The Space-Time Critical Dimension of an Open Parabosonic String

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Abstract

Using analytical properties of a 1-loop open parabosonic M-point transition amplitude, we show that the space-time critical dimension depends on the order of the paraquantization.

1. Introduction

One of the main goals of quantum mechanics (QM) is to provide a consistent and unified description of the so-called wave-particle duality which is a direct consequence of the Heisenberg equations of motion. It turns out that the canonical commutation relations - which guarantee the Heisenberg equations - are not unique [1]. The general framework in which the canonical commutation relations are generalized is called paraquantization and characterized by an order parameter Q [2-9]. Although it is, in principle, possible to study the paraquantum observables within the usual Hilbert space, it is often convenient to use a larger Hilbert space in which the operators satisfy simple bilinear relations [2], [10-12]. Traditionally, for Focks-type irreducible representation of paraquantum theories

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with a unique vacuum state, this is done by means of the Green decomposition [2], [10-12]:

$$a_n = \sum_{\beta=1}^{Q} a_n^\beta$$  \hspace{1cm} (1)

where $Q$ is the order of the paraquantization, $\beta$ the Green index and $a_n^\beta$ is the bosonic annihilation operators with Green components satisfying the following bilinear but anomalous commutation relations:

$$\left[ a_{n}^{(\beta)}, a_{m}^{+(\alpha)} \right]_+ = 0 \quad \alpha \neq \beta$$

$$\left[ a_{n}^{(\alpha)}, a_{m}^{+(\alpha)} \right]_- = \delta_{mn}. \hspace{1cm} (2)$$

The purpose of this paper is to derive the space-time critical dimension for an open para-

bosonic string by using the meromorphic properties of the $M$-point transition amplitude.

In Section 2, we describe the formalism and in Section 3 we derive the critical dimension

and finally in Section 4 we draw our conclusions.

2. Formalism

The Nambu-Goto classical action of a free relativistic open bosonic string is given by [13]:

$$S = -\frac{1}{2\pi \alpha'} \int d\tau d\sigma[(\dot{x} \cdot \dot{x})^2 - \dot{x}^2 x'^2]^{1/2}; \hspace{1cm} (3)$$

where $\tau$ and $\sigma$ are dimensionless word-sheet parameters and $\alpha'$ is the string tension (here, "\(\cdot\)" as in $x'$ and "\(\cdot\)" as in $\dot{x}$ denotes $\frac{\partial}{\partial \tau}$ and $\frac{\partial}{\partial \tau}$, respectively). The general solution of the equations of motion in the light cone gauge is [13]:

$$x^i(\sigma, \tau) = q^i + 2\alpha' p^i + 2\alpha' \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} [a_n^i e^{-in\tau} + a_n^{+i} e^{in\tau}] \cos n\sigma, \hspace{1cm} (4)$$

where $q^i$ and $p^i$ are the string centre of mass coordinates and momentum, respectively.

After quantization the physical states $|\Psi\rangle_{phy}$ are subject to the Virasoro conditions:

$$L_n |\Psi\rangle_{phy} = 0 \quad n > 1$$

and

$$[L_0 - \alpha(0)] |\Psi\rangle_{phy} = 0,$$  \hspace{1cm} (5)

(here, $\alpha(0) = 1$) where the Virasoro generators $L_n$ and $L_0$ are given by:

$$L_n = \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \alpha_{n-m}^{i} \alpha_{m}^{i};$$

$$L_0 = \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \alpha_{-m}^{i} \alpha_{m}^{i}. \hspace{1cm} (6)$$
with
\[ \alpha_0^i = 2\alpha'p^i, \quad \alpha_{-n}^i = \sqrt{2\alpha'na_n^i}, \quad \alpha_n^i = \sqrt{2\alpha'na_n^i}. \]

It is to be noted that the string dynamical variables \( q^i, p^i, q^-, p^+ \) and \( a^i \) verify the following non-vanishing canonical commutation relations:

\[
\begin{align*}
[q^i, p^j] &= i\delta_{ij} \\
[q^-, p^+] &= -i \\
[a_n^i, a_m^j] &= \delta_{mn}\delta_{ij}.
\end{align*}
\]

Now, for the paraquantization the commutation relations (2.5) become:

\[
\begin{align*}
[q^i, p^j] &= i\delta_{ij} \\
[q^-, p^+] &= -i \\
[a_n^{(\alpha)}, a_m^{(\beta)}]_+ &= 0, \quad \alpha \neq \beta \\
[a_n^{(\alpha)}, a_m^{(\beta)}]_- &= \delta_{mn}\delta_{ij},
\end{align*}
\]

where we have used the Green decomposition (1.1) for \( a_n^i \) and \( a_m^{+j} \) and the fact that the observables, like \( q^i, p^i, q^-, p^+ \), which describe the center of mass coordinates and momentum of the string, should not be affected by the paraquantization [14-17]. In other words, the space-time properties of the string remain unchanged. This can be achieved by choosing a specific direction in the Green para-space-like relations [14-17]:

\[
\begin{align*}
q^{i(\alpha)} &= q^i\delta_{\alpha 1}, \quad p^{i(\alpha)} = p^i\delta_{\alpha 1}, \quad q^{-}(\alpha) &= q^-\delta_{\alpha 1}, \quad p^{+}(\alpha) = p^\pm\delta_{\alpha 1}.
\end{align*}
\]

3. M-Point Transition Amplitude

The 1-loop open parabosonic string M-point transition amplitude for a planar diagram with M external tachyons, which is topologically equivalent to a disk with a hole quenched in the interior and external lines located on the exterior edge, can be written as:

\[
A(1, 2, \ldots, M) = \int \prod_{\beta=1}^{Q} d^D p^{(\beta)} Tr[V(k_1, 1)\Delta V(k_2, 1)\cdots\Delta V(k_M, M)].
\]

(Here, \( k_j = \sum_{\alpha} m^j \) is the jth external tachyon momentum and propagator \( \Delta \) has is expressed as

\[
\Delta = (L_0 - \alpha(0))^{-1}
\]

with \( L_0 \) as the paraquantum Virasoro operator [15-17] given by:

\[
L_0 = -\sum_{\beta=1}^{Q}\sum_{m=1}^{\infty} :a_{-m}^{(\beta)}a_{m}^{(\beta)}:
\]
(we take $2\alpha' = 1$) and
\[ \alpha(0) = Q(D - 2)/24. \]
(“: :” means normal ordering). The paraquantum vertex operator $V(K_r, 1)$ has the expression
\[ V(k_r, 1) = e^{iL_0} V(k_r, 0) e^{iL_0}, \]
where
\[ V(k_r, 0) = g : \exp \left[ \frac{i}{2} \sum_{\gamma = 1}^{Q} \sum_{i = 1}^{D - 2} K_i^{(\gamma)} q_i^{(\gamma)} \right] \tag{13} \]
where $g$ is the coupling.

It is to be noted that the propagator has the following useful integral representation:
\[ \Delta = \int dx x^{L_0 - \alpha(0) - 1}. \tag{14} \]

Now, using the integral representation (3.5) and the fact that
\[ x^{L_0} V(k_r, 1) = V(k_r, x) x^{L_0}, \tag{15} \]
where
\[ V(k_r, x) = e^{i x L_0} V_0(k_r, 0) e^{-i x L_0}, \tag{16} \]
the transition amplitude (3.1) can be rewritten as:
\[ A(1, 2, \ldots, M) = \int^{M} dx_1 \int^{M} dD \frac{p^{(\beta)}}{\beta} Tr \left[ V_0(k_1, x_1) \cdots V_0(k_M, x_1 \cdots x_M) w^{L_0 - \alpha(0)} \right] \tag{18} \]
with
\[ w = x_1 x_2 \cdots x_M. \tag{19} \]
Noticing that
\[ \prod_{i=1}^{M} dx_i = dw \prod_{i=1}^{M-1} \frac{d \rho_i}{\rho_i}, \tag{20} \]
where
\[ \rho_i = x_1 x_2 \cdots x_i, \tag{21} \]
Eq. (3.8) can be simplified to:
\[ A(1, 2, \ldots, M) = \int \frac{dw}{w^{1 + Q(D - 2)/24}} \int^{M-1} d\rho_r \frac{d \rho_r}{\rho_r} (\rho_r - \rho_{r+1}) I(1, 2, \ldots, M) \tag{22} \]
with
\[ I(1, 2, \ldots, M) = \int \frac{d\omega}{\beta = 1} d^{D-1} p^{(\beta)} \text{Tr} \left[ V_0(k_1, \rho_1)V_0(k_2, \rho_2) \cdots V_0(k_M, \rho_M) w^{k_n} \right]. \] (23)

The trace (3.13) can be easily calculated by using the paraquantum coherent state method [2]. In fact, using the identity
\[ \text{Tr} M = \sum_{\beta = 1}^{Q} \frac{1}{\pi} \int d\lambda_n^{(\beta)} d\lambda_m^{(\beta)} e^{-|\lambda_n^{(\beta)}|^2} \langle \lambda_n^{(\beta)} | M | \lambda_m^{(\beta)} \rangle, \] (24)

where
\[ |\lambda_n^{(\beta)}\rangle = \exp \left[ \lambda_n^{(\beta)} a_n^{+ (\beta)} \right] |0\rangle \] (25)

and
\[ \langle \mu_m^{(\alpha)} | \lambda_m^{(\beta)} \rangle = \delta_{\alpha \beta} \delta_{nm} \lambda_m^{(\beta)} |\lambda_m^{(\beta)}\rangle \] (26)

\[ \langle \mu_n^{(\alpha)} | \lambda_n^{(\beta)} \rangle = \exp \left[ \mu_n^{(\alpha)} a_n^{+ (\beta)} \right] \left[ \delta_{\alpha \beta} \delta_{nm} \right] \] (27)

(\mu_n^{(\alpha)} \text{ and } \lambda_m^{(\beta)} \text{ are arbitrary complex numbers}) and the fact that
\[ x \sum_{i=1}^{D-2} a_n^{(\beta)} a_m^{(\beta)} |\lambda_m^{(\beta)}\rangle = \delta_{nm} |\lambda_m^{(\beta)}\rangle x \] (28)

and
\[ (0 \exp \left( -K_l \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} a_n^{(\beta)} \right) + \sum_{k=2}^{D-2} \sum_{m=1}^{\infty} m \lambda_m^{(\beta)} a_n^{(\beta)} \exp \left( K_l \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} a_n^{(\beta)} \right) \right) |0\rangle = (1 - x)^{K_l} K_j \delta_{ij}, \] (29)

straightforward calculations give:
\[ I(1, 2, \ldots, M) = Q[f(w)]^{-Q(D-2)} \left( \frac{2\pi}{\ln w} \right)^{Q(D-2)/2} \prod_{l<j} [\Psi_{IJ}]^{k_l k_j}, \] (30)

where
\[ f(w) = \prod_{n=1}^{\infty} (1 - w^n) \] (31)

and
\[ \Psi_{IJ} = -2\pi i \exp \left[ \frac{\ln^2 C_{JI}}{2 \ln w} \right] \vartheta_1 \left( \frac{\ln C_{JI}}{2\pi i} |\ln w| \right) / \vartheta_1 \left( 0 \ln w \right) \] (32)

with
\[ C_{JI} = \rho_J / \rho_I \] (33)
and \( \vartheta_1 \) (resp. \( \vartheta_1' \)) being Jacobi function (resp. its derivative). Now, introducing new variables

\[
\nu_r = \frac{\ln \rho_r}{\ln w}
\]

and

\[
q = \exp\left(\frac{2\pi^2}{\ln w}\right),
\]

and using the identities

\[
\frac{d\rho_r}{\rho_r} \prod_{r=1}^{M-1} \vartheta_i(\rho_r - \rho_{r+1}) = \frac{1}{2\pi^2} (-\ln w)^{M+1} \frac{dq}{q} \prod_{r=1}^{M-1} \vartheta_i(\nu_{r+1} - \nu_r) d\nu_r
\]

and

\[
\frac{1}{w^{Q(D-2)/24}} [f(w)]^{-Q(D-2)} = \left(-\frac{\pi}{\ln q}\right)^{Q(D-2)/2} \frac{1}{q^{Q(D-2)/12}} [f(q^2)]^{-Q(D-2)},
\]

the transition amplitude (3.12) takes the form:

\[
A(1, 2, \ldots, M) = \frac{Q}{\pi} g^M \int_0^1 \prod_{i=1}^{M-1} \vartheta_i(\nu_{i+1} - \nu_i) d\nu_i \int_0^1 dq q^{-1+Q(2-D)/12} W^{-1-Q(2-D)/24} \times \left(-\frac{2\pi^2}{\ln q}\right)^M [f(q^2)]^{-Q(D-2)} \prod_{I<J} \Psi_{IJ}^{k_I k_J}.
\]

Now, by extracting the \( \ln q \) factor from (3.22) and using the kinematical relation

\[
\sum_{I<J} K_I K_J = -1/2 \sum_I K_I^2 = -M,
\]

and in order that the integrand in (3.28) can be a meromorphic function, i.e., the only existent singularities are a finite number of poles, the power of the \( W \) factor must vanish. Consequently, one deduces that the space time critical dimension must verify the relation

\[
D = \frac{24}{Q} + 2.
\]

4. Conclusion

We conclude that the meromorphic property of the \( M \)-point transition amplitude with external tachyons and the generalization of the quantization procedure are strongly related to the critical space-time dimension of the parabosonic string \( D = \frac{24}{Q} + 2 \). More details are being investigated [19].

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