THE ISOMETRIES OF THE BOCHNER SPACE $L^p(\mu, H)$

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Abstract

In this article, the known characterization of the surjective linear isometries of the Bochner space $L^p(\mu, H)$, for a $\sigma$-finite measure $\mu$ and an arbitrary Hilbert space $H$, in terms of regular set isomorphisms of the $\sigma$-algebra involved and strongly measurable families of surjective isometries of $H$, is extended to arbitrary measures.

Introduction

Let $(\Omega, A, \nu)$ be a positive measure space. Following [1], we call a mapping $\phi$ of $A$ into itself, defined modulo null sets, a regular set isomorphism if $\phi(A') = \phi(\Omega) \setminus \phi(A)$ for all $A \in A$, where $A'$ denotes the complement of $A$, $\phi(\cup_n A_n) = \bigcup_n \phi(A_n)$ for any disjoint sequence $\{A_n\}$ in $A$, and $\nu(\phi(A)) = 0$ if, and only if, $\nu(A) = 0$. Any such isomorphism induces a mapping which we shall call the induced mapping and denote by $\Phi$, on the space of vector-valued measurable functions with values in a Banach space $E$ characterized by $\Phi(\chi_A e) = \chi_{\phi(A)} e$, $A \in A$, $e \in E$, where $\chi_A$ denotes the characteristic function of $A$. This process is described for scalar functions in [2, pp. 453-454].

For $1 \leq p \leq \infty$ we shall denote the Bochner space $L^p(\Omega, A, \nu, E)$ by $L^p(\nu, E)$, and by $L^p(\nu)$ when $E$ is the scalar field, if there is no chance of ambiguity about the underlying measurable space. For definitions and properties of these spaces we refer to [3]. (We recall that a measurable function $F$ belongs to $L^\infty(\mu, E)$ if, and only if, there exists a number $a > 0$ such that the set $\{x \in \Omega : \|F(x)\| > a\}$ is locally null, i.e., its intersection with every set of finite measure is null.)

Let $\lambda$ denote the Lebesgue measure on $[0,1]$. Banach [4] proved that for every linear
isometry $T$ of $L^p(\lambda)$, $1 \leq p < \infty, p \neq 2$, there exists a measurable function $\sigma$ of $[0,1]$ almost onto itself and a scalar measurable function $h$ on $[0,1]$ such that for $f \in L^p(\lambda)$
\[(Tf)(x) = h(x)f(\sigma(x)) \ A.e. \ on \ [0,1].\]

If $\phi$ is the regular set isomorphism defined by $\phi(A) = \sigma^{-1}(A)$ on the Borel algebra of $[0,1]$, then the above representation becomes
\[(Tf)(x) = h(x)\Phi(f)(x) \ A.e. \ on \ [0,1]. \quad (0.1)\]

In [1], Lamperti proves that for any $\sigma$-finite measure space $(\Omega, A, \mu)$ the linear isometries of $L^p(\mu)$ onto itself, $1 \leq p < \infty, p \neq 2$, are indeed of the above form (1) except that the isomorphism $\phi$ of the $\sigma$-algebra $A$, need not be defined by a point mapping. Moreover, if the measure $\nu$ is defined by $\nu(A) = \mu[\phi^{-1}(A)], A \in A$, then
\[|h(x)|^p = d\nu/d\mu \ A.e. \ on \ \Omega. \quad (0.2)\]

In [5], Cambern generalizes this result to the Bochner spaces. He proves that if $(\Omega, A, \mu)$ is a $\sigma$-finite measure space and $H$ is a separable Hilbert space, then for any linear isometry of $L^p(\mu, H)$ onto itself, $1 \leq p < \infty, p \neq 2$, in addition to the maps $h$ and $\Phi$ in Lamperti’s characterization now there also exists a weakly measurable operator-valued function $U$ defined on $\Omega$, where $U_x = U(x)$ is an isometry of $H$ onto itself for almost all $x \in \Omega$, such that for $F \in L^p(\mu, H),
\[T(F)(x) = U_x(h(x)\Phi(F)(x)) \ A.e. \ on \ \Omega. \quad (0.3)\]

In [6], Greim and Jamison obtain the same representation for an arbitrary Hilbert space, but the measure is still $\sigma$-finite. In this article we shall show that even for an arbitrary measure space $(\Omega, A, \mu)$ and an arbitrary Hilbert space $H$, the linear isometries of $L^p(\mu, H)$ have the same form in a somewhat different set-up.

The Results

Following [7], we shall call a Borel measure $\mu$ on an extremally disconnected locally compact Hausdorff space $X$, perfect if
\[(i) \ every \ nonempty \ clopen \ (closed \ and \ open) \ set \ has \ positive \ measure, \]
\[(ii) \ every \ nowhere \ dense \ Borel \ set \ has \ measure \ zero, \ and \]

390
(iii) every nonempty clopen set contains another clopen set with finite measure

In [8], Cengiz proves that an arbitrary measure space \((\Omega, \mathcal{A}, \nu)\) can be replaced by a perfect measure space \((X, \sum, \mu)\) which does not affect the spaces \(L^p(\nu, E)\), that is, 
\[ L^p(\nu, E) \cong L^p(\mu, E) \] 
(isometric to), for all \(1 \leq p < \infty\), but may enlarge \(L^\infty(\nu, E)\). This new measure space also has the following additional properties:

(iv) \(X\) is the topological direct sum of a family \(\{X_i : i \in I\}\) of extremally disconnected compact Hausdorff spaces, i.e., 
\[ X = \sum_i \oplus X_i; \]

(v) the algebra \(\sum\) contains the Borel algebra and a subset \(A \subset X\) is measurable if, and only if, \(A \cap X_i\) is measurable for all \(i \in I\);

(vi) for each \(i \in I\), the restriction of \(\mu\) to the \(\sigma\)-algebra of \(X_i\) is a regular Borel measure on \(X_i\);

(vii) each \(\sigma\)-finite measurable set is contained a.e. in the union of a countable subfamily of \(\{X_i : i \in I\}\);

(viii) for each \(A \in \sum, \mu(A) = \sum_i \mu(A \cap X_i)\);

(ix) \(\mu(U) = \mu(U)\) for every open set \(U\), where \(\bar{U}\) denotes the closure of \(U\);

(x) each measurable set \(A\) is equivalent to a clopen set \(C\), i.e., 
\[ \mu(A \Delta C) = \mu(A \setminus C) + \mu(C \setminus A) = 0; \]

(xi) every locally null set is actually null; and

(xii) the measure \(\mu\) is complete.

Hence, in the light of this discussion, we may and will assume that any measure space is a perfect one satisfying all the above conditions.

In the proof of the main result of this article we shall use the following theorem which is proved in a forthcoming paper [9].

**Theorem 1.** Let \(H\) be an arbitrary Hilbert space, \((X, \sum_\mu)\) be an arbitrary perfect measure space, \(1 \leq p < \infty\), and define \(q\) to be that extended real number such that 
\[ \frac{1}{p} + \frac{1}{q} = 1. \] 
Then, for each \(G \in L^q(\mu, H)\), the map 
\[ F \to \int_X \langle F(.), G(.) \rangle d\mu, \quad F \in L^p(\mu, H) \] 
is a bounded linear functional on \(L^p(\mu, H)\) whose norm is \(\| G \|_q\), and conversely every bounded functional on \(L^p(\mu, H)\) is of this form, i.e., 
\[ L^p(\mu, H)^* = L^q(\mu, H). \]
This theorem is proved in [8] for arbitrary $\mu$ but separable $H$.

Now we fix a measure space $(X, \sum, \mu)$ satisfying the conditions (i)-(xii) and prove the following theorem.

**Theorem 2.** Let $H$ be an arbitrary Hilbert space and let $1 \leq p < \infty$, $p \neq 1$. Then, for any linear isometry $T$ of $L^p(\mu, H)$ onto itself there exists a regular set isomorphism $\phi$ of $\sum$ onto itself, defined modulo null sets, a measurable scalar function $h$ on $X$ and a locally strongly measurable operator-valued function $U$ on $X$ such that for each $x \in X, U_x = U(x)$ is a surjective linear isometry of $H$ and for every $F \in L^p(\mu, H)$,

$$T(F)(x) = U_x(h(x)\Phi(F)(x)) \text{ a.e. on } X.$$

Moreover, $|h|^p = \frac{d(\mu \circ \phi^{-1})}{d\mu}$ on the ring of $\sigma$-finite measurable sets.

First we prove a chain of lemmas.

Throughout the rest of this article $T$ will denote a fixed surjective linear isometry of $L^p(\mu, H)$.

For any $G \in L^q(\mu, H)$, where $q$ is the extended real number such that $1/p + 1/q = 1$, the mapping

$$F \mapsto \int_X \langle T^{-1}F, G \rangle \, d\mu$$

is a bounded linear functional on $L^p(\mu, H)$, and therefore, by Theorem 1, there exists a function $G^*$ in $L^q(\mu, H)$ such that

$$\int_X \langle T^{-1}F, G \rangle \, d\mu = \int_X \langle F, G^* \rangle \, d\mu$$

for all $F \in L^p(\mu, H)$.

By substituting $T(F)$ for $F$ in the above definition of $G^*$ we get

$$\int_X \langle F, G \rangle \, d\mu = \int_X \langle T(F), G^* \rangle \, d\mu$$

for all $F \in L^p(\mu, H)$.
For $G \in L^p(\mu, H)$ if we let $\Psi_G$ denote the map on $L^p(\mu, H)$ defined by

$$\Psi_G(F) = \int_x \langle F, G \rangle d\mu,$$

then, we get $\Psi_{G^*} = \Psi_G \circ T^{-1}$. Since $T$ is a surjective isometry, it follows that $\|G^*\|_q = \|G\|_q$.

We shall fix a measurable subset $X_0$ of $X$ of finite measure and denote its characteristic function by $\chi$.

When we refer to a function in $L^p(\mu, H)$ we shall mean a specific function, rather than an equivalence class. The support of a function $F$ is the set $\{x : F(x) \neq 0\}$.

**Lemma 1.** Let $e \in H, ||e|| = 1$, and let $E(x) = T(\chi e)(x)/\|T(\chi e)(x)\|$ for $x \in Y_0$ and $E(x) = 0$ elsewhere, where $Y_0$ is the support of $T(\chi e)$. Then

$$(\chi e)^*(x) = \|T(\chi e)(x)\|^{p-1} E(x),$$
a.e. on $X$ if $p > 1$, and on $Y_0$ if $p = 1$. Consequently, $(\chi e)^*$ and $T(\chi e)$ have a.e. the same support.

**Proof.** Let $F = \chi e$ and define $K(x) = F^*(x)/\|F^*(x)\|$ if $F^*(x) \neq 0$ and $K(x) = 0$ otherwise.

Since $\langle T(F)(x), F^*(x) \rangle = \|T(F)(x)\| \|F^*(x)\| \langle E(x), K(x) \rangle, |\langle E(x), K(x) \rangle | \leq 1$ a.e., and $\|F\|_p = [\mu(X_0)]^{1/p}$, we have

$$\mu(X_0) = \int_X \langle F(.), F(.) \rangle d\mu = \int_X \langle T(F)(.), F^*(.) \rangle d\mu$$

$$= \int_X \|T(F)(.)\| \|F^*(.)\| \langle E(\cdot), K(\cdot) \rangle d\mu$$

$$\leq \int_X \|T(F)(.)\| \|F^*(.)\| d\mu$$

$$\leq \|T(F)\|_p \|F^*\|_q = \mu(X_0).$$

Thus we have equality throughout.

For $p > 1$, from the equality

393
\[
\int_X \| T(F)(.) \| \| F^*(.) \| \, d\mu = \| T(F) \|_p \| F^* \|_q
\]

and a result in [10, p. 121], it follows that

\[
\| T(F)(x) \|^p = \left( \| T(F) \|_p^p / \| F^* \|_q^p \right) \| F^*(x) \|^q = \| F^*(x) \|^q
\]
a.e. from which we obtain \( \| F^*(x) \| = \| T(F)(x) \|^{p-1} \) a.e., which in turn implies that the support of \( K(x) \) equals \( Y_0 \) a.e.

It is easy to show that if \( f(x) > 0 \) and \( |g(x)| \leq 1 \) a.e. on a measurable set \( A \) and if \( fg \) and \( g \) have the same integral on \( A \) then \( g(x) = 1 \) a.e. on \( A \). From this observation and the equality

\[
\int_X \| T(F)(.) \| \| F^*(.) \| \langle E(x), K(x) \rangle \, d\mu = \int_X \| T(F)(.) \| \| F^*(.) \| \, d\mu
\]

it follows that \( \langle E(x), K(x) \rangle = 1 \) a.e. on \( Y_0 \).

If \( u, v \in H, \| u \| \leq 1, \| v \| \leq 1 \) and \( \langle u, v \rangle = 1 \), then \( u = v \). So, from the preceding result we conclude that \( E(x) = K(x) \) a.e. on \( Y_0 \), and hence a.e. on \( X \). Consequently,

\[
F^*(x) = \| F^*(x) \| K(x) = \| F^*(x) \| E(x)
\]

\[
= \| T(F)(x) \|^{p-1} E(x)
\]
a.e. on \( X \), proving our lemma for \( p > 1 \).

Now let us assume that \( p = 1 \). Since \( \| F^* \|_\infty = \| F \|_\infty = 1 \), from the equality

\[
\int_X \| T(F)(.) \| \| F^*(.) \| \, d\mu = \| T(F) \|_1
\]

we obtain

\[
\int_{Y_0} \| T(F)(.) \| (1 - \| F^*(.) \|) \, d\mu = 0
\]

394
from which, since \( \| F^*(x) \| \leq 1 \) a.e., it follows that \( F^*(x) = 1 \) a.e. on \( Y_0 \). This completes the proof of our lemma.

\[ \square \]

**Lemma 2.** Let \( e_1, e_2 \in H, \| e_1 \| = \| e_2 \| = 1 \) and \( \langle e_1, e_2 \rangle = 0 \). Let \( Y_i \) for \( i = 1, 2 \), be the support of \( T(\chi e_i) \) and define \( E_i(x) = T(\chi e_i)(x)/ \| T(\chi e_i)(x) \| \) on \( Y_i \) and \( E_i(x) = 0 \) elsewhere. Then \( \langle E_1(x), E_2(x) \rangle = 0 \) a.e. on \( X \).

**Proof.** Let \( F_i = T(\chi e_i) \), \( i = 1, 2 \), and let \( A \subseteq Y_i \cap Y_2 \) be any measurable set of finite measure. Since

\[
\chi e_1 = T^{-1}(F_1) = T^{-1}(\chi A F_1) + T^{-1}(\chi (Y_1 \setminus A) F_1),
\]

and since \( T^{-1} \) maps functions of disjoint support into functions of disjoint support (see [5, p. 11]) the terms on the right have disjoint supports. Multiplying both sides by \( \chi_B \) we get

\[
\chi_B(\chi e_1) = \chi_B T^{-1}(\chi A F_1) = T^{-1}(\chi A F_1),
\]

where \( B \) is the support of \( T^{-1}(\chi A F_1) \).

Since \( \langle e_1, e_2 \rangle = 0 \), by Lemma 1, we obtain

\[
0 = \int_X \langle \chi_B \chi e_1, \chi e_2 \rangle d\mu = \int_X \langle T(\chi_B \chi e_1), (\chi e_2)^* \rangle d\mu = \int_X \langle \chi A F_1, (\chi e_2)^* \rangle d\mu = \int_A \langle \| F_1(\cdot) \| E_1(\cdot), \| F_2(\cdot) \|^p E_2(\cdot) \rangle d\mu = \int_A \| F_1(\cdot) \| \| F_2(\cdot) \|^{p-1} \langle E_1(\cdot), E_2(\cdot) \rangle d\mu.
\]

Since this is true for every measurable subset \( A \) of \( Y_1 \cap Y_2 \) of finite measure and \( \| F_1(x) \| \| F_2(x) \| > 0 \) on \( Y_1 \cap Y_2 \), we conclude that \( \langle E_1(x), E_2(x) \rangle = 0 \) a.e. on \( Y_1 \cap Y_2 \), and hence, it is zero a.e. on \( X \). This completes the proof of the lemma. 

\[ \square \]
Lemma 3. For any two nonzero vectors $e_1, e_2$ in $H$, $T(\chi e_1)$ and $T(\chi e_2)$ have a.e. the same support.

Proof. Let $u, v$ be two nonzero orthogonal vectors in $H$, $w = au + bv$, $a, b \in C$, and let $Y_u, Y_v$ and $Y_w$ denote the supports of $T(\chi u), T(\chi v)$ and $T(\chi w)$ respectively. We will show that $Y_w = Y_u = Y_v$ for any $a \neq 0, b \neq 0$. By Lemma 2, $\langle aT(\chi u)(x), bT(\chi v)(x) \rangle = 0$ a.e. on $X$. Since the sum of two orthogonal vectors is different from zero if, and only if, at least one of them is different from zero, $T(\chi w)(x) = aT(\chi u)(x) + bT(\chi v)(x) \neq 0$ if, and only if, either $T(\chi u)(x) \neq 0$ or $T(\chi v)(x) \neq 0$, and therefore,

$$Y_w = Y_u \cup Y_v.$$  \hspace{1cm} (0.4)

Now let $w_1 = au + v, w_2 = -au + v$ where $a = -\| v \| / \| u \|$. Since $w_1, w_2$ are orthogonal, by the preceding discussion we have $Y_{w_1} = Y_u \cup Y_v = Y_w$, and $Y_v = Y_{w_1}$ where $Y_v = Y_w$. Similarly, $Y_u = Y_{w_1}$. Now from (4) we get $Y_w = Y_u = Y_v$ proving our claim.

Thus, we have proved that for any two nonzero vectors $w$ and $z$ in the span of $u$, and $v$, the functions $T(\chi w)$ and $T(\chi z)$ have a.e. the same support which actually completes the proof of our lemma, for $e_1$ and $e_2$ are contained in the span of two orthogonal vectors.

Lemma 4. Let $e \in H, e \neq 0$. Then $T$ maps $L^p(X_0, H)$ onto $L^p(Y_0, H)$ where $Y_0$ is the support of $T(\chi e)$.

Proof. Let $A \subset Y_0$ be a measurable set with finite measure, and let $F \in L^p(\mu, H)$ such that $T(F) = \chi_A e$. Now let $S_1 = \text{supp}(F) \cap X_0, S_2 = \text{supp}(F) \cap X_0, B_1 = \text{supp} T(\chi S_1 F)$ and $B_2 = \text{supp} T(\chi S_2 F)$. Then

$$\chi_A e = T(\chi S_1 F) + T(\chi S_2 F).$$

Since $T$ maps functions of disjoint support into functions of disjoint support, $A = B_1 \cup B_2$ is an a.e. disjoint union. On the other hand, since $\chi S_1 F$ and $\chi e$ have disjoint supports, $B_1$ is contained a.e. in the complement of $Y_0$. Thus, we conclude that $B_1$ is null which in turn implies that $S_1$ is null, i.e. $\text{supp}(F) \subset X_0$ a.e.
The above argument shows that $T(L^p(X_0, H))$ contains all simple functions in $L^p(Y_0, H)$ and consequently, all of $L^p(Y_0, H)$.

Now for the reverse inclusion let $B \subset X_0$ be a measurable set, and $z \in H, z \neq 0$. Then from $T(\chi z) = T(\chi_B z) + T(\chi_{(X_0\setminus B)} z)$ we get $\text{supp } T(\chi_B z) \subset \text{supp } T(\chi z) = Y_0$ (Lemma 3). Thus, $T(\varphi) \in L^p(Y_0, \mu, H)$ for every simple function $\varphi \in L^p(X_0, H)$, and consequently, $T(L^p(X_0, H)) \subset L^p(Y_0, H)$. Hence we have equality.

**Proof of the Theorem** Now, to complete the proof of the theorem, for each $i \in I$ we let $Y_i$ denote the support of $T(\chi_i e)$ where $\chi_i$ is the characteristic function of $X_i$, and $e$ is any nonzero vector in $H$. By Lemma 3, $Y_i$ does not depend on $e$. Being the support of an integrable function each $Y_i$ is finite. We may and will assume that it is clopen. Since $T$ maps functions of disjoint support into functions of disjoint support, $Y_i$’s are mutually disjoint.

By Lemma 4, for each $i \in I$, $T$ maps $L^p(X_i, H)$ onto $L^p(Y_i, H)$. Therefore, by the Greim-Jamison theorem [6, p. 513] there exists a regular set isomorphism $\phi_i$ of the $\sigma$-algebra $\sum(X_i)$ onto the $\sigma$-algebra $\sum(Y_i)$, defined modulo null sets, a scalar measurable function $h_i$, and a strongly measurable operator-valued function $U(i)$ on $Y_i$ such that for each $y \in Y_i$, $U(i)_y = U(i)(y)$ is a surjective isometry of $H$, and for every $F \in L^p(X_i, \mu, H)$

$$T(F)(y) = U(i)_y(h(y)\Phi_i(F)(y)) \text{ a.e. on } Y_i.$$

Let $Y = \bigcup_i Y_i$ and extend each $h_i$ and $U(i)$ to all $Y$ by defining them to be zero in the complement of $Y_i$. Let $h = \sum_i h_i, U = \sum_i U(i)$ and let $\phi$ be the map from the $\sigma$-algebra $\sum$ to the $\sigma$-algebra $\sum(Y)$ defined by

$$\phi(A) = \sum_i \phi_i(A \cap X_i), \ A \in \sum.$$

Clearly $\phi$ is defined modulo null sets since each $\phi_i$ is defined so. Since a subset $B \subset Y$ belongs to $\sum(Y)$ if, and only if, for each $i, B \cap Y_i$ belongs to $\sum(Y_i)$, it follows that $\phi$ is surjective.

Let $\Phi$ denote the induced map. Then it can be shown easily that for each $F \in L^p(\mu, H)$,

$$T(F)(y) = U_y(h(y)\phi(F)(y)) \text{ a.e. on } Y.$$
Next we show that $T$ maps $L^p(\mu, H)$ onto $L^p(Y, H)$ and that $Y'$ is null. Let $G \in L^p(Y, H)$. Then $\text{supp}(G) \subset \bigcup_k Y_k$ for some countable subfamily $\{Y_k : k = 1, 2, \ldots\}$ of $\{Y_i\}$. Therefore, $G = \sum_k G_k$ where $G_k = G\chi_{Y_k}$. For each $k = 1, 2, \ldots$, let $F_k \in L^p(X_i, H)$ such that $TF_k = G_k$ and let $F = \sum_k F_k$. Then, $F \in L^p(\mu, H)$ and $TF = G$.

To prove that $Y'$ is null we let $B \subset Y', \mu(B) < \infty$, $e \in H$, $e \neq 0$. Then there exists $G \in L^p(Y, H)$ such that $T^{-1}(\chi Be) = T^{-1}(G)$, which implies that $G = \chi Be$ a.e. and since they have disjoint supports we conclude that $B$ is null. Hence $Y'$ is null.

Now we extend $h$ and $U$ to all of $X$ by defining $h(x) = 0$ and $U_x = I$ (the identity map on $H$) for $x \in Y'$. Clearly $U$ is locally strongly measurable on $X$, i.e., its restriction to each set of finite measure is strongly measurable, for any such set is contained a.e. in a countable union of $X_i$’s. The measurability of $h$ follows from the fact that a set $B \subset X$ is measurable if, and only if, $B \cap Y_j$ is measurable for all $j \in I$. To see this, fix $i \in I$. Then, since $\mu(X_i) < \infty$ and $X_i \cap Y_j, j \in I$, are disjoint clopen sets in $X_i$ it follows that only countably many of these sets are nonempty, $X_i \cap Y_{jk} \neq \emptyset, k = 1, 2, \ldots$ say. Now assume that $B \cap Y_j$ is measurable for all $j \in I$. Then, since $\mu$ is complete, $X_i \cap Y' \cap B$ is measurable, and so, $X_i \cap B = [\bigcup_k (X_i \cap Y_{jk} \cap B)] \cup (X_i \cap Y' \cap B)$ is measurable for each $i \in I$. Hence $B$ is measurable. This ends the proof of the theorem.

Remark A natural question is whether or not the characterization obtained for the linear isometries of the Bochner space $L^p(\mu, H)$ onto itself, $1 \leq p < \infty p \neq 2$, holds, should a Banach space replace $H$ as the range space. In general, the answer is in the negative; however, Sourour [10] was able to replace $H$ by a suitable Banach space.

References


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