}_{e_1}\omega^\sharp, e_i) + f\widetilde{g}(\widetilde{\nabla}_{e_i}\omega^\sharp, \widetilde{\nabla}_N e_i), \\
&= -2\lambda_1 f\widetilde{g}(\widetilde{\nabla}_{e_i}\omega^\sharp, e_i) + f\widetilde{g}(\widetilde{R}(N, e_i)\omega^\sharp, e_i) + f\widetilde{g}(\widetilde{\nabla}_{e_i}\delta\omega^\sharp, e_i) \\
&\quad + f\widetilde{g}(\widetilde{\nabla}_{[N,e_i]}, e_i) + f\mathsf{Hess}_\sigma(e_i, \widetilde{\nabla}_N e_i),
\end{aligned} \tag{3.4}$$

where $(\delta\widetilde{g})(X,Y) = -2f\widetilde{g}(AX,Y)$ for $X,Y \in \chi(M)$ see([7]). Also,

$$\begin{aligned}
\delta\|\omega^\sharp\|^2_{\widetilde{g}} &= (\delta\widetilde{g})(\omega^\sharp, \omega^\sharp) + \frac{1}{2}\widetilde{g}(\delta\omega^\sharp, \omega^\sharp), \\
&= -2f\widetilde{g}(A\omega^\sharp, \omega^\sharp) + \frac{1}{2}f\widetilde{g}(\widetilde{\nabla}_N\omega^\sharp, \omega^\sharp), \\
&= -2f\omega(A\omega^\sharp) + \frac{f}{2}B(N, \omega^\sharp) = -\frac{3}{2}f\omega(A\omega^\sharp),
\end{aligned} \tag{3.5}$$

where (2.2) yields $B(N,\omega^\sharp) = -\omega(\widetilde{\nabla}_{\omega^\sharp}N) = \omega(A\omega^\sharp)$. Equations (3.3) together with (3.4) and (3.5), lead to

$$\begin{aligned}
\delta\widetilde{K} &= \frac{1}{2}f\big(\lambda_1\widetilde{g}(\widetilde{\nabla}_{e_1}\omega^\sharp, e_1) + \lambda_2\widetilde{g}(\widetilde{\nabla}_{e_2}\omega^\sharp, e_2)\big) \\
&\quad -\frac{5}{8}fH\|\omega^\sharp\|^2_{\widetilde{g}} - \frac{1}{2}\mathsf{div}(\delta\omega^\sharp) - f\omega(\frac{A\nabla f}{f}),
\end{aligned} \tag{3.6}$$

because of

$$\begin{aligned}
\omega(A\omega^\sharp) &= \widetilde{g}(\omega^\sharp, \widetilde{g}(\omega^\sharp, e_1)Ae_1 + \widetilde{g}(\omega^\sharp, Ae_2)e_2) \\
&= \lambda_1\omega(e_1)^2 + \lambda_2\omega(e_2)^2 = H\|\omega^\sharp\|^2_{\widetilde{g}}.
\end{aligned} \tag{3.7}$$

Further, equation (2.15) notes that

$$\begin{aligned}
\delta(\lambda_1\lambda_2) &= 2fH(\lambda_1\lambda_2) + \lambda_1\mathsf{Hess}_f(e_2, e_2) + \lambda_2\mathsf{Hess}_f(e_1, e_1) \\
&\quad + f\lambda_2\big(-\frac{1}{2}(\widetilde{g}(\widetilde{\nabla}_{e_1}\omega^\sharp, e_1) + B(N,N)) + \frac{1}{4}\omega(e_1)^2\big) \\
&\quad + f\lambda_1\big(-\frac{1}{2}(\widetilde{g}(\widetilde{\nabla}_{e_2}\omega^\sharp, e_2) + B(N,N)) + \frac{1}{4}\omega(e_2)^2\big), \\
&= \lambda_1\mathsf{Hess}_f(e_2, e_2) + \lambda_2\mathsf{Hess}_f(e_1, e_1) \\
&\quad -\frac{f}{2}\big(\lambda_2\widetilde{g}(\widetilde{\nabla}_{e_1}\omega^\sharp, e_1) + \lambda_1\widetilde{g}(\widetilde{\nabla}_{e_2}\omega^\sharp, e_2)\big) \\
&\quad + f\big(2H(\lambda_1\lambda_2) - H\omega(\frac{\nabla f}{f}) + \frac{3}{4}H\|\omega^\sharp\|^2_{\widetilde{g}}\big).
\end{aligned} \tag{3.8}$$

Hence from equations (3.2), (3.6) and (3.8), we end the proof of this theorem. □

We can take into account that under a variation of a surface M, the $\mathsf{Area}(\varphi_t(M)) = \int d\Omega_t$. Indeed, the volume element of the variation $d\Omega_t = u_t d\Omega$, where u_t is the area of the parallelogram spanned by $d(\varphi_t(X))$ and $d(\varphi_t(Y))$ for $X,Y \in \Gamma(TM)$. Hereof, the variational vector field is decomposed to $\xi = \xi^t + fN$, it is known that $\delta d\Omega_t = (\mathsf{div}\xi^t - 2fH)d\Omega$, where H denotes the mean curvature of M.

Theorem 3.2 *Let M be an isometrically immersed closed surface with the induced metric g in the 3-dimensional harmonic conformally flat space $(\widetilde{M},\widetilde{g})$. Then M is a minimal surface if sign of the mean curvature does not change.*

Proof Let $\varphi_t : M \to (\widetilde{M}^3,\widetilde{g})$ be a normal variation, where $\xi = fN$, of a immersed closed surface (M,g) in the 3-dimensional harmonic conformally flat space $(\widetilde{M},\tilde{g})$. Then directly by applying the Gauss-Bonnet theory together with Theorem 3.1 and Proposition 2.3 we obtain

$$\begin{aligned} 0=\delta\int_M K d\Omega &= \int_M \delta K d\Omega + \int_M K(\delta d\Omega) \\ &= \int_M \big\{\frac{f}{2}\big((\lambda_1-\lambda_2)\widetilde{g}(\widetilde{\nabla}_{e_1}\omega^\sharp,e_1)+(\lambda_2-\lambda_1)\widetilde{g}(\widetilde{\nabla}_{e_2}\omega^\sharp,e_2)\big) \\ &\quad +\lambda_1\mathsf{Hess}_f(e_2,e_2)+\lambda_2\mathsf{Hess}_f(e_1,e_1)-\frac{1}{2}\mathsf{div}(\delta\omega^\sharp) \\ &\quad +f\big(2H(\lambda_1\lambda_2)-H\omega(\frac{\nabla f}{f})-2HK \\ &\quad +\frac{1}{8}H\|\omega^\sharp\|^2_{\widetilde{g}}-\omega(\frac{A\nabla f}{f})\big)\big\}d\Omega \\ &= \int_M f\big\{\lambda_1\widetilde{g}(\widetilde{\nabla}_{e_1}\omega^\sharp,e_1)+\lambda_2\widetilde{g}(\widetilde{\nabla}_{e_2}\omega^\sharp,e_2) \\ &\quad -\frac{3}{8}H\|\omega^\sharp\|^2_{\widetilde{g}}-\omega(\frac{A\nabla f}{2f})-H\omega(\frac{\nabla f}{f})\big\}d\Omega, \end{aligned} \tag{3.9}$$

on the other hand we have

$$\begin{aligned} \lambda_2\widetilde{g}(\nabla_{e_2}\omega^\sharp,e_2)+\lambda_1\widetilde{g}(\nabla_{e_1}\omega^\sharp,e_1) &= -\big(\lambda_2\widetilde{g}(\widetilde{\nabla}_{e_1}\omega^\sharp,e_1)+\lambda_1\widetilde{g}(\widetilde{\nabla}_{e_2}\omega^\sharp,e_2)\big), \\ &\quad +2H\mathsf{div}\omega^\sharp \end{aligned} \tag{3.10}$$

so we replace (3.10) in (3.9) which together with the Proposition 2.4 follow

$$\begin{aligned} &\int_M \big\{-\big(\sigma\lambda_2\widetilde{g}(\widetilde{\nabla}_{e_1}\nabla f,e_1)+\sigma\lambda_1\widetilde{g}(\widetilde{\nabla}_{e_2}\nabla f,e_2)\big)+\frac{\sigma}{2}\omega(A\nabla f) \\ &+f\big(2H\mathsf{div}\omega^\sharp-\frac{7}{8}H\|\omega^\sharp\|^2_{\widetilde{g}}-\omega(\frac{A\nabla f}{2f})-H\omega(\frac{\nabla f}{f})\big)\big\}d\Omega=0. \end{aligned} \tag{3.11}$$

Hereof, $\widetilde{\nabla}_N N = -\frac{\nabla f}{f}$ [7] and notice the assumption that

$$\begin{aligned} 0=\mathsf{Div}\omega^\sharp &= \mathsf{div}\omega^\sharp+\widetilde{g}(\widetilde{\nabla}_N\omega^\sharp,N) \\ &= \mathsf{div}\omega^\sharp+\widetilde{g}(\omega^\sharp,\frac{\nabla f}{f}), \end{aligned} \tag{3.12}$$

it follows $\mathsf{div}\omega^\sharp = -\omega(\frac{\nabla f}{f})$. From here, (3.11) yields

$$\begin{aligned} &\int_M \big\{-f\big(\frac{7}{8}H\|\omega^\sharp\|^2_{\widetilde{g}}+\omega(\frac{A\nabla f}{2f})+3H\omega(\frac{\nabla f}{f})\big) \\ &-\sigma\big(\lambda_2\widetilde{g}(\widetilde{\nabla}_{e_1}\nabla f,e_1)+\lambda_1\widetilde{g}(\widetilde{\nabla}_{e_2}\nabla f,e_2)\big)+\frac{\sigma}{2}\omega(A\nabla f)\big\}d\Omega=0. \end{aligned} \tag{3.13}$$

After all, under the normal variation $\xi = N$, where $f \equiv 1$ the equation (3.13) follows

$$\int_M H\|\omega^\sharp\|^2_{\tilde{g}} d\Omega = 0,$$

hence as respect to the assumption that the sign of mean curvature does not change, also $\omega^\sharp \neq 0$, it obtains that $H = 0$. □

By regarding the mean curvature and considering the critical points of its integrating we get the following results.

Theorem 3.3 *Let $\varphi : M \to \widetilde{M}^3$ be an isometric immersion of a closed complete oriented surface M with the induced metric g in $3-$dimensional harmonic conformally flat space $(\widetilde{M}, \tilde{g})$. Let for $\delta \int H d\Omega = 0$, the immersion φ be a critical point of the mean curvature functional, then M is homeomorphic to a sphere and the associated Euler-Lagrange equation holds*

$$\|\omega^\sharp\|^2_{\tilde{g}} - \lambda_1\lambda_2 - K = 0. \tag{3.14}$$

Proof Let $\varphi_t : M \to \widetilde{M}^3$ denotes a normal variation of a closed complete oriented surface (M, g) in the 3-dimensional harmonic conformally flat space $(\widetilde{M}^3, \tilde{g})$ with the variational vector field $\xi = fN$. Let e_i for $i = 1, 2$ and N be the principal directions corresponding to the eigenvalues λ_i and local unit normal vector field on M, respectively. Then, from (2.15) and Proposition 2.3 by integrating of the mean curvature $H = \frac{\lambda_1+\lambda_2}{2}$ on M we get

$$\begin{aligned}
\delta \int_M H d\Omega &= \int_M \delta H d\Omega + \int_M H(\delta d\Omega) \\
&= \frac{1}{2}\int_M \big\{f\big(\tilde{g}(\widetilde{R}(e_1, N)N, e_1) + \tilde{g}(\widetilde{R}(e_2, N)N, e_2) + \lambda_1^2 + \lambda_2^2\big) \\
&\quad + \mathsf{Hess}_f(e_1, e_1) + \mathsf{Hess}_f(e_2, e_2) - 2fH^2\big\} d\Omega, \\
&= \int_M f\big\{-\lambda_1\lambda_2 + \frac{1}{2}\widetilde{\mathsf{Ric}}(N, N)\big\} d\Omega.
\end{aligned} \tag{3.15}$$

From (2.3), the second terms of (3.15), tensor $\widetilde{\mathsf{Ric}}$ associated to $\widetilde{M}$ satisfies

$$\begin{aligned}
\widetilde{\mathsf{Ric}}(N, N) &= -B(N, N) - \frac{1}{2}\big((\tilde{g}(\widetilde{\nabla}_{e_1}\omega^\sharp, e_1) + \tilde{g}(\widetilde{\nabla}_{e_2}\omega^\sharp, e_2)\big) + \frac{\|\omega^\sharp\|^2_{\tilde{g}}}{4}, \\
&= -\frac{1}{2}\omega(\frac{\nabla f}{f}) + \frac{3}{4}\|\omega^\sharp\|^2_{\tilde{g}},
\end{aligned} \tag{3.16}$$

since, from (2.2) and (3.12) it follows that $B(N, N) = \omega(\frac{\nabla f}{f}) - \frac{\|\omega^\sharp\|^2_{\tilde{g}}}{2}$ and $\sum_{i=1}^2 \tilde{g}(\widetilde{\nabla}_{e_i}\omega^\sharp, e_i) = \mathsf{div}\omega^\sharp = -\omega(\frac{\nabla f}{f})$, respectively. From here and (2.8) we get

$$K = \frac{1}{2}\omega(\frac{\nabla f}{f}) + \frac{1}{4}\|\omega^\sharp\|^2_{\tilde{g}} + \lambda_1\lambda_2, \tag{3.17}$$

thus (3.15) together with (3.16) and (3.17) imply

$$\delta \int_M H d\Omega = -\frac{1}{2} \int_M f\big(\lambda_1\lambda_2 + K - \|\omega^\sharp\|^2_{\tilde{g}}\big) d\Omega. \tag{3.18}$$

According to the assumption $\delta \int_M H d\Omega = 0$, hence (3.18) follows that the critical points of the functional hold

$$\lambda_1\lambda_2 + K - \|\omega^\sharp\|^2_{\tilde{g}} = 0. \tag{3.19}$$

Further, equation (3.17) yields $\lambda_1\lambda_2 = K - \frac{1}{2}\omega(\frac{\nabla f}{f}) - \frac{1}{4}\|\omega^\sharp\|^2_{\tilde{g}}$ that together with (3.18) imply

$$\delta \int_M H d\Omega = -\frac{1}{2} \int_M f\big(2K - \frac{1}{2}\omega(\frac{\nabla f}{f}) - \frac{5}{4}\|\omega^\sharp\|^2_{\tilde{g}}\big) d\Omega, \tag{3.20}$$

such that under the normal variation $\xi = N$, that is, $f \equiv 1$ by taking into account the assumption and making use of the Gauss-Bonnet theory, equation (3.20) follows that the Euler number $\chi(M)$ holds

$$\chi(M) = \frac{5}{16\pi} \int \|\omega^\sharp\|^2_{\tilde{g}} d\Omega > 0.$$

□

Finding the critical points of a functional is a major problem in the calculus of variation. In the rest of this section for a given topological space we consider closed surfaces with induced metric in 3-dimensional harmonic conformally flat spaces, in which $\delta \int H^2 d\Omega = 0$, that is known as the Willmore surfaces.

Theorem 3.4 *Let $\varphi : M \to \widetilde{M}^3$ be an isometric immersion of a closed surface M with the induced metric g in the 3-dimensional harmonic conformal flat space $(\widetilde{M}^3, \tilde{g})$. Then M is a Willmore surface if and only if*

$$\Delta_g H + H\big(2H^2 - K + \|\omega^\sharp\|^2_{\tilde{g}} - \lambda_1\lambda_2\big) = 0, \tag{3.21}$$

Proof Analogously, we are in the hypotheses of the last theorem such that under a normal variation where $\xi = fN$ and equation (2.15) we obtain

$$\begin{aligned} \delta \int_M H^2 d\Omega &= \int_M 2H(\delta H) d\Omega + \int_M H^2 (\delta d\Omega), \\ &= \int_M \big\{2H\big(f\widetilde{K} + \frac{f}{2}\widetilde{\mathsf{Ric}}(N,N) + \frac{\Delta f}{2} \\ &\qquad + f(2H^2 - K)\big) - 2fH^3\big\} d\Omega, \\ &= \int_M fH\big(\|\omega^\sharp\|^2_{\tilde{g}} - \lambda_1\lambda_2 - K\big) + H\Delta_g f + 2fH^3 d\Omega \end{aligned} \tag{3.22}$$

where $\widetilde{\mathsf{Ric}}(\mathsf{N},\mathsf{N})$ satisfies (3.16). Now, the divergence theorem can be applied such that equation (3.22) becomes

$$\delta \int H^2 d\Omega = \int f\big(\|\omega^\sharp\|^2_{\tilde{g}} H - \lambda_1\lambda_2 H + \Delta_g H + H(2H^2 - K)\big) d\Omega. \tag{3.23}$$

Consequently, the critical points of the above function hold

$$\Delta_g H + H\big(\|\omega^\sharp\|^2_{\tilde{g}} - \lambda_1\lambda_2 + 2H^2 - K\big) = 0. \tag{3.24}$$

□

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