Exact solutions for the double sinh-Gordon and generalized form of the double sinh-Gordon equations by using \((G'/G)\)-expansion method

Hossein KHEIRI and Azizeh JABBARI
Faculty of Mathematical Sciences, University of Tabriz, Tabriz-IRAN
e-mails: h-kheiri@tabrizu.ac.ir, azizeh.jabbari@gmail.com

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Abstract

In this paper, the \((G'/G)\)-expansion method is applied to seek traveling wave solutions to the double sinh-Gordon and the generalized form of the double sinh-Gordon equations. With the aid of a symbolic computation system, two types of more general traveling wave solutions (including hyperbolic functions and trigonometric functions) with free parameters are constructed. Solutions concerning solitary and periodic waves are also given by setting the two arbitrary parameters, involved in the traveling waves, as special values.

Key Words: \((G'/G)\)-expansion method, double sinh-Gordon equation, generalized form of the double sinh-Gordon equation, traveling wave solutions

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1. Introduction

The sinh-Gordon equation,

\[ u_{tt} - u_{xx} + \sinh u = 0, \]  

(1)
gained its importance because of the kink and antikink solutions with the collisional behaviors of solitons that arise from this equation. The equation appears in integrable quantum field theory, kink dynamics, and fluid dynamics [1–8].

Many powerful methods, such as Bäcklund transformation, inverse scattering method, Hirota bilinear forms, pseudo spectral method, the tanh method, tanh-sech method, the sine-cosine method [9–14], and many others were successfully applied to nonlinear equations. Recently, Wang et al. [15] proposed the \((G'/G)\)-expansion method and showed that it is powerful for finding analytic solutions of PDEs. Next, Bekir [16]
applied the method to some nonlinear evolution equations gaining traveling wave solutions. More recently, Zhang et al. [17] proposed a generalized \( (G'/G) \)-expansion method to improve and extend Wang et al.’s work [15] for solving variable coefficient equations and high dimensional equations. Kheiri et al. applied this method for solving the combined and the double combined sinh-cosh-Gordon equations [18]. Also, Zhang [17] solved the equations with the balance numbers of which are not positive integers, by this method.

The \( (G'/G) \)-expansion method is based on the explicit linearization of nonlinear differential equations for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Computations are performed with a computer algebra system such as Maple to deduce the solutions of the nonlinear equations in an explicit form. The solution process of the method is direct, effective and convenient due to solving the auxiliary equation of second-order differential equation with constant coefficients. We should mention that this technique is restricted to the search for single soliton solutions. To formally derive N-soliton solutions of any completely integrable equation, authors mainly used the Cole-Hopf transformation method combined with the Hirota’s bilinear method, where it was shown that soliton solutions are just polynomials of exponentials [19].

In this paper we apply the \( (G'/G) \)-expansion method to the double sinh-Gordon equation and its generalized form.

2. Description of the \( (G'/G) \)-expansion method

We suppose that the given nonlinear partial differential equation for \( u(x,t) \) has the form

\[
P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ... ) = 0,
\]

where \( P \) is a polynomial in its arguments. The essence of the \( (G'/G) \)-expansion method is the following steps:

**Step 1.** Seek traveling wave solutions of equation (2) by taking \( u(x,t) = U(\xi) \), \( \xi = x - ct \), and transform equation (2) to the ordinary differential equation

\[
Q(U, U', U'', ...) = 0,
\]

where prime denotes the derivative with respect to \( \xi \).

**Step 2.** If possible, integrate equation (3) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

**Step 3.** Introduce the solution \( U(\xi) \) of equation (3) in the finite series form

\[
U(\xi) = \sum_{i=0}^{N} a_i \left( \frac{G'(\xi)}{G(\xi)} \right)^i,
\]

where \( a_i \) are real constants with \( a_N \neq 0 \) to be determined, \( N \) is a positive integer to be determined. The function \( G(\xi) \) is the solution of the auxiliary linear ordinary differential equation

\[
G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0,
\]

where \( \lambda \) and \( \mu \) are real constants to be determined.

**Step 4.** Determine \( N \). This, usually, can be accomplished by balancing the linear term(s) of highest order
with the highest order nonlinear term(s) in equation (3).

**Step 5.** Substituting (4) together with (5) into equation (3) yields an algebraic equation involving powers of \((G'/G)\). Equating the coefficients of each power of \((G'/G)\) to zero gives a system of algebraic equations for \(a_i\), \(\lambda\), \(\mu\) and \(c\). Then, we solve the system with the aid of a computer algebra system, such as Maple, to determine these constants. On the other hand, depending on the sign of the discriminant \(\Delta = \lambda^2 - 4\mu\), the solutions of equation (5) are well known to us. So, as a final step, we can obtain exact solutions of equation (2).

3. **The double sinh-Gordon equation**

We first solve the double sinh-Gordon equation

\[
u_{tt} - ku_{xx} + 2\alpha \sinh u + \beta \sinh(2u) = 0,
\]

where \(\alpha\) and \(\beta\) are nonzero real constants. Making the transformation \(u(x, t) = u(\xi), \xi = x - ct\) and integrating once with respect to \(\xi\), we get

\[(c^2 - k)u'' + 2\alpha \sinh u + \beta \sinh(2u) = 0.\]

By applying the Painlevé transformation,

\[v = e^u,\]

or equivalently,

\[u = \ln v,\]

we have

\[
sinh(u) = \frac{v - v^{-1}}{2}, \quad \sinh(2u) = \frac{v^2 - v^{-2}}{2}, \quad \cosh(u) = \frac{v + v^{-1}}{2},
\]

then

\[u = \text{arccosh}\left[\frac{v + v^{-1}}{2}\right].\]

Consequently, we can write the double sinh-Gordon equation (7) as the ODE

\[
\beta v^4 + 2\alpha v^3 - 2\alpha v - \beta + 2(c^2 - k)vv'' - 2(c^2 - k)(v')^2 = 0.
\]

Balancing \(v^4\) with \(vv''\) gives

\[M = 1.\]

The \((G'/G)\)-expansion method allows us to use the finite expansion

\[v(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right), \quad a_1 \neq 0.\]
Substituting (14) into (12), setting coefficients of \( \frac{G_i}{G} \) \((i = 0, 1, \ldots, 4)\) to zero, we obtain the following underdetermined system of algebraic equations for \( a_0, a_1, c, \lambda, \) and \( \mu \):

\[
\begin{align*}
\left( \frac{G'}{G} \right)^0 & : -\beta + \beta a_0^4 + 2\alpha a_0^3 - 2\alpha a_0 - 2c^2 a_1^2 \mu^2 + 2ka_0^2 a_1 \lambda \mu - 2ka_0 a_1 \lambda \mu = 0, \\
\left( \frac{G'}{G} \right)^1 & : -2\alpha a_1 + 4\beta a_1 a_0^3 + 6\alpha a_1 a_0^2 + 2c^2 a_0 a_1 \lambda^2 + 4c^2 a_0 a_1 \mu - 2c^2 a_1^2 \lambda \mu - 2ka_0 a_1 \lambda^2 \\
& \quad -4ka_0 a_1 \mu + 2ka_1^2 \lambda \mu = 0, \\
\left( \frac{G'}{G} \right)^2 & : 6\beta a_1^2 a_0^2 + 6\alpha a_1^2 a_0 + 6c^2 a_0 a_1 \lambda - 6ka_0 a_1 \lambda = 0, \\
\left( \frac{G'}{G} \right)^3 & : 2\alpha a_1^3 + 4\beta a_1^3 a_0 + 4c^2 a_0 a_1 + 2c^2 a_1^2 \lambda - 4ka_0 a_1 - 2ka_1^2 \lambda = 0, \\
\left( \frac{G'}{G} \right)^4 & : \beta a_1^4 + 2c^2 a_1^2 - 2ka_1^2 = 0.
\end{align*}
\]

Solving this system using Maple gives

\[
\begin{align*}
a_0 & = \frac{-\alpha}{\beta} \pm \frac{\lambda}{\beta} \sqrt{\frac{\alpha^2 - \beta^2}{\lambda^2 - 4\mu}}, \\
a_1 & = \pm 2 \frac{\lambda}{\beta} \sqrt{\frac{\alpha^2 - \beta^2}{\lambda^2 - 4\mu}}, \\
c & = \pm \sqrt{k - 2 \frac{\alpha^2 - \beta^2}{\beta(\lambda^2 - 4\mu)}}, \\
\alpha & > \beta, \quad c^2 < k,
\end{align*}
\]

where \( \lambda \) and \( \mu \) are arbitrary constants. Substituting equation (15) into equation (14) yields

\[
v(\xi) = \frac{-\alpha}{\beta} \pm \frac{\lambda}{\beta} \sqrt{\frac{\alpha^2 - \beta^2}{\lambda^2 - 4\mu}} \pm \frac{2}{\beta} \sqrt{\frac{\alpha^2 - \beta^2}{\lambda^2 - 4\mu}} \left( \frac{G'}{G} \right),
\]

where \( \xi = x - \left( \pm \sqrt{k - 2 \frac{\alpha^2 - \beta^2}{\beta(\lambda^2 - 4\mu)}} \right) t \).

Substituting general solutions of equation (5) into equation (16), we have two types of traveling wave solutions of the double sinh-Gordon equation as follows.

- When \( \lambda^2 - 4\mu > 0 \),

\[
v_1(\xi) = \frac{-\alpha}{\beta} \pm \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \left( \frac{c_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + c_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{c_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + c_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right), \quad \alpha > \beta,
\]

where \( \xi = x - \left( \pm \sqrt{k - 2 \frac{\alpha^2 - \beta^2}{\beta(\lambda^2 - 4\mu)}} \right) t \).
Recall that if \(c^\alpha < \beta\) for where \(2 \neq 0\), In particular, if \(c^\alpha \cdot \lambda > 0\), then the solutions, for \(\alpha > \beta, k > c^2\),

\[
v_2(\xi) = \frac{-\alpha}{\beta} + \frac{\beta^2 - \alpha^2}{\beta} \left( \frac{-c_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + c_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{c_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + c_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right), \quad \alpha < \beta, \tag{18}\]

where \(\xi = x - (\pm \sqrt{k - 2\frac{c^2 - \beta^2}{\lambda^2 - 4\mu}}) t\). In solutions (20) and (21), \(c_1\) and \(c_2\) are left as free parameters. In particular, if \(c_1 \neq 0, c_2 = 0\), then \(u_1\) becomes

\[
u_1(\xi) = \text{arccosh} \left\{ \frac{1}{2} \left( \frac{-\alpha}{\beta} \pm \frac{\beta^2 - \alpha^2}{\beta} \cdot \tan \left[ \frac{\alpha^2 - \beta^2}{2\beta(\lambda^2 - 4\mu)} (x - ct) \right] \right) \right\}, \tag{22}\]

if \(c_2 \neq 0, c_1 = 0\),

\[
u_1(\xi) = \text{arccosh} \left\{ \frac{1}{2} \left( \frac{-\alpha}{\beta} \pm \frac{\beta^2 - \alpha^2}{\beta} \cdot \cot \left[ \frac{\alpha^2 - \beta^2}{2\beta(\lambda^2 - 4\mu)} (x - ct) \right] \right) \right\}, \tag{23}\]
where $\alpha > \beta$ and $k > c^2$, which are the solitary solutions of the double sinh-Gordon equation.

If $c_1 \neq 0$, $c_2 = 0$, $u_2$ becomes

$$u_2(\xi) = \text{arccosh} \left\{ \frac{1}{2} \left( \frac{-\alpha}{\beta} \pm \sqrt{\frac{\beta^2 - \alpha^2}{\beta}} \tan \left[ \sqrt{\frac{\beta^2 - \alpha^2}{2\beta(k - c^2)}} (x - ct) \right] \right) \right\},$$

(24)

if $c_2 \neq 0$, $c_1 = 0$,

$$u_2(\xi) = \text{arccosh} \left\{ \frac{1}{2} \left( \frac{-\alpha}{\beta} \pm \sqrt{\frac{\beta^2 - \alpha^2}{\beta}} \cot \left[ \sqrt{\frac{\beta^2 - \alpha^2}{2\beta(k - c^2)}} (x - ct) \right] \right) \right\},$$

(25)

where $\beta > \alpha$ and $k > c^2$. These are the periodic solutions of the double sinh-Gordon equation.

These special results show that our method obtains general solutions and it can be seen that the solutions obtained by using tanh function method [20, 21], are special cases of our solutions.

4. Generalized form of the double sinh-Gordon equation

In this section we consider generalized form of the double sinh-Gordon equation [8] given by

$$u_{tt} - ku_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu) = 0, \quad n \geq 1.$$  

(26)

Here, $\alpha$ and $\beta$ are nonzero real constants. Making the transformation $u(x, t) = u(\xi) = x - ct$, and integrating once with respect to $\xi$, we get

$$(c^2 - k)u'' + 2\alpha \sinh(nu) + \beta \sinh(2nu) = 0.$$  

(27)

Proceeding as before, we use the transformation,

$$v = e^{nu},$$

(28)

or equivalently

$$u = \frac{1}{n} \ln v,$$

(29)

we have

$$\sinh(nu) = \frac{v - v^{-1}}{2}, \quad \sinh(2nu) = \frac{v^2 - v^{-2}}{2}, \quad \cosh(nu) = \frac{v + v^{-1}}{2},$$

(30)

then

$$u = \frac{1}{n} \text{arccosh} \left[ \frac{v + v^{-1}}{2} \right].$$

(31)
As a result, this transformation will change the generalized form of double sinh-Gordon equation (27) to the ODE
\[ \beta n v' + 2 \alpha n v^3 - 2 \alpha n v - \beta n + 2(c^2 - k)v v'' - 2(c^2 - k)(v')^2 = 0. \] (32)

Balancing the \( v'^4 \) with \( vv'' \) gives
\[ M = 1. \] (33)

Consequently, we have
\[ v(\xi) = a_0 + a_1 \left( \frac{G'}{G} \right), \quad a_1 \neq 0. \] (34)

Substituting (34) into (32), setting coefficients of \( \left( \frac{G'}{G} \right)^i (i = 0, 1, \ldots, 4) \) to zero, we obtain the following under-determined system of algebraic equations for \( a_0, a_1, c, \lambda \) and \( \mu \):
\[
\begin{align*}
\left( \frac{G'}{G} \right)^0 & : - \beta n + 2 \alpha a_0^4 + 2 \alpha a_0^3 - 2 \alpha a_0 - 2 c^2 a_1^2 \mu^2 + 2 k a_1^2 \mu^2 + 2 c^2 a_0 a_1 \lambda \mu - 2 k \alpha a_0 a_1 \lambda \mu = 0, \\
\left( \frac{G'}{G} \right)^1 & : - 2 \alpha a_1 + 4 \beta na_1 a_0^3 + 6 \alpha na_1 a_0^2 + 2 c^2 a_0 a_1 \lambda^2 + 4 c^2 a_0 a_1 \mu - 2 c^2 a_1^2 \lambda \mu - 2 k a_0 a_1 \lambda^2 \\
& \quad - 4 k a_0 a_1 \mu + 2 k a_1^2 \lambda \mu = 0, \\
\left( \frac{G'}{G} \right)^2 & : 6 \beta n a_1^2 a_0^2 + 6 \alpha a_1^2 a_0 + 6 c^2 a_0 a_1 \lambda - 6 k a_0 a_1 \lambda = 0, \\
\left( \frac{G'}{G} \right)^3 & : 2 \alpha a_1^3 + 4 \beta na_1^3 a_0 + 4 c^2 a_0 a_1 + 2 c^2 a_1^2 \lambda - 4 k a_0 a_1 - 2 k a_1^2 \lambda = 0, \\
\left( \frac{G'}{G} \right)^4 & : \beta n a_1^4 + 2 c^2 a_1^2 - 2 k a_1^2 = 0.
\end{align*}
\]

Solving this system by Maple, gives
\[
\begin{align*}
a_0 &= \frac{-\alpha}{\beta} \pm \frac{\lambda}{\beta} \sqrt{\frac{\alpha^2 - \beta^2}{\lambda^2 - 4 \mu}}, \\
a_1 &= \frac{2}{\beta} \sqrt{\frac{\alpha^2 - \beta^2}{\lambda^2 - 4 \mu}}, \\
c &= \pm \sqrt{k - 2n \frac{\alpha^2 - \beta^2}{\beta(\lambda^2 - 4 \mu)}},
\end{align*}
\]
\[ \alpha > \beta, \quad c^2 < k, \] (35)

where \( \lambda \) and \( \mu \) are arbitrary constants. Substituting equation (35) into equation (34) yields
\[ v(\xi) = \frac{-\alpha}{\beta} \pm \frac{\lambda}{\beta} \sqrt{\frac{\alpha^2 - \beta^2}{\lambda^2 - 4 \mu}} \pm \frac{2}{\beta} \sqrt{\frac{\alpha^2 - \beta^2}{\lambda^2 - 4 \mu}} \left( \frac{G'}{G} \right), \] (36)
where \( \xi = x - (\pm \sqrt{k - 2n\frac{\alpha^2 - \beta^2}{\beta(\alpha^2 - 4\mu^2)}}) t \).

Substituting general solutions of equation (5) into equation (36), we have two types of traveling wave solutions of the generalized form of the double sinh-Gordon equation as follows.

- When \( \lambda^2 - 4\mu > 0 \),
  \[
  v_1(\xi) = \frac{-\alpha}{\beta} \pm \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \left( \frac{c_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + c_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{c_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + c_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right), \quad \alpha > \beta, \tag{37}
  \]

where \( \xi = x - (\pm \sqrt{k - 2n\frac{\alpha^2 - \beta^2}{\beta(\alpha^2 - 4\mu^2)}}) t \).

- When \( \lambda^2 - 4\mu < 0 \),
  \[
  v_2(\xi) = \frac{-\alpha}{\beta} \pm \frac{\sqrt{\beta^2 - \alpha^2}}{\beta} \left( \frac{-c_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + c_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{c_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + c_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right), \quad \alpha < \beta, \tag{38}
  \]

where \( \xi = x - (\pm \sqrt{k - 2n\frac{\alpha^2 - \beta^2}{\beta(\alpha^2 - 4\mu^2)}}) t \).

Recall that
\[
  u = \frac{1}{n} \arccosh \left[ \frac{v + v^{-1}}{2} \right],
\]
therefore, we obtain the solutions, for \( \alpha > \beta, \ k > c^2 \),

\[
  u_1(\xi) = \frac{1}{n} \arccosh \left\{ \frac{1}{2} \left( \frac{-\alpha}{\beta} \pm \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \left( \frac{c_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + c_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{c_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + c_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right) \right) \right\},
\]

\[
  \left. \left. \left( \frac{-\alpha}{\beta} \pm \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \left( \frac{c_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + c_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{c_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + c_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right) \right) \right]^{-1} \right\},
\]

for \( \alpha < \beta, \ k > c^2 \),

\[
  u_2(\xi) = \frac{1}{n} \arccosh \left\{ \frac{1}{2} \left( \frac{-\alpha}{\beta} \pm \frac{\sqrt{\beta^2 - \alpha^2}}{\beta} \left( \frac{-c_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + c_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{c_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + c_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right) \right) \right\},
\]

\[
  \left. \left. \left( \frac{-\alpha}{\beta} \pm \frac{\sqrt{\beta^2 - \alpha^2}}{\beta} \left( \frac{-c_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + c_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{c_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + c_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right) \right) \right]^{-1} \right\},
\]

where \( \xi = x - (\pm \sqrt{k - 2n\frac{\alpha^2 - \beta^2}{\beta(\alpha^2 - 4\mu^2)}}) t \).

In solutions (40), (41), \( c_1 \) and \( c_2 \) are left as free parameters.
In particular, if \( c_1 \neq 0, c_2 = 0 \), then \( u_1 \) becomes

\[
\begin{align*}
    u_1(\xi) &= \frac{1}{n} \arccosh \left\{ \frac{1}{2} \left( \frac{-\alpha}{\beta} \pm \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \tanh \left[ \sqrt{\frac{n(\alpha^2 - \beta^2)}{2\beta(k^2 - c^2)}} (x - ct) \right] \right) \right. \\
    & \quad \left. + \left( \frac{-\alpha}{\beta} \pm \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \tanh \left[ \sqrt{\frac{n(\alpha^2 - \beta^2)}{2\beta(k^2 - c^2)}} (x - ct) \right] \right) \right\} ,
\end{align*}
\]

(42)

If \( c_2 \neq 0, c_1 = 0 \),

\[
\begin{align*}
    u_1(\xi) &= \frac{1}{n} \arccosh \left\{ \frac{1}{2} \left( \frac{-\alpha}{\beta} \pm \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \coth \left[ \sqrt{\frac{n(\alpha^2 - \beta^2)}{2\beta(k^2 - c^2)}} (x - ct) \right] \right) \right. \\
    & \quad \left. - \left( \frac{-\alpha}{\beta} \pm \frac{\sqrt{\alpha^2 - \beta^2}}{\beta} \coth \left[ \sqrt{\frac{n(\alpha^2 - \beta^2)}{2\beta(k^2 - c^2)}} (x - ct) \right] \right) \right\} ,
\end{align*}
\]

(43)

where \( \alpha > \beta \) and \( k > c^2 \). These are the solitary solutions of the generalized form of the double sinh-Gordon equation.

If \( c_1 \neq 0, c_2 = 0 \), \( u_2 \) becomes

\[
\begin{align*}
    u_2(\xi) &= \frac{1}{n} \arccosh \left\{ \frac{1}{2} \left( \frac{-\alpha}{\beta} \pm \frac{\sqrt{\beta^2 - \alpha^2}}{\beta} \tan \left[ \sqrt{\frac{n(\beta^2 - \alpha^2)}{2\beta(k^2 - c^2)}} (x - ct) \right] \right) \right. \\
    & \quad \left. - \left( \frac{\alpha}{\beta} \pm \frac{\sqrt{\beta^2 - \alpha^2}}{\beta} \tan \left[ \sqrt{\frac{n(\beta^2 - \alpha^2)}{2\beta(k^2 - c^2)}} (x - ct) \right] \right) \right\} ,
\end{align*}
\]

(44)

If \( c_2 \neq 0, c_1 = 0 \),

\[
\begin{align*}
    u_2(\xi) &= \frac{1}{n} \arccosh \left\{ \frac{1}{2} \left( \frac{-\alpha}{\beta} \pm \frac{\sqrt{\beta^2 - \alpha^2}}{\beta} \cot \left[ \sqrt{\frac{n(\beta^2 - \alpha^2)}{2\beta(k^2 - c^2)}} (x - ct) \right] \right) \right. \\
    & \quad \left. - \left( \frac{\alpha}{\beta} + \frac{\sqrt{\beta^2 - \alpha^2}}{\beta} \cot \left[ \sqrt{\frac{n(\beta^2 - \alpha^2)}{2\beta(k^2 - c^2)}} (x - ct) \right] \right) \right\} ,
\end{align*}
\]

(45)

where \( \beta > \alpha \) and \( k > c^2 \). These are the periodic solutions of the generalized form of the double sinh-Gordon equation.

Again, comparing these special results with Wazwaz’s results [20] shows that our results are more general.

5. Conclusions

In this paper, the \((\frac{G'}{G})\)-expansion method is used to conduct an analytic study on the double sinh-Gordon and generalized form of the double sinh-Gordon equations. The exact traveling wave solutions being determined
in this study are more general, and it is not difficult to arrive at some known analytic solutions for certain choices of the parameters \(c_1\) and \(c_2\). Comparing the proposed method with the methods used in [1–8, 15], show that the \((G'/G)\)-expansion method is not only simple and straightforward, but also avoids tedious calculations.

References