Coproduct of $SU_q(2)$, Coherent States and Four Point Function with Logarithmic Regge Trajectories

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Abstract

The representation of the operators belonging to a $2 \times 2$ $SU_q(2)$ matrix yield a Hilbert space $H$. The coproducts of these operators define a Hilbert space $H^{(2)}$ isomorphic to $H$ and canonically embedded in $H \otimes H$. The four-point function obtained by taking the scalar product of the ground state of $H^{(2)}$ with the coherent states in $H \otimes H$ is uniquely defined, is meromorphic and has Regge behaviour.

Key Words: Dual amplitudes, Quantum groups, Regge behaviour, Hopf algebras, Coherent states.

Dual amplitudes with logarithmic trajectories are the widest class of dual amplitudes having simple analyticity structure that is unchanged under a linear transformation of the Mandelstam variables $s$ and $t$. They were discovered in the 1970’s [1, 2] and were rediscovered [3] in 1990 after the discovery of quantum groups [4, 5]. Considering the scattering amplitude as a function of two variables $\sigma$ and $\tau$, where $\sigma$ and $\tau$ have a linear dependence on $s$ and $t$, respectively, then the most general meromorphic dual four-point function with Regge behaviour is given by [6]

$$M(\sigma, \tau) = \sum_{m, n=0}^{\infty} h_{mn} A(q^m, q^n, \sigma, \tau),$$

where

$$A(\sigma, \tau) = \frac{G_q(\sigma \sigma')}{\alpha_q(\sigma) \alpha_q(\tau)} = \sum_{m, n=0}^{\infty} \frac{a_m^m a_n^n}{f_m(q) f_n(q)}$$

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is the Coon-Baker [1] four-point function,

\[
G_q(x) = \prod_{l=0}^{\infty} (1 - q^l x) = \sum_{l=0}^{\infty} (-1)^l \frac{g^{(l-1)/2}}{f_l(q)} \ x^l,
\]

\[
f_l(q) = (1 - q) \cdots (1 - q^l) = \frac{G_q(q) G_q(q^{l+1})}{G_q(q^l)}
\]

and \(h_{mn}\) are entire function coefficients, i.e. the function

\[
H(w, z) = \sum_{m,n=0}^{\infty} h_{mn} \ w^m \ z^n
\]

is an entire function of both variables. Equation (1) can also be expressed in terms of the Cremmer-Nuyts [2] four-point function which itself does not have Regge behaviour for \(\tau < q\):

\[
a(\sigma, \tau) = \sum_{m,n=0}^{\infty} a_{mn} \ q^m \ \tau^n
\]

\[
= \sum_{n=0}^{\infty} \frac{\tau^n}{1 - q^\sigma}
\]

by

\[
A(\sigma, \tau) = \sum_{m,n=0}^{\infty} C_{mn} a(q^m \ \sigma, q^n \ \tau),
\]

where

\[
C_{mn} = (-1)^{m+n} \frac{q^{m(m+1)/2}}{f_m(q)} \frac{q^{n(n+1)/2}}{f_n(q)}.
\]

The four-point function in (1) has a great deal of arbitrariness associated with the arbitrariness of the coefficients \(h_{mn}\). In particular, taking any finite number of the coefficients \(h_{mn}\) to be nonzero defines a meromorphic four-point function with Regge behaviour. Thus a physical or mathematical guiding principle is needed to determine the coefficients \(h_{mn}\) in (1). In this paper, we propose such a principle using the Hilbert space associated with the operators defined by the Hopf algebra generated by the matrix elements of \(SU_q(2)\) [5]. The well-known \(SU_q(2)\) quantum matrix group is composed of the matrices

\[
U = \begin{pmatrix} a & b^* \\ -b & a^* \end{pmatrix}.
\]

Let \(A\) be the Hopf algebra over the complex numbers generated by the elements \(a, a^*, b\) and \(b^*\), satisfying the Hermiticity conditions \((a^*)^* = a, (b^*)^* = b\) and the commutation relations

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\[ a a^* + q^2 b b^* = 1 \]
\[ a^* a + b^* b = 1 \]
\[ ab = qba \]
\[ a b^* = q b^* a \]
\[ b b^* = b^* b. \] (8)

The coproduct \( \Delta : A \to A \otimes A \), the antipode \( S : A \to A \) and the counit \( \varepsilon : A \to C \) are defined by

\[ \Delta(a) = a \otimes a - q b \otimes b^* \]
\[ \Delta(b) = a \otimes b + b \otimes a^* \]
\[ \Delta(a^*) = (\Delta(a))^* , \quad \Delta(b^*) = (\Delta(b))^* \]
\[ \varepsilon(a) = \varepsilon(a^*) = 1 \]
\[ \varepsilon(b) = \varepsilon(b^*) = 0 \] (9)

\[ S(a) = a^* \]
\[ S(a^*) = a \]
\[ S(b) = -q^{-1} b \]
\[ S(b^*) = -q b^*. \]

It can easily be seen that the defining relations (8) of \( A \) are invariant under \( b \leftrightarrow b^* \). Therefore, there exists a related second coproduct with

\[ \Delta(a) = a \otimes a - q b^* \otimes b \]
\[ \Delta(b^*) = a \otimes b^* + b^* \otimes a^* \] (10)
\[ \Delta(a^*) = (\Delta(a))^* , \quad \Delta(b) = (\Delta(b^*))^*. \]
We look for a representation of $A$ on a Hilbert space such that $b$ is invertible. If $b$ is not invertible, then its zero eigenvalue subspace is a trivial irreducible representation where $b = 0$ and $a$ is any unitary operator. If $b$ is invertible, then $a^* a$ and the phase of $b$ form a commuting set. $a^*$ and $a$ act as creation and annihilation operators, respectively. In other words, we have

$$a \mid n, \alpha \rangle = (1 - q^{2n})^{1/2} \mid n - 1, \alpha \rangle$$
$$a^* \mid n, \alpha \rangle = (1 - q^{2n+2})^{1/2} \mid n + 1, \alpha \rangle$$
$$b \mid n, \alpha \rangle = q^\alpha e^{i\alpha} \mid n, \alpha \rangle$$
$$b^* \mid n, \alpha \rangle = q^\alpha e^{-i\alpha} \mid n, \alpha \rangle,$$

where $\alpha \in [0, 2\pi)$ and we have chosen the normalization $\langle n, \alpha \mid m, \alpha \rangle = \delta_{nm}$ for a fixed $\alpha$. We denote the Hilbert space spanned by $\mid n, \alpha \rangle$ by $H_\alpha$. The algebra generated by $\Delta(a)$ and $\Delta(b)$ has a unique representation on $H_\alpha \otimes H_\alpha$. In other words, there is only one $\vert \otimes \alpha \rangle \in H_\alpha \otimes H_\alpha$ such that the following relations hold:

$$\Delta(a) \vert \otimes \alpha \rangle = 0$$
$$\Delta(b) \vert \otimes \alpha \rangle = e^{i\alpha} \vert \otimes \alpha \rangle,$$

where $\vert \otimes \alpha \rangle = \sum_{n,m} C_{nm} \mid n, \alpha \rangle \otimes \mid m, \alpha \rangle$. Note that $\vert \otimes \alpha \rangle$ is given by:

$$\vert \otimes \alpha \rangle = \frac{(\Delta(a^*))^n}{\sqrt{f_n(q^2)}} \mid \otimes \alpha \rangle.$$

Using the coproduct relations (9) together with (12), one finds the following recursion relations for $C_{n,m}$:

$$C_{n,m} = q^m (1 - q^{2(n+1)})^{1/2} C_{n+1,m} + q^n (1 - q^{2m})^{1/2} C_{n,m-1}$$
$$C_{n+1,m+1} = q^{n+m+1} (1 - q^{2(n+1)})^{-1/2} (1 - q^{2(m+1)})^{-1/2} C_{n,m}.$$

Setting $C_{0,0} = 1$, these recursion equations have the unique solution

$$C_{n,m} = \frac{q^{nm}}{\sqrt{f_n(q^2) f_m(q^2)}}.$$

This can easily be verified by putting $C_{n,m} = \frac{q^{nm}}{\sqrt{f_n(q^2) f_m(q^2)}} A_{n,m}$ into (14) which leads to $A_{n,m}$ being independent of $n$ and $m$. Thus, considering
\[ | n, \alpha \rangle \equiv \frac{(a^*)^n}{\sqrt{f_n(q^2)}} | 0, \alpha \rangle \]  

(16)

and (15) one obtains

\[ | 0, \alpha \rangle = \sum_{n,m=0}^{\infty} \frac{q^{nm}}{f_n(q^2) f_m(q^2)} ((a^*)^n \otimes (a^*)^m) | 0, \alpha \rangle. \]  

(17)

Let \( | \sigma, \alpha \rangle \) be the coherent state of the annihilation operator \( a \); that is

\[ a | \sigma, \alpha \rangle = \sigma | \sigma, \alpha \rangle, \]  

(18)

then

\[ | \sigma, \alpha \rangle = \sum_{n=0}^{\infty} \frac{\sigma^n}{\sqrt{f_n(q^2)}} | n, \alpha \rangle. \]  

(19)

Recall that \( | 0, \alpha \rangle \rangle \) is the ground state of \( \Delta(a) \) where \( \Delta \) is the Hopf algebra coproduct of \( SU_q(2) \) defined in (9). Consider the inner product in \( H_\alpha \otimes H_\alpha \) defined by

\[ M(\sigma, \tau) = \langle \langle 0, \alpha \| (| \sigma, \alpha \rangle \otimes | \tau, \alpha \rangle \rangle \]  

(20)

where \( | \sigma, \alpha \rangle \) and \( | \tau, \alpha \rangle \) are the coherent states of \( a \) satisfying (18). Then \( M(\sigma, \tau) \) in (20) is the transition amplitude from the ground state of the coproduct of \( a \) to the tensor product of the coherent states of \( a \). We will show that the function \( M(\sigma, \tau) \) in (20) defines a unique, Regge behaved, meromorphic four-point function. Using (3) enables us to express \( f_m(q^2) \) in the form

\[ f_m(q^2) = f_m(q) \frac{G_q(-q)}{G_q(-q^{m+1})}. \]

If we consider the expression above together with (17) and (20), then we get

\[ M(\sigma, \tau) = \frac{1}{G_q(-q)} \sum_{k,l=0}^{\infty} \frac{q^{kl(k+1)/2} q^{l(l+1)/2}}{f_k(q) f_l(q)} A(q^k \sigma, q^l \tau), \]  

(21)

where \( A(\sigma, \tau) \) is given by (2). The function in (4) then becomes

\[ H(w, z) = \sum_{k,l=0}^{\infty} \frac{q^{kl(k+1)/2} q^{l(l+1)/2}}{f_k(q) f_l(q)} w^k z^l \]

\[ = G_q(-qw) G_q(-qz). \]

Since \( H(w, z) \) is an entire function of both variables, then \( M(\sigma, \tau) \) in (20) is Regge behaved. The nicest feature of our derivation of the four-point function (21) is its uniqueness. We have shown that quantum group considerations and in particular the \( SU_q(2) \)

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Hopf algebra coproduct can be used to construct Regge behaved, meromorphic scattering amplitudes. A deeper physical understanding of our construction and construction of multiparticle scattering amplitudes are worth further investigation.

References