TYURINA COMPONENTS AND RATIONAL CYCLES
FOR RATIONAL SINGULARITIES

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Abstract

In this paper, we give a geometric proof of Pinkham’s theorem on the positive cycles supported on the exceptional divisor of a rational singularity. In order to do this, we give several properties of the Tyurina components of the exceptional divisor and of the points of blowing-up surface of a rational singularity.

Introduction

An isolated singularity of a complex surface $S$ is rational if the stalk at the singularity of the coherent sheaf $R^1 \pi_* \mathcal{O}_X$ is equal to zero where $\pi : X \to S$ is a resolution of $S$ at the singularity. The numerical characterization of a rational singularity, given by M. Artin in [1] (see theorem 2.4 below), permits us to study these singularities by the exceptional divisor of a resolution of the singularity (see [15] or [17]). Here we are interested in the positive cycles supported on the exceptional divisor of a resolution divisor of a rational singularity. In [1] and [11], it has been shown that these cycles correspond to some special functions on $S$. In section 4, we use this correspondance to prove Pinkham’s theorem given in [12] (see theorem 4.4 below).

We start our paper by introducing some notations. In section 3, following the general case of a theorem of M. Artin [1] (see theorem 2.4 below), we give a proof on the nature of the exceptional divisor of a resolution of a rational singularity (see corollary 3.2 below). After giving some properties on the blowing up surface of a rational singularity, we finish the section by giving a bound on the non-Tyurina components of the exceptional divisor.
of a resolution of a rational singularity.

Recall

Let \((S, \xi)\) be a germ of a normal analytic surface embedded in \(\mathbb{C}^N\). Denote by \(S\) a sufficiently small representative of the germ \((S, \xi)\). A resolution of \(S\) is a complex analytic surface \(X\) and a proper holomorphic map \(\pi : X \rightarrow S\) such that its restriction to \(X - \pi^{-1}(\xi)\) is a biholomorphic map and \(X - \pi^{-1}(\xi)\) is dense in \(X\). By the Main Theorem of Zariski, the exceptional divisor \(E := \pi^{-1}(\xi)\) is connected and of dimension 1. Let \(E_1, \ldots, E_n\) denote its irreducible components.

We call positive cycle a formal sum of the irreducible components \(E_i\) of \(E\) with non-negative integral coefficients and with at least one non-zero coefficient. We denote by \(E^+\) the set of the positive cycles \(Y\) such that \((Y, E_i) \leq 0\) for all \(i\) (see [11], §18). The existence of such cycles is due to O. Zariski (see [19]). We define a partial ordering on \(E^+\) as following: For \(Y, Y' \in E^+\) with \(Y = \sum_{i=1}^{n} m_i E_i\), we have \(Y \geq Y'\) if \(m_i \geq m'_i\) for all \(i, (i = 1, \ldots, n)\). Since \(E\) is connected, we have:

**Remark 2.1.** For a positive cycle \(Y = \sum_{i=1}^{n} m_i E_i\), if \((Y, E_i) \leq 0\) for all \(i\), then we have \(m_i \geq 1\) for all \(i, (i = 1, \ldots, n)\).

**Definition 2.2** Let \(A\) be a set of positive cycles \(Y = \sum_{i=1}^{n} m_i E_i\). We define \(\lnf A\) as \(Z_0 = \sum_{i=1}^{n} a_i E_i\) with

\[
a_i = \inf_{Y \in A} \{ m_i \mid m_i = \text{mult}_Y E_i \}
\]

where \(\text{mult}_Y E_i\) is the coefficient of \(E_i\) in \(Y\). The cycle \(Z_0\) is a positive cycle since \(m_i \in \mathbb{N}^+\) for all \(i\).

**Theorem 2.3** For all subset \(A\) of \(E^+\), we have \(\lnf A \in E^+\).

**Proof.** Let \(Z_0 = \sum_{i=1}^{n} a_i E_i = \lnf A\). We will show that \((Z_0, E_i) \leq 0\) for all \(i\):

\[
(Z_0, E_j) = a_j (E_j, E_j) + \sum_{i \neq j} a_j (E_i, E_j)
\]

Let \(Y^0 = \sum_{i=1}^{n} m_i^0 E_i\) be a positive cycle in \(A\) such that \(m_i^0 = a_j\). We have then
\[ (Z_0, E_j) = m_j^0 (E_j, E_j) + \sum_{i \neq j} a_i (E_i, E_j) \]

\[ (Z_0, E_j) \leq m_j^0 (E_j, E_j) + \sum_{i \neq j} m_i^0 (E_i, E_j) = (Y_0^0, E_j) \leq 0. \]

Hence we have \( Z_0 \in \mathcal{E}^+ \).

**Theorem 2.4** For a resolution of \( S \), the arithmetic genus of \( \text{lnf} \mathcal{E}^+ \) is greater than or equal to zero. In particular, the singularity \( E \) of \( S \) is rational if and only if the arithmetic genus of \( \text{lnf} \mathcal{E}^+ \) is zero.

By [1] and [19], the cycle \( Z = \text{lnf} \mathcal{E}^+ = \sum_{i=1}^n a_i E_i \) is called the fundamental cycle of the resolution \( \pi \) and it can be computed by Laufer algorithm (see [8], proposition 4.1).

**Tyurina Components**

The purpose of this section is to understand the nature of the exceptional divisor of a resolution of a rational singularity and the points of the blowing-up surface of a rational singularity. We will finish this section by giving a bound on the non-Tyurina components of the exceptional divisor of a resolution of a rational singularity.

From the proof of proposition 1 in [1], we deduce following theorem:

**Theorem 3.1** Let \( S \) be a sufficiently small representation of a germ \((S, \xi)\) of a complex analytic normal surface having rational singularity at \( \xi \). Let \( S' \) be a normal surface and let \( \rho : S' \rightarrow S \) be a bimeromorphic proper map which is not the identity map. Let \( Y \) be a positive cycle supported on the divisor \( \rho^{-1}(\xi) \). Then we have \( H^1(|Y|, \mathcal{O}_Y) = 0 \).

**Proof.** The Main Theorem of Zariski says that the divisor \( \rho^{-1}(\xi) \) is connected and of dimension 1. Let \( E_1, \ldots, E_n \) be the irreducible components of \( \rho^{-1}(\xi) \) and let \( Y_r = \sum r_j E_j \) with \( r = (r_1, \ldots, r_k) \) be a positive cycle supported on \( \rho^{-1}(\xi) \). By the analytic comparison theorem of H. Grauert ([2], p.15-02), we have:

\[ (R^1 \rho_* \mathcal{O}_{S'})^\wedge_{\xi} = \lim_{(r) \leftarrow (\infty)} H^1(|Y_r|, \mathcal{O}_{Y_r}) \]
where \((R^1\rho_*\mathcal{O}_S)^\wedge\) is the completion of the module of finite type \((R^1\rho_*\mathcal{O}_S)\) on \(\mathcal{O}_{S,\xi}\) for the \(\mathcal{M}\)-adic topology where \(\mathcal{M}\) is the maximal ideal of the local ring \(\mathcal{O}_{S,\xi}\). Since the spaces \(|Y_{(r)}|\) have dimension 1, the map

\[ H^1(|Y_{(r)}|, \mathcal{O}_{Y_{(r)}}) \to H^1(|Y_{(r')}|, \mathcal{O}_{Y_{(r')}}) \]

is surjective when \(r_j \geq r'_j\) for all \(j\). This gives \(H^1(|Y_{(r)}|, \mathcal{O}_{Y_{(r)}}) = 0\) for all \((r)\).

Since for all positive cycles \(Y\) supported on the divisor \(\rho^{-1}(\xi)\), there exists \((r)\) such that \(Y \subset Y_{(r)}\), we have \(H^1(Y, \mathcal{O}_Y) = 0\).

\[ \square \]

**Corollary 3.2**  With the same hypothesis as in the theorem 3.1, all irreducible components of the divisor \(\rho^{-1}(\xi)\) are non-singular rational curves.

**Proof.** We have to prove that \(E_i\) is non-singular and \(p(E_i) = 0\) where \(p(E_i)\) is the arithmetic genus of \(E_i\). By the theorem 3.1, we have \(H^1(E_i, \mathcal{O}_{E_i}) = 0\), and since \(E_i\) is a reduced irreducible curve, \(H^0(E_i, \mathcal{O}_{E_i}) = 1\) (see [6], theorem I.3.4). Since \(1-p(E_i) = \chi(\mathcal{O}_{E_i}) = \dim H^0(E_i, \mathcal{O}_{E_i}) - \dim H^1(E_i, \mathcal{O}_{E_i}) = 1\) where \(X(\mathcal{O}_{E_i})\) is the Euler characteristic of \(\mathcal{O}_{E_i}\), we obtain \(p(E_i) = 0\).

Let \(n: E_i \to E_i\) be the normalization of \(E_i\). Since \(\tilde{E}_i\) is non-singular, we have the following exact sequence of coherent sheaves on \(E_i\) (see [6], exercise IV.1.8):

\[ 0 \to \mathcal{O}_{E_i} \to n_* \mathcal{O}_{E_i} \to \sum_{p \in E_i} \mathcal{O}_p/\mathcal{O}_p \to 0 \]

where \(\mathcal{O}_p\) is the integral closure sheaf of \(\mathcal{O}_p\) is a coherent sheave concentrated on the singular points of \(E_i\). Moreover we have \(X(n_* \mathcal{O}_{E_i}) = X(\mathcal{O}_{E_i})\) (see [6], exercise III.4.1).

Then

\[ X(\mathcal{O}_{E_i}) = -\dim H^1(\tilde{E}_i, \mathcal{O}_{E_i}) \]

Let \(\delta_p = \text{length } (\mathcal{O}_p/\mathcal{O}_p)\). We have:

\[ X(\sum_{p \in E_i} \mathcal{O}_p/\mathcal{O}_p) = \dim_C H^0(E_i, \sum_{p \in E_i} \mathcal{O}_p/\mathcal{O}_p) = \sum_{p \in E_i} \delta_p \]

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Hence $p(E_i) = p(\hat{E}_i) + \sum_{p \in E_i} \delta_p$. Here $p(E_i) \geq 0$ (see [6], p.181) and $\delta_p \geq 0$ for all $p \in E_i$. Since $p(E_i) = 0$ we have $p(\hat{E}_i) = 0$ and $\sum_{p \in E_i} \delta_p = 0$. Then $E_i$ is a non-singular curve.

By the theorem 3.1 and corollary 3.2, we obtain:

\[\square\]

**Corollary 3.3** With the same notations above, we have:

(i) $(E_i, E_j) = 0$ or 1 if $i \neq j$.

(ii) $E_i \cap E_j \cap E_k = \emptyset$ if, $i$, $j$ and $k$ are three integers pairwise distincts,

(iii) $E = \cup E_i, (1 \leq i \leq n)$ doesn’t contain any cycle.

**Proof.**

(i) If $(E_i \cap E_j) = \emptyset$ we have $(E_i, E_j) = 0$. If $(E_i \cap E_j) \neq \emptyset$ we have:

\[H^0(E_i + E_j, \mathcal{O}_{E_i+E_j}) \cong \mathbb{C}\]

and, by theorem 3.1, $H^1(E_i + E_j, \mathcal{O}_{E_i+E_j}) = 0$. So $p(E_i + E_j) = 0$. Moreover, by Riemann Roch theorem, we have:

\[p(E_i + E_j) = p(E_i) + (E_i, E_j) - 1\]

Then we obtain $(E_i, E_j) = 1$.

(ii) Let $E_i, E_j$ and $E_k$ be three components of $\rho^{-1}(\xi)$ which are pairwise distincts. We have :

\[p(E_i + E_j + E_k) = p(E_i) + p(E_j + E_k) + (E_i, (E_j + E_k)) - 1\]

Assume $E_i \cap E_j \cap E_k \neq \emptyset$. This implies $(E_i, E_j) \neq 0$ and $(E_i, E_k) \neq 0$, which means, by (i) above, $(E_i, E_j) = (E_i, E_k) = 1$. Since $p(E_i) = 0$, and $E_i + E_j + E_k$ and $E_j + E_k$ are connected, by the proof of the corollary 3.2, we obtain:

\[p(E_i + E_j + E_k) = p(E_i) = p(E_j + E_k) = 0\]

Hence $(E_i, E_j) + (E_i, E_k) - 1 = 0$. This contradicts $(E_i, E_j) = 1$ and $(E_i, E_k) = 1$. So we have $E_i \cap E_j \cap E_k = \emptyset$. 

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(iii) Let $E_{i_1}, \ldots, E_{i_p}$ the irreducible components of $\rho^{-1}(\xi)$ pairwise distincts. Assume $(E_{i_1}.E_{i_{1+1}}) \neq 0$ and $(E_{i_1}.E_{i_p}) \neq 0$. By (i) above, $(E_{i_1}.E_{i_{1+1}}) = (E_{i_1}.E_{i_p}) = 1$. Now consider

$$p(E_{i_1} + \ldots + E_{i_p}) = p(E_{i_1}) + p(E_{i_2} + \ldots + E_{i_p}) + (E_{i_1} + (E_{i_2} + \ldots + E_{i_p})) - 1$$

Since $E_{i_1} + \ldots + E_{i_p}$ and $E_{i_2} + \ldots + E_{i_p}$ are connected, we obtain $p(E_{i_1} + \ldots + E_{i_p}) = p(E_{i_1} = p(E_{i_2} + \ldots + E_{i_p}) = 0$. Hence $(E_{i_1}.E_{i_2} + \ldots + E_{i_p}) - 1 = 0$

Since $(E_{i_1}.E_{i_2}) = (E_{i_1}.E_{i_p}) = 1$ and $(E_{i_1}.E_{ij}) \geq 0$ for $2 \leq j \leq p$, this is a contradiction. 

\[\square\]

3.4 Let $S$ be a normal surface having a singularity at $\xi$ (not necessarily rational). Let $J$ be the ideal of $\mathbb{C}[x_1, \ldots, x_N]$ which defines the surface $S$ in a neighbourhood $U$ in $\mathbb{C}^N$. Let $f \in J$. We call initial form of $f$ at $\xi$, noted by $ln_\xi f$, the homogeneous polynomial of lowest degree in the Taylor expansion of $f$ at $\xi$. Let $J$ be the ideal of $\mathbb{C}[x_1, \ldots, x_N]$ generated by the set $\{ln_\xi f \mid f \in J\}$. The tangent cone $C_{S, \xi}$ of $S$ at $\xi$ is the 2-dimensional algebraic subvariety of $\mathbb{C}^N$ defined by the homogeneous ideal $\mathcal{L}$. We denote by $\text{Proj}C_{S, \xi}$ the projectivized curve in $\mathbb{P}^{N-1}$ associated with $C_{S, \xi}$. The set $\text{Proj} | C_{S, \xi} | = | \text{Proj}C_{S, \xi}$ is the set of lines of the tangent cone $C_{S, \xi}$. If $\sigma : S \rightarrow S$ denotes the blowing up of $S$ at $\xi$ then we have $\sigma^{-1}(\xi) \cong \text{Proj}C_{S, \xi} \subset \mathbb{P}^{N-1}$ (see [18], theorem 5.8).

In what follows, we assume that $S$ has a rational singularity at $\xi$. Then $\mathcal{MO}_X$ is locally principal (see [1], theorem 4). Let $\sigma : S \rightarrow S$ be the blowing up of $S$ at $\xi$. The surface $S$ is normal (see [16], theorem 1). If $\pi : X \rightarrow S$ denotes a resolution of $S$, by the universal property of blowing up, there exists a map $\tilde{\pi} : X \rightarrow \hat{S}$ such that $\sigma \circ \tilde{\pi} = \pi$. Let us denote by $E = \hat{\cup}E_i$ and by $Z = \sum_{i=1}^{n} a_i E_i$ the fundamental cycle of $\pi$.

**Definition 3.5** A Tyurina component of $E$ is a maximal connected set $B$ of irreducible components of $E$ such that $(Z.E_i) = 0$ for all irreducible components $E_i$ in $B$.

By [15] and [16], the Tyurina components have the following geometric interpretation: Consider the non-Tyurina components of $E$, i.e. the irreducible components $E_i$ of $E$ such that $(Z.E_i) < 0$. Suppose that the number of these components is equal to $s$. We have $s \leq n$. We may assume that $(Z.E_i) < 0$ for all $1 \leq i \leq s$. Now we consider the closure
of the curve $E \setminus (E_1 \cup \ldots \cup E_s)$; it is not necessarily connected. Denote by $B_1, \ldots, B_k$ its connected components. So each $B_j, (1 \leq j \leq k)$, is a Tyurina component of $E$. As in [15] (remark 3.2), we have:

**Proposition 3.6** An irreducible component $E_i$ of $E$ is contained in a Tyurina component of $E$ if and only if $\hat{\pi}(E_i)$ is a point of $\tilde{S}$.

**Proof.** Let $Z_1$ be the positive cycle defined by $\mathcal{MO}_S$ where $\mathcal{M}$ is the maximal ideal of the local ring $\mathcal{O}_{\tilde{S}, \xi}$. We have $\mathcal{MO}_X = \hat{\pi}^*(\mathcal{MO}_S)$. Since $\hat{\pi}$ is a proper map, the projection formula (see [3], paragraph 2.6) says that:

$$(Z.E_i) = (Z_1.\hat{\pi}(E_i)).$$

It is clear that if $\hat{\pi}(E_i)$ is a point in $\tilde{S}$ then we have $(Z.E_i) = 0$.

If $\hat{\pi}(E_i)$ is not a point, then it is an irreducible component of $|Z_1|$ where $|Z_1|$ is the reduced curve associated with $Z_1$. Let $C_0$ be a generic hyperplane section of $S$ at $\xi$ defined by the equation $(h = 0)$ with $h \in \mathcal{M}/\mathcal{M}^2$ such that its strict transform $h'$ by $\sigma$ intersects $|Z_1|$ transversely. Let $h''$ be the strict transform by $\pi$ of $C_h$. The divisor of $h$ in $X$ can be written as $(\pi^*h) = Z + h''$. Since $((\pi^*h).E_i) = 0$ for all $i$, the projection formula gives:

$$(h''.E_i) = (h'.\hat{\pi}(E_i)).$$

Since $(h''.\hat{\pi}(E_i)) > 0$, we deduce $(Z.E_i) < 0$. This implies that $E_i$ is a non-Tyurina component of $E$.

Then the normal surface $\tilde{S}$ has $k$ singularities each of which is obtained by the contraction of a Tyurina component $B_j, (1 \leq j \leq k)$, of $E$ by $\hat{\pi}$ to a point of $\tilde{S}$. Let us denote by $\xi_1, \ldots, \xi_k$ these singularities. Let $V_j$ be a small neighbourhood of $\xi_j$ in $\tilde{S}$. We have:

**Corollary 3.7** With the preceding notations, we have:

1. The restriction map $\hat{\pi} \mid_{\pi^{-1}(V_j)}$ is a resolution of the germ $(\tilde{S}, \xi_j)$.
(2) If \( X \) is the minimal resolution of \( S \) then \( X \) is the minimal resolution of \( \tilde{S} \), and a Tyurina component \( B_j \) of \( E \) is the exceptional divisor \( \pi^{-1}(\xi_j) \) of the minimal resolution of \( \tilde{S} \) at \( \xi_j \).

(3) A point of \( \tilde{S} \) which is not the contraction of a Tyurian component is a non-singular point of \( \tilde{S} \).

(4) The singular points of \( \sigma^{-1}(\xi) \) are necessarily the intersection points of irreducible components of \( \sigma^{-1}(\xi) \) (see corollary 3.2).

Moreover we have:

**Proposition 3.8**  The singularities \( \xi_1, \ldots, \xi_k \) of \( \tilde{S} \) are all rational.

**Proof.** The contraction of each \( B_j \) gives a normal surface singularity (see [7], lemma 5.11 and [5], theorem 1). By the theorem 2.4, we have \( p(Z_{B_j}) \geq 0 \) where \( Z_{B_j} \) is the fundamental cycle associated with \( B_j \). Moreover, theorem 3.1 implies \( p(Z_{B_j}) \leq 0 \). This gives the proposition.

**Proposition 3.9**  Suppose that \( \xi \) is a rational singularity and the multiplicity of \( S \) at \( \xi \) is \( m \). Let \( E \) be the exceptional divisor of a resolution \( \pi : X \to S \) of \( S \). Then the number of the non-Tyurina components of \( E \) is less than or equal to \( m \).

**Proof.** Let \( E_1, \ldots, E_n \) be the irreducible components of \( E \). Assume that \( (Z.E_i) < 0 \) if and only if \( i \in \{1, \ldots, s\} \). It is well known that \( (Z.Z) = -m \) (see [1], corollary 6). If we denote \( (Z.E_i) = -d_i \) where \( d_i \) is a positive integer for all \( i, (1 \leq i \leq s) \), we obtain:

\[
(Z.Z) = \sum_{i=1}^{n} a_i (Z.E_i) = -\sum_{i=1}^{s} d_i a_i = -m
\]

By definition, we have \( a_i \geq 1 \) and \( d_i \geq 1 \); so \( a_i d_i \geq 1 \). Then \( \sum_{i=1}^{s} a_i d_i \geq s \).

The non-Tyurina components of the exceptional divisor of a resolution of \( S \) correspond exactly to the strict transform by \( \tilde{\pi} \) of the components of \( \text{Proj} \ | C_{S,\xi} | \). We will use this fact in the next section to associate the Tyurina components with the functions on \( S \).
Rational cycles

We will call *rational cycle* an element of $\mathcal{E}^+$. In this section, we will construct the elements of $\mathcal{E}^+$ by using the fundamental cycle $Z$ (see [16] or [12]), and some special functions on $S$ which correspond to these rational cycles. In order to do this, we will prove theorem 4.2 that we call Pinkham’s theorem.

Let $\pi : X \to S$ be a resolution of $S$ having a rational singularity at $\xi$. By [11], there is a one-to-one correspondence between the $\mathcal{M}$-primary complete ideals $I$ in the local ring $\mathcal{O}_{S,\xi}$ such that $I\mathcal{O}_X$ is invertible and the rational cycles. In other words, there exists a rational cycle $D$ on $X$ and a $\mathcal{M}$-primary ideal $I$ in $\mathcal{O}_{S,\xi}$ such that $I\mathcal{O}_X = \mathcal{O}(-D)$. In particular, we have $\mathcal{M}\mathcal{O}_X = \mathcal{O}(-Z)$ where $Z$ is the smallest rational cycle of $\pi$.

Here we will speak on the functions on $S$ rather than on the $\mathcal{M}$-primary complete ideals. In fact, the rational cycles correspond to the elements of these ideals in $\mathcal{O}_{S,\xi}$. By [1], a positive cycle $D$ supported on $E$ is an element of $\mathcal{E}^+$ if and only if there exists a function $f$ in $\mathcal{M}$ on $S$ such that the compact part of the divisor in $X$ corresponding to $f$ is $D$ i.e. $(\pi^*f) = D + f''$, where $f''$ is the strict transform by $\pi$ of $f$. Since $(\pi^*f).E_i = 0$ for all irreducible components $E_i$ of the exceptional divisor $E$ of $\pi$, we obtain $(f''.E_i) = -(D.E_i)$. This means that $D$ is a rational cycle since $(f''.E_i) \geq 0$. To understand the function on $S$ which corresponds to the smallest rational cycle $Z$, we define:

**Definition 4.1** [9] A line of the tangent cone $\mathcal{C}_{S,\xi}$ is called exceptional tangent of $S$ at $\xi$ if it corresponds to a singular point of $S$ or a singular point of $\text{Proj} | \mathcal{C}_{S,\xi}$.

**Definition 4.2** [14] A function $f$ on $S$ is called generic if it is defined by a non-singular function $F$ defined in a neighbourhood of $\xi$ in $\mathbb{C}^N$ so that the tangent hyperplane at $\xi$ to the hypersurface $(F = 0)$ is not tangent to the tangent cone $\mathcal{C}_{S,\xi}$ and doesn’t contain any exceptional tangent of $S$ at $\xi$.

Let $h$ be a function on $S$ defined by a hyperplane $H_1$. Assume that $h$ is a generic function on $S$. This implies that the zero locus of $h \circ \sigma$ intersects the components of $\text{Proj} | \mathcal{C}_{S,\xi}$ transversely at the non-singular points of $\text{Proj} | \mathcal{C}_{S,\xi}$ and of $S$. By [4], the compact part of the total transform of such a function $h$ by a resolution $\pi$ of $S$ is exactly the fundamental cycle of the resolution (i.e. we have $(\pi^*h) = Z + h''$ where $h''$ is the strict transform by $\pi$ of $h$).
Now to construct the elements of $E^+$, choose a component $E_{io}$ of $E$. Let $Z_0 = Z + E_{io}$.

Consider the sequence of the positive cycles

$$Z_0 = Z + E_{io}, \ldots, Z_{i+1} = Z_i + E_{m(i)}$$

if there exists $m(i)$ such that $(Z_i, E_{m(i)}) > 0$ and $Z_{i+1} = Z_i$ otherwise. This process is finite (see [8] or [12], proposition 1.2).

**Lemma 4.3** Let $Y$ be a positive cycle supported on $E$ such that there is a component $E_i$ for which $(Y, E_i) > 0$. Then there exists a cycle $Z_1 > Y$ such that $Z_1$ is in $E^+$ and verify $Z_1(Y : E_i) > 0$ and $Z_1 = Z_i$ otherwise.

**Proof.** Let $Z = \sum_{i=1}^{n} a_i E_i$ be the fundamental cycle and $Y + E_i = \sum_{i=1}^{n} m_i E_i$ a positive cycle supported on $E$. For all $i$, there exists $n_i \in \mathbb{N}^*$ such that $n_i a_i \geq m_i$. Let $s = \sup_{i=1, \ldots, n_i}$. We have then $s Z \geq Y + E_i$. Moreover $s Z \in E^+$. So this gives the existence of a positive cycle $Z_1$ in $E^+$ which verify $Z_1 \geq Y + E_i$.

Let us consider $A_i = \{ Z_1 \in E^+ \mid Z_1 \geq Z + E_i \}$. Denote $\inf A_i = \tilde{Z}(E_i)$. By theorem 2.3, $\tilde{Z}(E_i) \in E^+$. By [16], if $E_i$ is contained in a Tyurina component $B_j$ of $E$, then $\tilde{Z}(E_i) = Z + \Delta Z_i$ where $\Delta Z_i$ is a linear combination of the irreducible components of the Tyurina component $B_j$. In that case, H. Pinkham ([12], proposition of section 14) gives precisely $\Delta Z_i$. We give this result in the following theorem:

**Theorem 4.4** (Pinkham’s theorem) Let $E^j_i$ be an irreducible component of $E$ which is contained in a Tyurina component $B_j$ of $E$. Then the smallest $\tilde{Z}(E^j_i)$ in $E^+$ such that $\tilde{Z}(E^j_i) \geq Z + E^j_i$ is equal to $Z + Z(B_j)$ where $Z(B_j)$ is the fundamental cycle associated with $B_j$.

We shall speak later about $\Delta Z_i$ in the case where $E_i$ is a non-Tyurina component. First we prove theorem 4.4. The aim of our proof is to see the relation between the Tyurina components of the exceptional divisor and the functions on the surface $S$.

**Proof.** Let $f$ be a function in the maximal ideal $\mathcal{M}$ of $\mathcal{O}_{S, X}$. Let $(\pi^* f) = Y + f''$ where $f''$ is the strict transform by $\pi$ of $f$ where $Y$ is the compact part of the divisor $(\pi^* f)$. Let us denote $E^j_i$ an irreducible component $E_i$ of $E$ which belongs to a Tyurina component $B_j(1 \leq j \leq k)$. As in Section 2, let $A_i = \{ D \mid D \geq Z + E^j_i \}$. Assume that $Y$ is an element of $A_i$. Let us denote by $Z_1$ (resp. $Y_1$) the positive cycle supported on the $\text{Proj} C_{S, \delta}$ such
that $\pi^*(Z_1) = Z$ (resp. $\pi^*(Y_1) = Y$) (see proof of proposition 3.6). We have $Y_1 \geq Z_1$. Since the case $Y_1 > Z_1$ will be an obvious consequence of the case $Y_1 = Z_1$, we assume that $Y_1 = Z_1$. The total transform by $\pi$ of $f$ can be written in the form $(\pi^* f) = (\pi^* Z_1)$ where $f'$ is the strict transform by $\sigma$ of $f$. This give $(\pi^* f) = Z + F + f''$ where $F$ is a positive cycle supported on the components of the divisor $\pi^{-1}(f' \cap \text{Proj} C_{S, \xi})$. We notice that $Y = Z + F$.

Let $\xi_j$ be the singular point of $\tilde{S}$ obtained by the contraction of the Tyurina component $B_j$ of $E$. If $f'$ doesn’t pass through $E_j$ in $\tilde{S}$ then $B_j$ is not contained in $F$ (see remark 3.7-(4)). So we exclude this case because $Y$ is not contained in $A_i$. If $f'$ pass through $\xi_j$ in $\tilde{S}$ then $B_j$ is contained in $F$. Then we can write $Y \geq Z + Z(B_j)$. (Notice that, if $Y_1 > Z_1$ above, we have $Y \geq Z + Z(B_j)$). Now in order to prove $\lnf A_i = Z + Z(B_j)$, we will show that $Z + Z(B_j)$ is an element of $A_i$. This is equivalent to show that $Z + Z(B_j)$ is a rational cycle. So it is sufficient to prove $(Z + Z(B_j)).E_i \leq 0$ for all irreducible components $E_i$ of $E_i (1 \leq i \leq n)$. We prove it in the following two cases:

1. If $E_i$ is contained in a Tyurina component of $E$, we have $(Z.E_i) = 0$ and $(Z(B_j).E_i) \leq 0$, so $(Z + Z(B_j)).E_i \leq 0$.

2. If $E_i$ is a non-Tyurina component of $E$ we have two cases: If $E_i \cap B_j = \emptyset$, we have $(Z(B_j).E_i) = 0$; this gives $(Z + Z(B_j)).E_i \leq 0$ since $(Z.E_i) < 0$. If $E_i \cap B_j \neq \emptyset$, we have $(Z(B_j).E_i) > 0$. This gives $(Z(B_j).E_i) = a^i_m$ where $a^i_m$ is the multiplicity of the component $E^i_m$ in $Z(B_j)$ attached to $E_i$. By [10], theorem 4.6), this multiplicity is equal to one. Then we obtain $(Z + Z(B_j)).E_i \leq 0$.

Therefore $Z + Z(B_j)$ is a rational cycle. Since $Z + Z(B_j) \geq Z + E^i_1$, it is an element of $A_i$. By definition this gives $\lnf A_i = (Z + Z(B_j))$.

Hence a rational cycle $D$ is an element of $A_i$ if and only if there exists a function $g$ in $M$ such that $(\pi^* g) = D + g''$ where $g''$ is the strict transform by $\pi$ of $g$ and the strict transform $g'$ by $\sigma$ in $\tilde{S}$ of $g$ passe through the singular point $\xi_j$.

In particular, $D$ is the smallest element of $A_i$ if and only if the strict transform $g'$ by $\sigma$ of $g$ intersects $\text{Proj} \mid C_{S, \xi}$ at the nonsingular points of $\text{Proj} \mid C_{S, \xi}$ and of $\tilde{S}$ except $\xi_1$ and the branch of $g'$ passing through $\xi_j$ is a generic function on $\tilde{S}$.

Remark 4.5 If $E^i_1$ and $E^i_m$ are two irreducibles components in the Tyurina component
If \( B_i \) of \( E \) then we have \( \tilde{Z}(E_i^j) = \tilde{Z}(E_i^m) \).

4.6 Now let us consider the set \( A_i = \{ Z_1 \in E^i \mid Z_1 \geq Z + E_i \} \) when \( E_i \) is a non-Tyurina component of \( E \). In order to give precisely the smallest cycle \( \tilde{Z}(E_i) = Z + \Delta Z_i \) of this set, we introduce some notations:

Let us denote by \( B \) a Tyurina component of \( E \) and by \( E_1 \) an irreducible component of \( E \) which is contained in \( B \). Let \( B^0 = B \) and let \( Z(B^0) \) be the fundamental cycle of \( B^0 \). If \( (Z(B^0), E_1) < 0 \) then we put \( B^l = B^0 \) for \( l \geq 1 \). If \( (Z(B^0), E_1) = 0 \) then we denote by \( B^1 \) the Tyurina component of \( Z(B^0) \) which contains \( E_1 \). We have \( B^0 \supset B^1 \). By induction, we define the sequence \( B^0 = B, B^1, \ldots, B^p \) such that \( B^l \) is a Tyurina component of \( B^{l-1} E_1 \), is contained in \( B^l \) and \( E_1 \) is a non-Tyurina component of \( Z(B^0), 1 \leq l \leq p \). As in [10], we define:

**Definition 4.7** [10] We call \( B^0 = B, B^1, \ldots, B^p \) desingularization sequence of \( E_1 \) and \( p \) desingularization depth of \( E_1 \).

**Theorem 4.8** Let \( E \) be a non-Tyurina component of \( E \). Let us denote by \( B_1, \ldots, B_q \) the Tyurina components of \( E \) attached to \( E \) and by \( F_1, \ldots, F_q \) the irreducible components of \( B_1, \ldots, B_q \) respectively such that \( (F_i \cap E) \neq \emptyset \) for all \( t, (1 \leq t \leq q) \). Then the smallest cycle \( \tilde{Z}(E) \) in \( E^+ \) which is greater than \( Z + E \), is

\[
\tilde{Z}(E) = Z + E + \sum_{t=1}^{q} \left( \sum_{l=0}^{p} Z(B^l_t) \right)
\]

Here \( l = 0, \ldots, p \) is the desingularization depth of \( F^+ \) in \( B_t, (1 \leq t \leq q) \).

In particular, if \( E \) is not attached to any Tyurina component then we have \( \tilde{Z}(E) = Z + E \).

This result has been proved during the proofs and it will appear in a forthcoming paper.

**References**

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