Turkish Journal of Mathematics

Volume 46 | Number 1 Article 2

1-1-2022

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 24 – 35
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doi:10.3906/mat-2101-56

$k-$uniformly multivalent functions involving Liu-Owa $q-$integral operator

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Received: 12.01.2021 • **Accepted/Published Online:** 17.11.2021 • **Final Version:** 19.01.2022

Abstract: In this paper, we introduce $q-$analogue of Liu-Owa integral operator and define a subclass of $k-$uniformly multivalent starlike functions of order $\gamma, (0 \leq \gamma < p; p \in \mathbb{N})$ by using the Liu-Owa $q-$integral operator. We examine coefficient estimates, growth and distortion bounds for the functions belonging to the subclass of $k-$uniformly multivalent starlike functions of order γ. Moreover, we determine radii of $k-$uniformly starlikeness, convexity and close-to-convexity for the functions belonging to this subclass.

Key words: Analytic functions, starlike functions, growth theorem, distortion theorem, coefficient estimate

1. Introduction

After Jackson [9, 11] introduced the useful version of the $q-$derivative operator, quantum calculus (or $q-$calculus) has found applications in physics, quantum mechanics, analytic number theory, Sobolev spaces, representation theory of groups, theta functions, gamma functions, operator theory, and more recently in geometric function theory. In fact, $q-$calculus methodology is centered on the idea of deriving $q-$analogues results without the use of limits.

Let $q \in (0,1)$. The $q-$derivative (or $q-$difference) operator, which was introduced by Jackson [9] and may go back to Heine [7], is defined by

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \ (z \neq 0).$$

We note that $\lim_{q\to 1^-}(D_q f)(z) = f'(z)$ if f is differentiable at z.

For the function $f(z) = z^n$, we observe that

$$D_q z^n = [n]_q z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}$. Clearly, for $q \to 1^-$, $[n]_q \to n$.

The $q-$integral of any function f was defined by Jackson [10] as:

$$\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(xq^n).$$

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2010 *AMS Mathematics Subject Classification:* 30C45, 30C50

In [5], for the case $q \in (0,1)$, researchers found that $q-$gamma function is equivalent to the following

$$\Gamma_q(x) := (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{x+n}}, \quad (x>0).$$

It is also easy to see that

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(x+1) = [x]_q!, \tag{1.1}$$

where the $q-$factorial $[x]_q!$ is given by

$$[x]_q! = \begin{cases} [x]_q[x-1]_q \ldots [2]_q[1]_q, & \text{if } x \geq 1 \\ 1, & \text{if } x = 0. \end{cases}$$

Note that $\Gamma_q(x) \to \Gamma(x)$ when $q \to 1^-$. For more details of gamma and $q-$gamma functions, one may refer to [1, 2, 5].

The $q-$beta function has the $q-$integral representation, which is a $q-$analogue of Euler's formula:

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x, \quad (t,s>0). \tag{1.2}$$

Jackson also showed that the $q-$beta function defined by the usual formula

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}, \tag{1.3}$$

which tends to $B(t,s)$ as $q \to 1^-$.

Finally, the Gauss $q-$binomial coefficient is shown by

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{1.4}$$

For the definitions and properties of $q-$derivative and $q-$calculus, one may refer to [4, 5, 9, 11, 13].

Before starting our main results, we would like to draw attention to some basic concepts needed for building our main results. Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \ (p \in \mathbb{N}) \tag{1.5}$$

that are analytic and $p-$valent (or multivalent) in the open unit disc $\mathbb{D} := \{z : |z| < 1\}$. Let Ω be the family of Schwarz functions w which are analytic in $\mathbb{D}$ and satisfy the conditions $w(0) = 0$, $|w(z)| < 1$ for all $z \in \mathbb{D}$. If f_1 and f_2 are analytic functions in $\mathbb{D}$, then we say that f_1 is subordinate to f_2, written as $f_1 \prec f_2$, if there exists a Schwarz function $w \in \Omega$ such that $f_1(z) = f_2(w(z))$. We also note that if f_2 is univalent in $\mathbb{D}$, then

$$f_1 \prec f_2 \Leftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\mathbb{D}) \subset f_2(\mathbb{D}), \ (z \in \mathbb{D}).$$

A function f belonging to $\mathcal{A}_p$ is said to be $p-$valently starlike of order γ, denoted by $\mathcal{S}_p^*(\gamma)$, if it satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \ (0 \leq \gamma < p; z \in \mathbb{D}).$$

A function f belonging to $\mathcal{A}_p$ is said to be $p-$valently convex of order γ, denoted by $\mathcal{K}_p(\gamma)$, if it satisfies

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\gamma,\ (0\le\gamma<p; z\in\mathbb{D}).$$

A function f belonging to $\mathcal{A}_p$ is said to be $p-$valently close-to-convex of order γ, denoted by $\mathcal{C}_p(\gamma)$, if it satisfies

$$Re\left(\frac{f'(z)}{z^{p-1}}\right)>\gamma,\ (0\le\gamma<p; z\in\mathbb{D}).$$

Note that, a function $f(z)\in\mathcal{K}_p(\gamma)$ if and only if $\frac{zf'(z)}{p}\in\mathcal{S}_p^*(\gamma)$. For comprehensive details, one may refer to [6].

For $0\le\gamma<p$, $k\ge0$ and $z\in\mathbb{D}$, denote by $k-\mathcal{US}_p^*(\gamma)$, Kharsani [14] defined the class of $k-$uniformly $p-$valent starlike function of order γ by

$$Re\left(\frac{zf'(z)}{f(z)}-\gamma\right)>k\left|\frac{zf'(z)}{f(z)}-p\right|.$$

Corresponding to a conic domain $\Omega_{p,k,\gamma}$ defined by

$$\Omega_{p,k,\gamma}=\left\{u+iv: u>k\sqrt{(u-p)^2+v^2}+\gamma\right\},$$

we define the function $h_{p,k}$ which maps $\mathbb{D}$ onto the conic domain $\Omega_{p,k,\gamma}$ such that $1\in\Omega_{p,k,\gamma}$ as the following ([14]):

$$h_{p,k}(z)=\begin{cases}\frac{1+(1-2\gamma)z}{1-z}, & (k=0)\\ \frac{p-\gamma}{1-k^2}\cos\left\{\frac{2}{\pi}(\cos^{-1}k)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\frac{k^2p-\gamma}{1-k^2}, & (0<k<1)\\ p+\frac{2(p-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & (k=1)\\ \frac{p-\gamma}{k^2-1}\sin\left\{\frac{\pi}{2K(x)}\int_0^{u(z)/\sqrt{x}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-x^2t^2}}\right\}+\frac{k^2p-\gamma}{k^2-1}, & (k>1),\end{cases}\tag{1.6}$$

where $u(z)=\frac{z-\sqrt{x}}{1-\sqrt{xz}}, x\in(0,1)$ and $K(x)$ is such that $k=\cosh\frac{\pi K'(x)}{4K(x)}$. By virtue of the properties of the conic domain $\Omega_{p,k,\gamma}$ we have

$$Re(h_{p,k}(z))>\frac{kp+\gamma}{k+1}.$$

Making use of principle subordination between analytic functions and the definition $h_{p,k}$, the class $k-\mathcal{US}_p^*(\gamma)$ can be written as ([14]):

$$k-\mathcal{US}_p^*(\gamma)=\left\{f\in\mathcal{A}_p:\frac{zf'(z)}{f(z)}\prec h_{p,k}(z)\right\}.$$

In literature, some important integral operators were defined by using multivalent functions. In 2004, Liu and Owa [15] introduced $p-$valent integral operator $Q^{\alpha}_{p,\beta}:\mathcal{A}_p\to\mathcal{A}_p$ by

$$\begin{aligned} Q^{\alpha}_{p,\beta}f(z) &= \binom{p+\alpha+\beta-1}{p+\beta-1}\frac{\alpha}{z^{\beta}}\int_0^z\left(1-\frac{t}{z}\right)^{\alpha-1}t^{\beta-1}f(t)dt \\ &= z^p+\sum_{n=1}^{\infty}\frac{\Gamma(\beta+p+n)}{\Gamma(\alpha+\beta+p+n)}\frac{\Gamma(\alpha+\beta+p)}{\Gamma(\beta+p)}a_{n+p}z^{n+p},\ (\alpha>0;\beta>-1). \end{aligned}$$

Motivated by Liu-Owa integral operator, we introduce the Liu-Owa $q-$integral operator as below:

Definition 1.1 *Let $\alpha>0,\beta>-1,p\in\mathbb{N},q\in(0,1)$, then the Liu-Owa $q-$integral operator $\mathcal{L}^{\alpha,p}_{\beta,q}:\mathcal{A}_p\to\mathcal{A}_p$ is defined by*

$$\mathcal{L}^{\alpha,p}_{\beta,q}f(z)=\binom{p+\alpha+\beta-1}{p+\beta-1}_q\frac{[\alpha]_q}{z^{\beta}}\int_0^z\left(1-\frac{qt}{z}\right)_q^{\alpha-1}t^{\beta-1}f(t)d_qt. \tag{1.7}$$

Using (1.1), (1.2), (1.3) and (1.4), we obtain power series of the operator $\mathcal{L}^{\alpha,p}_{\beta,q}f(z)$ by

$$\mathcal{L}^{\alpha,p}_{\beta,q}f(z)=z^p+\sum_{n=1}^{\infty}\frac{\Gamma_q(\beta+p+n)}{\Gamma_q(\alpha+\beta+p+n)}\frac{\Gamma_q(\alpha+\beta+p)}{\Gamma_q(\beta+p)}a_{n+p}z^{n+p}.$$

For special values of the parameters, we obtain the following famous integral operators as special cases:

(i) For $q\to1^-$, the operator $\mathcal{L}^{\alpha,p}_{\beta,q}f(z)$ reduces to the $p-$valent integral operator introduced by Liu and Owa in [15].

(ii) For $p=1$, the operator $\mathcal{L}^{\alpha,p}_{\beta,q}f(z)$ reduces to the $q-$integral operator introduced by Mahmood et al. in [16].

(iii) For $p=1,q\to1^-$, the operator $\mathcal{L}^{\alpha,p}_{\beta,q}f(z)$ reduces to the Jung-Kim-Srivastava integral operator introduced in [12].

(iv) For $p=1,\alpha=1$, the operator $\mathcal{L}^{\alpha,p}_{\beta,q}f(z)$ reduces to the $q-$Bernardi integral operator introduced by Noor [17].

(v) For $p=1,\alpha=1,q\to1^-$, the operator $\mathcal{L}^{\alpha,p}_{\beta,q}f(z)$ reduces to the Bernardi operator [3].

Making use of Liu-Owa $q-$integral operator $\mathcal{L}^{\alpha,p}_{\beta,q}f(z)$ and the class of $k-$uniformly $p-$valent starlike function of order γ, we define the following new subclass:

Definition 1.2 *Let $0\le\gamma<p,\ (p\in\mathbb{N}),\ q\in(0,1),\ k\ge0,\ \alpha>0,\beta>-1$. A function $f\in\mathcal{A}_p$ is in the class $k-\mathcal{US}^*_p(q;\gamma)$ if and only if for all $z\in\mathbb{D}$*

$$Re\left(\frac{zD_q(\mathcal{L}^{\alpha,p}_{\beta,q}f(z))}{\mathcal{L}^{\alpha,p}_{\beta,q}f(z)}-\gamma\right)>k\left|\frac{zD_q(\mathcal{L}^{\alpha,p}_{\beta,q}f(z))}{\mathcal{L}^{\alpha,p}_{\beta,q}f(z)}-[p]_q\right|. \tag{1.8}$$

Inequality in (1.8) can be written equivalently as:

$$\frac{zD_q(\mathcal{L}^{\alpha,p}_{\beta,q}f(z))}{\mathcal{L}^{\alpha,p}_{\beta,q}f(z)} \prec h_{p,k}(z), \tag{1.9}$$

where $h_{p,k}$ *is given by (1.6).*

The main idea of the paper is constructed on Liu-Owa $q-$integral operator and its applications in multivalent functions. We first define the subclass of $k-$uniformly $p-$valent starlike function of order γ by using the Liu-Owa $q-$integral operator. Next, we obtain integral representation formula, coefficient estimates, growth and distortion bounds and radii of $k-$uniformly starlikeness, convexity and close-to-convexity for the functions belonging to this subclass.

Unless otherwise stated, we assume in the reminder of the article that $q \in (0,1)$, $0 \le \gamma < p$, $(p \in \mathbb{N})$, $k \ge 0$, $\alpha > 0, \beta > -1$ and $z \in \mathbb{D}$.

2. Main results

We first begin with integral representation formula for functions belonging to the class $k-\mathcal{US}^*_p(q;\gamma)$.

Theorem 2.1 *Let* $f \in k-\mathcal{US}^*_p(q;\gamma)$. *Then*

$$\mathcal{L}^{\alpha,p}_{\beta,q}f(z) \prec z\exp\int_0^z \frac{h_{p,k}(w(\xi))-1}{\xi}d\xi, \tag{2.1}$$

where w *is analytic in* $\mathbb{D}$ *with* $w(0)=0$ *and* $|w(z)|<1$. *In addition, for* $|z|=\mu$ *we have*

$$\exp\left(\int_0^1 \frac{h_{p,k}(w(-\mu))-1}{\mu}d\mu\right) \le \left|\frac{\mathcal{L}^{\alpha,p}_{\beta,q}f(z)}{z}\right| \le \exp\left(\int_0^1 \frac{h_{p,k}(w(\mu))-1}{\mu}d\mu\right), \tag{2.2}$$

where $h_{p,k}$ *is defined by (1.6).*

Proof If $f \in k-\mathcal{US}^*_p(q;\gamma)$, then there exists a Schwarz function w in $\mathbb{D}$ with $w(0)=0$ and $|w(z)|<1$ such that (1.9) can be written as

$$\frac{zD_q(\mathcal{L}^{\alpha,p}_{\beta,q}f(z))}{\mathcal{L}^{\alpha,p}_{\beta,q}f(z)} = h_{p,k}(w(z)),$$

which implies that

$$\frac{D_q(\mathcal{L}^{\alpha,p}_{\beta,q}f(z))}{\mathcal{L}^{\alpha,p}_{\beta,q}f(z)} - \frac{1}{z} = \frac{h_{p,k}(w(z))-1}{z}.$$

Integrating both sides of the last expression, we get

$$\log\mathcal{L}^{\alpha,p}_{\beta,q}f(z) - \log z = \int_0^z \frac{h_{p,k}(w(\xi))-1}{\xi}d\xi.$$

Consequently, we obtain

$$\log\frac{\mathcal{L}^{\alpha,p}_{\beta,q}f(z)}{z} = \int_0^z \frac{h_{p,k}(w(\xi))-1}{\xi}d\xi. \tag{2.3}$$

Principle of subordination shows that

$$\mathcal{L}^{\alpha,p}_{\beta,q}f(z) \prec z \exp \int_0^z \frac{h_{p,k}(w(\xi))-1}{\xi} d\xi.$$

Moreover, we note that the function $h_{p,k}$ maps the disk $0<|z|<\mu<1$ onto a domain which is convex and symmetric with respect to the real axis, that is;

$$h_{p,k}(-\mu|z|) \leq Re(h_{p,k}(w(z\mu))) \leq h_{p,k}(\mu|z|),\ \ \mu \in (0,1),$$

which satisfies

$$h_{p,k}(-\mu) \leq h_{p,k}(-\mu|z|), \quad h_{p,k}(\mu|z|) \leq h_{p,k}(\mu)$$

and

$$\int_0^1 \frac{h_{p,k}(w(-\mu|z|))-1}{\mu} d\mu \leq Re\biggl(\int_0^1 \frac{h_{p,k}(w(\mu))-1}{\mu} d\mu\biggr) \leq \int_0^1 \frac{h_{p,k}(w(\mu|z|))-1}{\mu} d\mu.$$

By using (2.3) in the above relation we get

$$\int_0^1 \frac{h_{p,k}(w(-\mu|z|))-1}{\mu} d\mu \leq \left| \log \frac{\mathcal{L}^{\alpha,p}_{\beta,q}f(z)}{z} \right| \leq \int_0^1 \frac{h_{p,k}(w(\mu|z|))-1}{\mu} d\mu.$$

Consequently, we obtain

$$\exp\biggl(\int_0^1 \frac{h_{p,k}(w(-\mu))-1}{\mu} d\mu\biggr) \leq \left| \frac{\mathcal{L}^{\alpha,p}_{\beta,q}f(z)}{z} \right| \leq \exp\biggl(\int_0^1 \frac{h_{p,k}(w(\mu))-1}{\mu} d\mu\biggr).$$

This completes the proof. □

Next, we prove sufficient coefficient condition of the class $k-\mathcal{US}^*_p(q;\gamma)$ as below:

Theorem 2.2 *Let the function* f *defined by (1.5) satisfies the inequality*

$$\sum_{n=1}^{\infty} \bigl([n+p]_q(k+1)-(k[p]_q+\gamma)\bigr)\psi_{n+p}|a_{n+p}| \leq [p]_q-\gamma, \tag{2.4}$$

where

$$\psi_{n+p} = \frac{\Gamma_q(\beta+p+n)}{\Gamma_q(\alpha+\beta+p+n)} \frac{\Gamma_q(\alpha+\beta+p)}{\Gamma_q(\beta+p)}, \tag{2.5}$$

then $f \in k-\mathcal{US}^*_p(q;\gamma)$.

Proof To show that $f \in k-\mathcal{US}^*_p(q;\gamma)$, it is enough to prove that

$$k\left|\frac{zD_q(\mathcal{L}^{\alpha,p}_{\beta,q}f(z))}{\mathcal{L}^{\alpha,p}_{\beta,q}f(z)} - [p]_q\right| - Re\biggl(\frac{zD_q(\mathcal{L}^{\alpha,p}_{\beta,q}f(z))}{\mathcal{L}^{\alpha,p}_{\beta,q}f(z)} - [p]_q\biggr) \leq [p]_q-\gamma.$$

Let

$$\left|\frac{zD_q(\mathcal{L}_{\beta,q}^{\alpha,p}f(z))}{\mathcal{L}_{\beta,q}^{\alpha,p}f(z)}-[p]_q\right| = \left|\frac{zD_q(\mathcal{L}_{\beta,q}^{\alpha,p}f(z))-[p]_q\mathcal{L}_{\beta,q}^{\alpha,p}f(z)}{\mathcal{L}_{\beta,q}^{\alpha,p}f(z)}\right|$$
$$= \left|\frac{\sum_{n=1}^{\infty}([n+p]_q-[p]_q)\psi_{n+p}a_{n+p}z^{n+p}}{z^p+\sum_{n=1}^{\infty}\psi_{n+p}a_{n+p}z^{n+p}}\right|$$
$$\le \frac{\sum_{n=1}^{\infty}\big([n+p]_q-[p]_q\big)\psi_{n+p}|a_{n+p}|}{1-\sum_{n=1}^{\infty}\psi_{n+p}|a_{n+p}|}. \tag{2.6}$$

Due to (2.4) it follows that

$$1-\sum_{n=1}^{\infty}\psi_{n+p}|a_{n+p}|>0.$$

From (2.6), we have

$$k\left|\frac{zD_q(\mathcal{L}_{\beta,q}^{\alpha,p}f(z))}{\mathcal{L}_{\beta,q}^{\alpha,p}f(z)}-[p]_q\right|-Re\left(\frac{zD_q(\mathcal{L}_{\beta,q}^{\alpha,p}f(z))}{\mathcal{L}_{\beta,q}^{\alpha,p}f(z)}-[p]_q\right)$$
$$\le k\left|\frac{zD_q(\mathcal{L}_{\beta,q}^{\alpha,p}f(z))}{\mathcal{L}_{\beta,q}^{\alpha,p}f(z)}-[p]_q\right|+\left|\frac{zD_q(\mathcal{L}_{\beta,q}^{\alpha,p}f(z))}{\mathcal{L}_{\beta,q}^{\alpha,p}f(z)}-[p]_q\right|$$
$$\le (k+1)\left|\frac{zD_q(\mathcal{L}_{\beta,q}^{\alpha,p}f(z))-[p]_q\mathcal{L}_{\beta,q}^{\alpha,p}f(z)}{\mathcal{L}_{\beta,q}^{\alpha,p}f(z)}\right|$$
$$\le (k+1)\left\{\frac{\sum_{n=1}^{\infty}\big([n+p]_q-[p]_q\big)\psi_{n+p}|a_{n+p}|}{1-\sum_{n=1}^{\infty}\psi_{n+p}|a_{n+p}|}\right\}$$
$$\le [p]_q-\gamma,$$

which proves (2.4). □

We now introduce an additional new subclass of $k-\mathcal{US}_p^*(q;\gamma)$ with negative coefficients. Denote by $\mathcal{T}_p$ the subclass of $\mathcal{A}_p$ consisting of functions of the form:

$$f(z)=z^p-\sum_{n=1}^{\infty}|a_{n+p}|z^{n+p}. \tag{2.7}$$

We also let

$$k-\mathcal{TUS}_p^*(q;\gamma):=\mathcal{T}_p\cap k-\mathcal{US}_p^*(q;\gamma).$$

In the following result, it is shown that the condition (2.4) is also necessary for the functions in the subclass $k-\mathcal{TUS}_p^*(q;\gamma)$.

Theorem 2.3 *Let the function f defined by (2.7) satisfies the inequality*

$$\sum_{n=1}^{\infty}\big([n+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{n+p}|a_{n+p}|\le[p]_q-\gamma, \tag{2.8}$$

where ψ_{n+p} is given by (2.5), then $f\in k-\mathcal{TUS}_p^*(q;\gamma)$.

Proof In view of Theorem 2.2, we need to prove the "only if" part of this theorem. Let $f \in k-\mathcal{TUS}_p^*(q;\gamma)$ and z be real, then from (1.8) we get

$$\frac{[p]_q - \sum_{n=1}^{\infty}[n+p]_q\psi_{n+p}a_{n+p}z^n}{1-\sum_{n=1}^{\infty}\psi_{n+p}a_{n+p}z^n} - \gamma \geq k\left|\frac{\sum_{n=1}^{\infty}([n+p]_q-[p]_q)\psi_{n+p}a_{n+p}z^n}{1-\sum_{n=1}^{\infty}\psi_{n+p}a_{n+p}z^n}\right|.$$

Letting $z \to 1$ along the real axis, we get

$$\frac{([p]_q-\gamma)-\sum_{n=1}^{\infty}([n+p]_q(k+1)-(k[p]_q+\gamma))\psi_{n+p}a_{n+p}}{1-\sum_{n=1}^{\infty}\psi_{n+p}a_{n+p}} \geq 0.$$

This inequality gives the desired result. The result in (2.8) is sharp for the function

$$f(z) = z^p - \frac{[p]_q-\gamma}{\big([n+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{n+p}}z^{n+p}. \tag{2.9}$$

□

Corollary 2.4 *Let the function f defined by (2.7) be in the class $f \in k-\mathcal{TUS}_p^*(q;\gamma)$, then*

$$|a_{n+p}| \leq \frac{[p]_q-\gamma}{\big([n+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{n+p}}, \ (n \geq 1).$$

In view of Theorem 2.3, we establish the following growth and distortion theorems, respectively:

Theorem 2.5 *Let the function f defined by (2.7) be in the class $k-\mathcal{TUS}_p^*(q;\gamma)$, then for $|z| = r < 1$ we have*

$$|f(z)| \leq r^p + \frac{[p]_q-\gamma}{\big([1+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{1+p}}r^{1+p}, \tag{2.10}$$

$$|f(z)| \geq r^p - \frac{[p]_q-\gamma}{\big([1+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{1+p}}r^{1+p}. \tag{2.11}$$

These bounds are sharp.

Proof Since $f \in k-\mathcal{TUS}_p^*(q;\gamma)$, then we obtain

$$\big([1+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{1+p}|a_{n+p}|\sum_{n=1}^{\infty} \leq$$

$$\sum_{n=1}^{\infty}\big([n+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{n+p}|a_{n+p}| \leq [p]_q-\gamma,$$

which gives the following inequality:

$$\sum_{n=1}^{\infty}|a_{n+p}| \leq \frac{[p]_q-\gamma}{\big([1+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{1+p}}. \tag{2.12}$$

Therefore, by using (2.7) and (2.12) we get

$$|f(z)| = \left| z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \right| \leq |z|^p + \sum_{n=1}^{\infty} |a_{n+p}||z|^{1+p}$$
$$\leq |z|^p + \frac{[p]_q - \gamma}{\big([1+p]_q(k+1) - (k[p]_q + \gamma)\big)\psi_{1+p}} |z|^{1+p}.$$

Similarly, we also get

$$|f(z)| \geq |z|^p - \sum_{n=1}^{\infty} |a_{n+p}||z|^{1+p}$$
$$\geq |z|^p - \frac{[p]_q - \gamma}{\big([1+p]_q(k+1) - (k[p]_q + \gamma)\big)\psi_{1+p}} |z|^{1+p}.$$

These results are sharp for the function

$$f(z) = z^p - \frac{[p]_q - \gamma}{\big([1+p]_q(k+1) - (k[p]_q + \gamma)\big)\psi_{1+p}} z^{1+p}. \tag{2.13}$$

□

Theorem 2.6 *Let the function f defined by (2.7) be in the class $k - \mathcal{TUS}_p^*(q;\gamma)$, then for $|z| = r < 1$ we have*

$$|D_q f(z)| \leq [p]_q r^{p-1} + \frac{[1+p]_q([p]_q - \gamma)}{\big([1+p]_q(k+1) - (k[p]_q + \gamma)\big)\psi_{1+p}} r^p, \tag{2.14}$$

$$|D_q f(z)| \geq [p]_q r^{p-1} - \frac{[1+p]_q([p]_q - \gamma)}{[1+p]_q\big((k+1) - (k[p]_q + \gamma)\big)\psi_{1+p}} r^p. \tag{2.15}$$

These bounds are sharp for the function (2.13).

Proof In view of $q-$derivative of (2.7), we have

$$D_q f(z) = [p]_q z^{p-1} - \sum_{n=1}^{\infty} [n+p]_q a_{n+p} z^{n+p-1}.$$

Thus, making use of (2.12) and inequalities given by

$$|D_q f(z)| = \left| [p]_q z^{p-1} - \sum_{n=1}^{\infty} [n+p]_q a_{n+p} z^{n+p-1} \right|$$
$$\leq [p]_q |z|^{p-1} + \sum_{n=1}^{\infty} [1+p]_q |a_{n+p}| |z|^p$$

and

$$|D_q f(z)| \geq [p]_q |z|^{p-1} - \sum_{n=1}^{\infty} [1+p]_q |a_{n+p}| |z|^p,$$

we get (2.14) and (2.15), respectively. □

3. Radii of $k-$ uniform starlikeness, convexity, close-to-convexity

Making use of $q-$derivative operator, the subclasses of $p-$valently starlike, $p-$valently convex and $p-$valently close-to-convex of order γ, respectively, denoted by $\mathcal{S}_p^*(q,\gamma)$, $\mathcal{K}_p(q,\gamma)$ $\mathcal{C}_p(q,\gamma)$ can be defined by

$$\mathcal{S}_p^*(q,\gamma)=\left\{f\in\mathcal{A}_p: Re\left(\frac{zD_q(z)}{f(z)}\right)>\gamma, z\in\mathbb{D}\right\}, \tag{3.1}$$

$$\mathcal{K}_p(q,\gamma)=\left\{f\in\mathcal{A}_p: Re\left(\frac{D_q(zD_qf(z))}{D_q(f(z))}\right)>\gamma, z\in\mathbb{D}\right\}, \tag{3.2}$$

$$\mathcal{C}_p(q,\gamma)=\left\{f\in\mathcal{A}_p: Re\left(\frac{D_qf(z)}{z^{p-1}}\right)>\gamma, z\in\mathbb{D}\right\}, \tag{3.3}$$

where $q\in(0,1)$ and $0\leq\gamma<p$, $p\in\mathbb{N}$. Also, for the functions f given by (2.7), we write

$$\mathcal{TS}_p^*(q,\gamma):=\mathcal{T}_p\cap\mathcal{S}_p^*(q,\gamma),$$

$$\mathcal{TK}_p(q,\gamma):=\mathcal{T}_p\cap\mathcal{K}_p(q,\gamma),$$

$$\mathcal{TC}_p(q,\gamma):=\mathcal{T}_p\cap\mathcal{C}_p(q,\gamma).$$

In this section, we introduce radii of $k-$uniformly starlikeness, convexity and close-to-convexity for functions belonging to the class $k-\mathcal{TUS}_p^*(q;\gamma)$.

Theorem 3.1 *Let the function f defined by (2.7) be in the class $k-\mathcal{TUS}_p^*(q;\gamma)$. Then, f is $p-$valently starlike of order γ, $(0\leq\gamma<p)$ in $|z|<r_1$, where*

$$r_1=\inf_{n\geq1}\left\{\frac{\big([n+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{n+p}}{[p]_q-\gamma}\times\left(\frac{[p]_q-\gamma}{[n+p]_q-\gamma}\right)\right\}^{1/n}. \tag{3.4}$$

The result is sharp for the function given by (2.9).

Proof In view of (3.1), equivalently, it suffices to show that

$$\left|\frac{zD_qf(z)}{f(z)}-[p]_q\right|\leq[p]_q-\gamma,\quad (|z|<r_1). \tag{3.5}$$

Indeed we have

$$\left|\frac{zD_qf(z)}{f(z)}-[p]_q\right|=\left|\frac{-\sum_{n=1}^{\infty}([n+p]_q-[p]_q)a_{n+p}z^n}{1-\sum_{n=1}^{\infty}a_{n+p}z^n}\right|$$
$$\leq\frac{\sum_{n=1}^{\infty}([n+p]_q-[p]_q)|a_{n+p}||z|^n}{1-\sum_{n=1}^{\infty}|a_{n+p}||z|^n}.$$

Hence inequality in (3.5) is true if

$$\frac{\sum_{n=1}^{\infty}([n+p]_q-[p]_q)|a_{n+p}||z|^n}{1-\sum_{n=1}^{\infty}|a_{n+p}||z|^n}\leq[p]_q-\gamma.$$

Thus, we have

$$\sum_{n=1}^{\infty} \frac{[n+p]_q-\gamma}{[p]_q-\gamma}|a_{n+p}||z|^n \leq 1.$$

Using (2.8), we obtain

$$\left(\frac{[n+p]_q-\gamma}{[p]_q-\gamma}\right)|z|^n \leq \frac{\big([n+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{n+p}}{[p]_q-\gamma}$$

or equivalently

$$|z| \leq \left\{\frac{\big([n+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{n+p}}{[p]_q-\gamma} \times \left(\frac{[p]_q-\gamma}{[n+p]_q-\gamma}\right)\right\}^{1/n}.$$

This implies that

$$r_1 = \inf_{n\geq 1}\left\{\frac{\big([n+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{n+p}}{[p]_q-\gamma} \times \left(\frac{[p]_q-\gamma}{[n+p]_q-\gamma}\right)\right\}^{1/n}.$$

This is the desired result. □

Theorem 3.2 *Let the function* f *defined by (2.7) be in the class* $k-\mathcal{TUS}_p^*(q;\gamma)$. *Then,* f *is* $p-$*valently convex of order* γ, $(0 \leq \gamma < p)$ *in* $|z| < r_2$, *where*

$$r_2 = \inf_{n\geq 1}\left\{\frac{\big([n+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{n+p}}{[p]_q-\gamma} \times \left(\frac{[p]_q([p]_q-\gamma)}{[n+p]_q([n+p]_q-\gamma)}\right)\right\}^{1/n}. \tag{3.6}$$

The result is sharp for the function given by (2.9).

Proof In order to prove (3.6), it is sufficient to show that

$$\left|\frac{D_q(zD_qf(z))}{D_q(f(z))}-[p]_q\right| \leq [p]_q-\gamma, \quad (|z|<r_2).$$

With the similar technique given in Theorem 3.1, we get the proof. □

Theorem 3.3 *Let the function* f *defined by (2.7) be in the class* $k-\mathcal{TUS}_p^*(q;\gamma)$. *Then,* f *is* $p-$*valently close-to-convex of order* γ, $(0 \leq \gamma < p)$ *in* $|z| < r_3$, *where*

$$r_3 = \inf_{n\geq 1}\left\{\frac{\big([n+p]_q(k+1)-(k[p]_q+\gamma)\big)\psi_{n+p}}{[p]_q-\gamma} \times \left(\frac{[p]_q-\gamma}{[n+p]_q}\right)\right\}^{1/n}. \tag{3.7}$$

The result is sharp for the function given by (2.9).

Proof In order to prove (3.7), it is sufficient to show that

$$\left|\frac{D_qf(z)}{z^{p-1}}-[p]_q\right| \leq [p]_q-\gamma, \quad (|z|<r_3).$$

Using same method given in Theorem 3.1, we completes the proof. □

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