Plane Symmetric Solutions of Gravitational Field Equations in Five Dimensions

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Abstract

We present the effective field equations obtained from a generalized gravity action with Euler-Poincaré term and a cosmological constant in a $D$ dimensional bulk space-time. A class of plane-symmetric solutions that describe a 3-brane world embedded in a $D = 5$ dimensional bulk space-time are given.

1. Introduction

Brane-world theories that receive a lot of interest recently are strictly motivated by string models [1]. They were mainly proposed to provide new solutions to the hierarchy problem and compactification of extra dimensions [2],[3]. The main content of the brane-world idea is that we live in a four dimensional world embedded in a higher dimensional bulk space-time. According to the brane-world scenarios, the gauge fields, fermions and scalar fields of the Standard Model should be localised on a 3-brane, while gravity may freely propagate into the higher dimensional bulk.

In our previous work [4] we derived covariant gravitational field equations on a 3-brane embedded in a five-dimensional bulk space-time with $\mathbb{Z}_2$ symmetry in a generalization that included a dilaton scalar as well as the second order Euler-Poincaré density in the action. We introduced a general ADM-type coordinate setting to show that the effective gravitational field equations on the 3-brane remain unchanged, however, the evolution equations off the brane are significantly modified due to the acceleration of normals to the brane surface in the non-geodesic, ADM slicing of space-time.

In the second part of this paper, using the language of differential forms, we present the field equations of a generalized gravity model with a dilaton 0-form and an axion 3-form in Einstein frame from an action that includes the second order Euler-Poincaré term and a cosmological constant in a $D$-dimensional bulk space-time. In the third part, we present some plane-symmetric solutions that generalize the well-known domain-wall solution [5].
2. Model

We consider a $D$-dimensional bulk space-time manifold $M$ equipped with a metric $g$ and a torsion-free, metric compatible connection $\nabla$. We determine our gravitational field equations by a variational principle from a $D$-dimensional action that includes the second order Euler-Poincaré term and a cosmological constant

$$I[e, \omega, \phi, H] = \int_M \mathcal{L}$$

where in the Einstein frame the Lagrangian density $D$-form [6]

$$\mathcal{L} = \frac{1}{2} R^{ab} \wedge * (e_a \wedge e_b) - \frac{\alpha}{2} d\phi \wedge * d\phi + \frac{\beta}{2} e^{-\beta_2 \phi} H \wedge * H + \Lambda e^{-\beta_1 \phi} * 1 + \eta \frac{1}{4} R^{ab} \wedge \bar{R}^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d) + (de^a + \omega^a \wedge e^b) \wedge \lambda_a + (dH - \frac{\varepsilon}{2} R_{ab} \wedge R^{ab}) \wedge \mu.$$ (2)

Here $\lambda_a$ and $\mu$ are Lagrange multiplier forms that upon variation impose the zero-torsion and anomaly-freedom constraints.

The final form of the variational field equations to be solved are the Einstein field equations

$$\frac{1}{2} R^{ab} \wedge * (e_a \wedge e_b \wedge e_c) = -\frac{\alpha}{2} \tau_a[\phi] + \frac{\beta}{2} e^{-\beta_2 \phi} \tau_a[H] + \Lambda e^{-\beta_1 \phi} * e_c$$

$$-\frac{\eta}{4} R^{ab} \wedge \bar{R}^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d)$$

$$-2\varepsilon \beta D(e^{-\beta_2 \phi} \tau_b (R_{c}^{b} \wedge * H)) - \frac{\varepsilon \beta}{2} e_c \wedge D(e^{-\beta_2 \phi} \tau_b (R_{c}^{b} \wedge * H)),$$ (3)

where the dilaton stress-energy forms

$$\tau_a[\phi] = \iota_a d\phi \wedge d\phi + d\phi \wedge \iota_a * d\phi$$

and the axion stress-energy forms

$$\tau_a[H] = \iota_a H \wedge * H + H \wedge \iota_a * H ,$$

the dilaton scalar field equation

$$\alpha d(*d\phi) = \frac{\beta_2 \beta}{2} e^{-\beta_2 \phi} H \wedge * H + \Lambda \beta_1 e^{-\beta_1 \phi} * 1 ,$$ (4)

and the axion field equations

$$dH = \frac{\varepsilon}{2} R_{ab} \wedge R^{ab} , \quad d(e^{-\beta_2 \phi} * H) = 0.$$ (5)

3. Plane symmetric solutions in $D = 5$

We investigate below a class of plane symmetric solutions in 5-dimensions. We consider the metric

$$g = -f^2(t,\omega) dt^2 + u^2(t,\omega) d\omega^2 + g^2(t,\omega) \left( \frac{dx^2 + dy^2 + dz^2}{(1 + \bar{k} r^2)^2} \right),$$ (6)

the dilaton scalar field

$$\phi = \phi(t,\omega)$$ (7)
and 3-form gauge field

\[ H = h(t, \omega) \frac{dx \wedge dy \wedge dz}{(1 + kr^2)^4} \]  \tag{8}

in terms of local coordinates

\[ x^M : \{x^0 = t, x^5 = \omega, x^i = x, x^2 = y, x^3 = z\}. \]

We choose our co-frame 1-forms as

\[ e^0 = f(t, \omega) dt, \quad e^5 = u(t, \omega) d\omega, \quad e^i = g(t, \omega) \frac{dx^i}{(1 + kr^2)^4}, \quad i = 1, 2, 3. \] \tag{9}

Then we calculate the Levi-Civita connection 1-forms

\[ \omega^0_i = \frac{g_u}{fu} e^i, \quad \omega^i_j = k \frac{u^2}{2g} (x^i e^j - x^j e^i), \] \tag{10}

\[ \omega^0_5 = \frac{u^5}{fu} e^5 + \frac{f_\omega}{fu} e^0, \quad \omega^i_5 = g_{\omega u} e^i. \] \tag{11}

and the corresponding curvature 2-forms

\[ R^{ij} = \frac{1}{fu} \left\{ k + \left( \frac{g_u}{fu} \right)^2 \right\} e^i \wedge e^j, \] \tag{12}

\[ R^{05} = \frac{1}{fu} \left\{ \left( \frac{g_u}{fu} \right)_\omega - \left( \frac{u^2}{u} \right)_t \right\} e^0 \wedge e^5, \] \tag{13}

\[ R^{0i} = \frac{1}{fu} \left\{ \left( \frac{g_u}{fu} \right)_t - \frac{f_\omega g_u}{u^2} \right\} e^0 \wedge e^i + \frac{1}{u g} \left\{ \left( \frac{g_u}{fu} \right)_\omega - \frac{u^2 g_u}{fu} \right\} e^5 \wedge e^i, \] \tag{14}

\[ R^{i5} = \frac{1}{fu} \left\{ \frac{f_\omega g_u}{fu} - \left( \frac{g_u}{fu} \right)_u \right\} e^i \wedge e^0 + \frac{1}{u g} \left\{ \left( \frac{g_u}{fu} \right)_\omega - \frac{g_u^2}{fu} \right\} e^5 \wedge e^i. \] \tag{15}

From these expressions we note that \( R_{ab} \wedge R^{ab} = 0 \). Therefore \( dH = 0 \) implying that

\[ H = \frac{Q}{g^6} e^1 \wedge e^2 \wedge e^3 \] \tag{16}

where \( Q \) may be identified as a magnetic charge. Now, for simplicity, we let \( k = 0 \) and take the functions \( g, f \) and \( u \) independent of time. Then we obtain the following system of coupled ordinary differential equations \((\prime\) denotes derivative with respect to \( \omega \)):

\[ 2G - 2C - B - A = -\eta(2CG - AB) - \frac{\alpha}{2} \left( \frac{\phi'}{u} \right)^2 - \frac{\beta Q^2}{2 g^6} e^{-\beta_1 \phi} + \Lambda e^{-\beta_2 \phi}; \] \tag{17}

\[ 3A - 3G = 3\eta GA + \frac{\alpha}{2} \left( \frac{\phi'}{u} \right)^2 - \frac{\beta Q^2}{2 g^6} e^{-\beta_2 \phi} - \Lambda e^{-\beta_1 \phi}; \] \tag{18}
\[ 3C + 3A = -3\eta CA - \frac{\alpha}{2} \left( \frac{\phi'}{u} \right)^2 - \frac{\beta}{2} Q^2 - \frac{\beta}{2} g^0 e^{-\beta_2 \phi} - \Lambda e^{-\beta_1 \phi}, \]  \hspace{1cm} (19)

\[ \alpha \left( \frac{\phi' g^3 f}{u} \right)' \frac{1}{g^3 f u} = \frac{\beta_2 \beta}{2} e^{-\beta_2 \phi} \frac{Q^2}{g^0} + \Lambda \beta \dot{e}^{-\beta_1 \phi}. \]  \hspace{1cm} (20)

where

\[ A = - \left( \frac{f'}{g} \right)^2 \frac{1}{u^2} \quad B = - \left( \frac{u'}{u} \right)^2, \]  \hspace{1cm} (21)

\[ C = - \frac{f' g'}{u^2 f g} \quad G = \left( \frac{g'}{u} \right)^2 \frac{1}{u g}. \]  \hspace{1cm} (22)

We will give below some special classes of solutions:

**Case:** \( \phi = \text{constant}, \) \( H = 0 \) and \( \eta = 0. \)

Here the Euler-Poincaré term is absent, \( H = 0 \) and the dilaton scalar is constant. We obtain the AdS solution in 5-dimensions that is also known as Randall-Sundrum model [3]:

\[ g = d\omega^2 + e^{\beta \phi} (-dt^2 + dx^2 + dy^2 + dz^2). \]  \hspace{1cm} (23)

where \( p^2 = \frac{\Lambda}{4}. \)

**Case:** \( \phi = \text{constant}, \) \( H = 0. \)

Here \( H = 0 \) and the dilaton scalar is constant. Solutions are given by the metric

\[ g = d\omega^2 + e^{\beta \phi} (-dt^2 + dx^2 + dy^2 + dz^2) \]  \hspace{1cm} (24)

where

\[ s^2 = \frac{1 + \sqrt{1 - \frac{\eta \Lambda}{4}}}{\eta} \]  \hspace{1cm} (25)

provided that \( \Lambda \eta \leq 3. \) When \( \eta \Lambda = 3, \) the solution may alternatively be given in AdS form as

\[ g = -4 \cosh^2 (l \omega) dt^2 + d\omega^2 + 4 \sinh^2 (l \omega) (dx^2 + dy^2 + dz^2) \]  \hspace{1cm} (26)

where \( l^2 = \frac{1}{\eta}. \)

**Case:** \( \eta = 0, \) \( H = 0. \)

Here the Euler-Poincaré term is absent and \( H = 0. \) We obtain the following solution:

\[ g = e^{\frac{\beta_1}{2} \phi (\omega)} d\omega^2 + e^{\frac{\beta_1}{2} \phi (\omega)} (-dt^2 + dx^2 + dy^2 + dz^2) \]  \hspace{1cm} (27)

with

\[ \phi(\omega) = \frac{1}{\left( \frac{1}{2} \beta_1 - \frac{8 \alpha}{3 \beta_1} \right)} \ln \left[ \frac{2 \beta_1 \Lambda}{\beta_1 \beta_1} - \frac{8 \alpha}{3 \beta_1} \right] \omega + C_0 \]  \hspace{1cm} (28)
where $C_0$ is an integration constant. When $\beta_1 = 2$, it reduces to a supersymmetric domain wall solution presented in [5].

Case: $\eta = 0$.

In this case the solution possesses a magnetic charge. It is given by

$$g = e^{\frac{4(\beta_1 - \beta_2)}{3}\phi(\omega)}d\omega^2 + e^{\frac{2(\beta_1 - \beta_2)}{3}\phi(\omega)}(-dt^2 + dx^2 + dy^2 + dz^2)$$

(29)

with

$$\phi(\omega) = \frac{6}{4\beta_2 - \beta_1} \ln \left(\frac{4\beta_2 - \beta_1}{6}\right) \sqrt{\frac{6\left(\frac{\beta_2^2 Q^2 + \Lambda \beta_1}{2}\right)}{(\beta_1 - 4\beta_2)\alpha}} \omega + C$$

(30)

provided that the constants satisfy

$$(\beta_1 - \beta_2) \left(\frac{\beta Q^2}{2} + \beta_1 \Lambda\right) = \left(\frac{\beta Q^2}{2} + 4\Lambda\right)\alpha.$$

(31)

$C$ is an integration constant. $H$ is given by

$$H = Q e^{\frac{(\beta_1 - 2\Lambda)}{4}\phi(\omega)} e^1 \wedge e^2 \wedge e^3$$

(32)

We note that when $Q = 0$ and the constants $\beta_1$ and $\beta_2$ satisfy $\beta_1 - \beta_2 = \frac{4\alpha}{\beta_1}$, the solutions reduce to (27) and (28).

We also note that an electric dual of solutions (29) and (30) may be given by defining a 2-form field

$$F = e^{3\phi} \wedge H.$$  

(33)

Then the solutions are identified as electrically charged solutions.

4. Conclusion

We have given a class of solutions to the variational field equations of a generalized theory of gravity in a $D$ dimensional bulk space-time derived from an action that includes the second-order Euler-Poincaré term and a cosmological constant. The theory describes a heterotic type first order effective string model in $D$ dimensions in the Einstein frame. The special class of plane-symmetric solutions of this model in 5-dimensions we gave refer to a 3-brane world also called a domain wall solution in the literature [5].

References


