Turk J Phys
34 (2010) , 173 – 180.

doi:10.3906/fiz-1007-2

On Robertson-Walker universe model with variable cosmological term and gravitational constant in cosmological relativity theory

Kangujam Priyokumar SINGH
Department of Mathematics, Ideal Girls' College, Imphal, 795 001, INDIA
e-mail: pk_mathematics@yahoo.co.in

Received 02.07.2010

Abstract

This paper presents cosmological models in which the gravitational and cosmological constants G and Λ are time-dependent. We find a variety of solutions for the variation of cosmological parameters. It is also found that, in the case of matter dominated Robertson-Walker Universe, if the cosmological constant $\Lambda \neq 0$, the Hubble's constant H is not constant at all but inversely proportional to the cosmic time. The conservation law for the energy momentum tensor is still valid with our law of variation: $\Lambda \propto \mathrm{R}^{-2}$ and $\mathrm{G}\rho \propto \frac{\Lambda}{8\pi}$.

Key Words: Cosmology, cosmological model, homogeneous isotropic, cosmic time, variable cosmological constant

PACS. 98.80. Jk . 98.80.Es.

Mathematics Subject Classification. (2010) 83F05.

1. Introduction

In general relativity the cosmological constant Λ may be regarded as the measure of energy density of the vacuum and can, in principle, lead to the avoidance of the big-bang singularity, which is a characteristic of other Friedmann-Robertson-Walker (FRW) models. The cosmological constant problem has a long history, and while there are many possible resolutions, none has gained widespread acceptance. In classical general relativity, the energy density and pressure of the vacuum obeys the relation $\rho c^2 = -p = \Lambda c^4/8\pi G$, where c is the speed of light and G is the gravitational constant. The astrophysically-determined value of Λ, for the present epoch at least, is small but in quantum theory, the vacuum (or zero-point) energies associated with particle interactions lead to a value of Λ which is big. This discrepancy may be large as 10^{120}. The authors of [1, 2] have recently reviewed this problem, and outlined a resolution wherein the classical value of Λ is essentially the statistical value "left over" from numerous stronger interactions described by quantum field theory.

Some of the recent discussions on the cosmological constant "problem" and consequence on cosmology with a time-varying cosmological constant are invested by [3]. [4] stressed that the relation $\Lambda = Bt^{-2}$ plays an important role in cosmology. Abdel-Rahman in [5] found the same relation. Chen and Wu in [6] suggested that $\Lambda \propto R^{-2}$ where R is the Scale factor in Robertson-Walker metric. Berman in [7, 8] discussed variable Λ, cosmological models with G as constant or variable using the law of variation $\Lambda \propto t^{-2}$ rather than the law recently suggested by [6] (i.e. $\Lambda \propto \mathrm{R}^{-2}$). The authors of [9] have constructed homogeneous isotropic cosmological models with variable cosmological constant and gravitational constant satisfying the present day observational data and the initial conditions as proposed by [10] with some additional requirements [11–19]. However, all vacuum decaying cosmological models do not predict acceleration. Workers in [20–22] have proposed a cosmological model with a cosmological constant of form $\Lambda = B\frac{\ddot{R}}{R}$, where B is a constant. Following the same decay law, recently [23, 24] has investigated cosmic acceleration with positive cosmological constant and analyze the implication of model built-in cosmological constant. one of the motivations for introducing Λ term is to reconcile the age parameter and the density parameter of the universe with recent observational data. Also, [25] obtained a cosmological model with cosmological term of the form $\Lambda \propto \frac{\ddot{R}}{R}$ and have also investigated well known astrophysical phenomenon.

In this paper we discuss a model with variables Λ and G while the conservation law for the energy-momentum tensor is still valid with the assumptions that $\Lambda \propto \mathrm{R}^{-2}$ and

$\mathrm{G}\rho \propto \frac{\Lambda}{8\pi}$ we find that in the case of matter dominated Robertson-Walker universe if the cosmological constant $\Lambda \neq 0$,the Hubble's constant is not a constant at all but it varies inversely with cosmic time. Also we found that the Einstein field equations with variables G and Λ with our assumption.

2. Mathematical formulation

The Einstein field equations with time-dependent cosmological and gravitational "constants" is given by the relation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, G and Λ being the variable gravitational and cosmological terms.

The usual Conservation law is

$$T^{\mu\nu}_{;\nu} = 0. \tag{2}$$

By applying equation (2) in (1), we get

$$8\pi G,_{\mu} T^{\mu\nu} + \Lambda,_{\mu} g^{\mu\nu} = 0. \tag{3}$$

It shows that Λ and G vary simultaneously. Workers in [7, 8] and [5] have constructed homogenous and isotropic cosmological models with the use of equations (1), (2), (3) and by assuming the variations of $G\rho$ and Λ as

$$G\rho = At^{-2}, \Lambda = Bt^{-2}. \tag{4}$$

Let us consider the line element of Robertson-Walker universe

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right], \tag{5}$$

where $R(t)$ is the scale factor and K is the curvature index which takes the values +1, 0, -1.

Also let us consider a particle at the origin $r = 0$ and another particle at r then the proper distance X between the two particles at a time t is given by

$$X = R(t)\int_0^T \frac{dr}{\sqrt{1-Kr^2}} = \begin{cases} R\sin h^{-1} r, & \text{for open model;} \\ Rr, & \text{for flat model;} \\ R\sin^{-1} r, & \text{for closed model,} \end{cases} \tag{6}$$

which implies $X \propto R(t)$.

By differentiating X with respect to t, we obtain the proper velocity V of the particle at r relative to the particle at the origin relining that r remains constant. Therefore,

$$V = X = R\int_0^T \frac{dr}{\sqrt{1-Kr^2}} = \frac{\dot{R}}{R}X. \tag{7}$$

By Hubble's law,

$$V = HX. \tag{8}$$

We know that

$$H \propto \frac{\dot{R}}{R}. \tag{9}$$

The energy momentum tensor is given by the equation

$$T^{\mu\nu} = (p+\rho)\, u^{\mu}u^{\nu} - pg^{\mu\nu}, \tag{10}$$

where p is the proper pressure and ρ is the proper density. The four velocity of the fluid particle u^{μ} is given by

$$u^{\mu} = (0,0,0,1). \tag{11}$$

As did Berman, we consider the Einstein field equation as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{12}$$

And the conservation law is

$$T^{\mu\nu}_{,\nu} = 0. \tag{13}$$

By inserting (13) into equation (12), we get

$$8\pi G_{,\mu}\, T^{\mu}_{\nu} + \Lambda_{,\mu}\, g^{\mu}_{\nu} = 0. \tag{14}$$

Inserting equations (5), (10) and (11) into (12), we get

$$-2\frac{\dot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{K}{R^2} = 8\pi G p - \Lambda, \tag{15}$$

and

$$3\frac{\dot{R}^2}{R^2} + \frac{3K}{R^2} = 8\pi G\rho + \Lambda. \tag{16}$$

Differentiating equation (16) with respect to t, we get

$$8\pi\left(\dot{G}\rho+G\dot{\rho}\right)=-3\frac{\dot{R}}{R}\left(-\frac{2\ddot{R}}{R}+2\frac{\dot{R}^2}{R^2}+\frac{2K}{R^2}\right)-\dot{\Lambda}. \tag{17}$$

Adding (15) and (16), we get

$$-2\frac{\ddot{R}}{R}+2\left(\frac{\dot{R}}{R}\right)^2+\frac{2K}{R^2}=8\pi G\left(p+\rho\right). \tag{18}$$

Then equation (17) becomes

$$\dot{G}\rho+G\dot{\rho}+3\frac{\dot{R}}{R}G\rho=-3\frac{\dot{R}}{R}Gp-\frac{\dot{\Lambda}}{8\pi}. \tag{19}$$

From equation (19), we get

$$R^3\left(\dot{G}\rho+\frac{\dot{\Lambda}}{8\pi}\right)+G\left[\frac{d}{dt}\left(\rho R^3\right)+3p\dot{R}R^2\right]=0. \tag{20}$$

By using equations (5), (10) and (11), the above equation (12) reduces to

$$\frac{d}{dt}\left(\rho R^3\right)+3pR^2\dot{R}=0. \tag{21}$$

Using (21) in (20), we get

$$\dot{G}\rho+\frac{\dot{\Lambda}}{8\pi}=0. \tag{22}$$

From equations (15) and (16), we get

$$3\ddot{R}=-4\pi GR\left[3p+\rho-\frac{\Lambda}{4\pi G}\right]. \tag{23}$$

When we consider the time dependence of various physical quantities in the expanding universe. These are conveniently parameterized in terms of the scale function $R(t)$. If ψ be the Baryon density, then

$$\Psi(t)R^3(t)=\text{Constant}. \tag{24}$$

The above result is intuitively obvious. The element of spatial volume is seen from equation (5) to be proportional to $R^3(t)$; hence the baryon number density must obey the law

$$J^\lambda=\Psi u^\lambda, \tag{25}$$

where u^λ is given by

$$u^\mu=0=u_\mu;u^o=u_o. \tag{26}$$

Since in the matter-dominated era, the dominant contribution to the density ρ comes from non-relativistic mass densities, $\rho\tilde{-}\Psi m$, where m is the nucleon rest mass.

Then equation (24) reduces to

$$\rho R^3=\text{constant}. \tag{27}$$

Moreover, the density of radiation in the universe follows the law $\rho \propto R^{-4}$ (R = radius of the universe) while that of its matter content obeys $\rho \propto R^{-3}$. At the present time the latter density greatly outweighs the former [26].Using this law we can obtain the same relation given by the equation (27).

Then equation (21) becomes

$$3pR^2\dot{R} = 0, \tag{28}$$

since the scale factor of the universe $R \neq 0$. Relation (28) can be survived only when $p = 0$.Thus the problem reduces to the case of dust distribution.

Now, from equation (23), we obtain

$$3\ddot{R} = -4\pi GR\left[\rho - \frac{\Lambda}{4\pi G}\right]. \tag{29}$$

Using equation (4) in equation (29), the latter reduces to

$$H \propto \frac{1}{t}. \tag{30}$$

The result given by equation (30) shows that the Hubble's Constant varies inversely as the cosmic time t. This result tallies if which result obtained in solving a problem of a flat matter dominated (Einstein- de Sitter) universe where the Cosmological constant $\Lambda = 0$, [27].

The difference obtained in the two solutions is that, in the present problem $\Lambda \neq 0$, while in the (Einstein-de Sitter) universe the Cosmological constant $\Lambda = 0$. Equations (16), (21), (22) and (23) are the fundamental equations governing a homogeneous and isotropic cosmological model of the universe. Equations (16), (21) and (22) are same as the corresponding field equations in general relativity but we have obtained additional Equation (23) due to variation of G and Λ.

3. Solutions with variation of G and Λ and their interpretations

We consider the dependence of the pressure in a homogeneous isotropic model of the universe as

$$p = \varepsilon\rho, (0 \leq \varepsilon \leq 1). \tag{31}$$

From equations (21) and (31), we obtain

$$\frac{\dot{\rho}}{\rho} = -3(1+\varepsilon)\frac{\dot{R}}{R}. \tag{32}$$

We assume,

$$G\rho \propto \frac{\Lambda}{8\pi}, \text{i.e.} G\rho = \frac{\alpha\Lambda}{8\pi}, \tag{33}$$

and

$$\Lambda \propto \frac{1}{R^2}, \text{ i.e. } \Lambda \propto \gamma\frac{1}{R^2}, \tag{34}$$

where α and γ are the constants of proportionality. From equation (34), we get

$$\dot{\Lambda} = -2\gamma R^{-3}\dot{R}, \tag{35}$$

then equation (22) reduces to

$$\dot{G}\rho = \frac{1}{4\pi}\gamma R^{-3}\dot{R}. \tag{36}$$

From equations (36) and (33), we get

$$\frac{\dot{G}}{G} = \frac{2}{\alpha} \cdot \frac{\dot{R}}{R}. \tag{37}$$

From equation (2), we obtain

$$\dot{G}\rho = -\frac{\dot{\Lambda}}{8\pi}. \tag{38}$$

From (33) and (38), we get

$$\frac{\dot{G}}{G} = -\frac{1}{\alpha} \cdot \frac{\dot{\Lambda}}{\Lambda}. \tag{39}$$

From (37) and (39), we get

$$\frac{\dot{\Lambda}}{\Lambda} = -2\frac{\dot{R}}{R}. \tag{40}$$

Using the conditions

$$\Lambda = \Lambda_0, t = t_0, G = G_0, \rho = \rho_0, \tag{41}$$

and integrating (32), (37) and (40), we get

$$\rho = \rho_0 \left(\frac{R}{R_0}\right)^{-3(1+\varepsilon)}, \tag{42}$$

$$G = G_0 \left(\frac{R}{R_0}\right)^{\frac{2}{\alpha}}, \tag{43}$$

and

$$\Lambda = \Lambda_0 \left(\frac{R_0}{R}\right)^2. \tag{44}$$

From equation (16) and (33), we have

$$\dot{R} = \sqrt{\left(\frac{1+\alpha}{3}\right)\Lambda R^2 - K}. \tag{45}$$

By using the relation (34), we get

$$\dot{R} = \sqrt{\left\{\frac{(1+\alpha)}{3}\gamma - K\right\}}. \tag{46}$$

From equations (23), (31) and (33), we get

$$3\ddot{R} = \left\{1 - \frac{\alpha(1+3\varepsilon)}{2}\right\}\frac{\gamma}{R}. \tag{47}$$

During evolution of the Universe, we propose that the density decreases from ρ to ρ_0 and G increases from G to G_0 as R expands from R to R_0.

From equations (42) and (43), we have $1 + \varepsilon > 0$ and $\alpha < 0$.

Since $G > 0$ and $\rho > 0$ during the expansion of the Universe and $\alpha < 0$, from equation (44) it is clear that $\Lambda < 0$.

Also, from equation (22) for $\rho > 0$ and $\dot{G} < 0$, we must have $\dot{\Lambda} > 0$, and hence during the expansion of the Universe Λ must be negative and increasing function of time by equation (40).

On the other hand if we assume that $\Lambda > 0$, $\dot{\Lambda} > 0$ and $\rho > 0$ then $\dot{G} > 0$ and $G < 0$ during the expansion of the Universe which is unphysical.

From equation (47) it is clear that for $\alpha < 0$, $\varepsilon > 0$, $\Lambda < 0$ implies that $\ddot{R} < 0$ i.e. the model is decelerating.

4. Concluding remarks

We have seen that in the case of matter dominated Robertson-Walker universe, if the cosmological constant $\Lambda \neq 0$, the Hubble's constant is not a constant at all, but varies inversely with cosmic time. Also we found that the Einstein field equations with variables G and Λ such that $G\rho = \frac{\alpha\Lambda}{8\pi}$ and $\Lambda \propto \frac{1}{R^2}$, it is found that the usual conservation law implies that $\alpha < 0$, for $G > 0$ and $G < 0$, $\Lambda < 0$ for $\dot{\Lambda} > 0$. We conclude that the relation $\Lambda \propto t^{-2}$, which was pointed out by [4], seems to play a major role in cosmology.

References

[1] T. Padamanabhan, *Class. Quantum Gravit.*, **9**, (2002), 5387.

[2] T. Padamanabhan, *Phys. Rep.*, **380, (**2003), 235.

[3] S. Carrol, W. Press, E. Turner, *Annu. Rev. Astron. Astrophys.*, **30, (**1992), 499.

[4] M. Berman, M. Som, *Gen. Relativ. Gravit*, **22**, (1990), 625.

[5] A.-M. M. Abdel-Rahman, *Gen. Relativ. Gravit.*, **22**, (1990), 655.

[6] W. Chen, Y. Wu, *Phy. Review*, **41**, (1990), 695.

[7] M. Berman, *Gen. Relativ. Gravit*, **23**, (1991a), 465.

[8] M. Berman, *Phy. review D*, **43**, (1991b), 1075.

[9] H. Pande, R. Chandra, R. Mishra, *Ind. J. Pure Appl. Maths.*, **32**, (2000), 161.

[10] C. Sivaram, K. Sinha, E. Lord, *Curr. Sci.*, **44**, (1975), 143.

[11] C. Tsagas, R. Maartens, *Class. Quantum Gravit.*, **17**, (2000), 2215.

[12] V. Sahni, A. Starobinsky, *Int. J. Mod. Phys.D*, **9**, (2000), 373.

[13] P. Peebles, B. Ratra, *Rev. Mod. Phys.*, **75,** (2003), 559.

[14] R. Vishwakarma, *Class. Quantum Gravit.*, **18**, (2001a), 1159.

[15] R. Vishwakarma, *Gen. Relativ. Gravit.*, **33, (**2001b) **,** 1973.

[16] R. Vishwakarma, *Mon. Not. Roy. Astron. Soc.*, **331**, (2002a), 776.

[17] R. Vishwakarma, *Class. Quantum Gravit.*, **119**,(2002b), 4747.

[18] A. Pradhan, A.Yadav, L. Yadav, *Cezch.J.Phys.*, **55**, (2005a), 503.

[19] A. Pradhan, S. Srivastava, M. Yadav, *Astrophys. Space Sci.*, **298**, (2005b), 419.

[20] A. Al-Rawaft, M. Taha, *Gen. Relativ. Gravit.*, **28**, (1996), 935.

[21] A. Al-Rawaft, *Gen. Relativ. Gravit.*, **28,** (1996), 935.

[22] J. Overduin, F. Cooperstock, *Phys. Rev. D*, **58**, **(**1998), 043506.

[23] A. Arbab, *J. Cosmol. Astropart. Phys.*, **05**, (2003a), 008.

[24] A. Arbab, *J. Class. Quantum Gravit.*, **20**, (2003b), 93.

[25] G. Khadekar, A. Pradhan, M. Molaei, *Int. J. mod. Phys.D*, **15**, (2006), 95.

[26] W. Rindler, *Relativity, Oxford University Press*, (2001), 141.

[27] R. Mould, *Basic Relativity, Springer-Verlago*, (New York, 1995), 407. ISBN 0-387-941886.