SOME THEOREMS INVOLVING INEQUALITIES ON
P-VALENT FUNCTIONS

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Abstract

In this paper, some theorems involving inequalities on p-valent functions (that is p-valent close-to-convex functions, p-valently starlike functions, and p-valently convex functions) are given. Moreover, some applications in the theorems which are important for geometric function theory are also included.

Keywords: Analytic, p-valent, p-valently close-to-convex functions, p-valently starlike functions, p-valently convex functions, open unit disk, and Jack’s Lemma.

1. Introduction and Definitions

Let \( T(p) \) denote the class of functions \( f(z) \) of the form

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in N = \{1, 2, 3, \ldots\}),
\]

which are analytic and p-valent in the open unit disk \( U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\} \). A function \( f(z) \in T(p) \) is said to be in the subclass \( TK(p) \) of p-valently close-to-convex functions with respect to the origin in \( U \) if it satisfies the inequality (cf.[1-3]):

\[
Re \left\{ \frac{f'(z)}{z^{p-1}} \right\} > 0, \quad (z \in U; p \in N).
\]

On the other hand, a function \( f(z) \in T(p) \) is said to be in the subclass \( TS(p) \) of p-valently starlike functions with respect to the origin in \( U \) if it satisfies the inequality...
(cf.[1-3]):

\[
\text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0, \quad (z \in U; p \in N).
\]  

Furthermore, a function \( f(z) \in T(p) \) is said to be in the subclass \( TC(p) \) of \( p \)-valently convex functions with respect to the origin in \( U \) if it satisfies the inequality (cf.[1-3]):

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad (z \in U; p \in N).
\]  

A function \( f(z) \in T(p) \) is said to be in the subclass \( V_q(p) \) if it satisfies the inequality:

\[
\left| \frac{(p-q)!}{p!} \frac{D_q^p f(z)}{z^{p-q}} - 1 \right| < 1,
\]  

\((z \in U; p > q; p \in N; q \in N_0 = N \cup \{0\})\),

and, a function \( f(z) \in T(p) \) is said to be in the subclass \( W_q(p) \) if it satisfies the inequality:

\[
\left| \frac{zD_q^{p+1} f(z)}{D_q^1 f(z)} - (p-q) \right| < p - q,
\]  

\((z \in U; p > q; p \in N; q \in N_0)\).

Here and throughout this paper, \( D_q^p \) denotes the \( q \)-th-order ordinary differential operator. For a function \( f(z) \in T(p) \),

\[
D_q^p f(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q}, \quad (p > q; p \in N; q \in N_0).
\]  

To establish our results, we need the following Lemma given by Jack [4] (also, by Miller and Mocanu [5]).
Lemma \ Let \( w(z) \) be analytic in \( U \) with \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value on the circle \( |z| = r < 1 \) at a point \( z_0 \), then

\[
z_0 w'(z_0) = cw(z_0),
\]

where \( c \) is a real number and \( c \geq 1 \).

2. Some Theorems Involving Inequalities on P-valent Functions

We first prove the following theorem.

Theorem 1. \ If \( f(z) \in T(p) \) satisfies the inequality

\[
\Re \left\{ q + \frac{zD^{q+1}_zf(z)}{D^q_zf(z)} - p \right\} < \frac{1}{2} \quad (z \in U; p > q; p \in \mathbb{N}; q \in \mathbb{N}_0),
\]

then \( f(z) \in V_q(p) \).

Proof. \ Let the function \( f(z) \in T(p) \). Then defining the function \( w(z) \) by

\[
\frac{(p-q)!}{p!} \cdot \frac{zD^{q+1}_zf(z)}{z^{p-q}} = 1 + w(z), \quad (z \in U; p > q; p \in \mathbb{N}; q \in \mathbb{N}_0),
\]

we have that \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). It follows from (10) that

\[
\frac{(p-q)!}{p!} \cdot \frac{zD^{q+1}_zf(z)}{z^{p-q}} = zw'(z) + (p-q)[1 + w(z)].
\]

Then, we have from (10) and (11) that

\[
F(z) = q + \frac{zD^{q+1}_zf(z)}{D^q_zf(z)} - p = \frac{zw'(z)}{1 + w(z)}.
\]

Now, suppose that there exists a point \( z_0 \in U \) such that

\[
\max_{|z| \leq |z_0|} \ |w(z)| = |w(z_0)| = 1.
\]

Then, using the Lemma and letting \( w(z_0) = e^{i\theta}(w(z_0) \neq -1) \) in the equation (12), we have
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\[ Re\{F(z_0)\} = Re\left\{ \frac{z_0 w'(z_0)}{1 + w(z_0)} \right\} = Re\left\{ \frac{c w(z_0)}{1 + w(z_0)} \right\} \]

\[ = c Re\left\{ \frac{e^{i\theta}}{1 + e^{i\theta}} \right\} = \frac{c}{2} \geq \frac{1}{2}, \]

which contradicts the hypothesis (9). Therefore, we conclude that \( |w(z)| < 1 \) for all \( z \in U \), and the definition (10) immediately yields the inequality

\[ \left| \frac{(p - q)!}{p!} \frac{D_q^p f(z)}{z^{p-q}} - 1 \right| < 1, \quad (z \in U; p > q; p \in N; q \in N_0), \]

that is, \( f(z) \in V_q(p) \). Thus, the proof is completed.

Next we prove the following theorem.

**Theorem 2.** If \( f(z) \in T(p) \) satisfies the inequality

\[ Re\left\{ 1 + z \left( \frac{D_q^{p+2} f(z)}{D_q^{p+1} f(z)} - \frac{D_q^{p+1} f(z)}{D_q^{p} f(z)} \right) \right\} < \frac{1}{2}, \quad (z \in U; p > q; p \in N; q \in N_0), \]

then \( f(z) \in W_q(p) \).

**Proof.** Let the function \( f(z) \in T(p) \). Now consider the function \( w(z) \) defined by

\[ \frac{ZD_q^{p+1} f(z)}{D_q^{p} f(z)} = (p - q)[1 + w(z)], \quad (z \in U; p > q; p \in N; q \in N_0). \]

Clearly the function \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). It follows from the definition of \( w(z) \) that

\[ 1 + \frac{ZD_q^{p+2} f(z)}{D_q^{p+1} f(z)} = (p - q) \left( 1 + w(z) + zw'(z) \frac{D_q^{p} f(z)}{D_q^{p+1} f(z)} \right) \]

\[ = (p - q)[1 + w(z)] + \frac{zw'(z)}{1 + w(z)}. \]
Now suppose that there exists a point \( z_0 \in U \) such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1. \tag{19}
\]

Then the Lemma gives \( w(z_0) = e^{i\theta} \) and \( z_0 w'(z_0) = cw(z_0)(c \geq 1) \). Therefore, we obtain from (17)-(20) that
\[
\text{Re} \left\{ 1 + z \left( \frac{D_{q+2} f(z)}{D_{q+1} f(z)} - \frac{D_{q+1} f(z)}{D_q f(z)} \right) \right\} |_{z=z_0} = \text{Re} \left\{ \frac{z_0 w'(z_0)}{1 + w(z_0)} \right\} = \text{Re} \left\{ \frac{cw(z_0)}{1 + w(z_0)} \right\} = c \text{Re} \left\{ \frac{e^{i\theta}}{1 + e^{i\theta}} \right\} = \frac{c}{2} \geq \frac{1}{2}, \tag{20}
\]
which contradicts the condition (16). Hence, we conclude that \( |w(z)| < 1 \) for all \( z \in U \), that is, that \( f(z) \in W_q(p) \). This completes the proof of Theorem 2.

By taking \( q = 0 \) in Theorems 1 and 2, we have the following corollaries.

**Corollary 1.** If \( f(z) \in T(p) \) satisfies the inequality
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} - p \right\} < \frac{1}{2}, \quad (z \in U; p \in N), \tag{21}
\]
then
\[
\left| \frac{f(z)}{z^p} - 1 \right| < 1, \quad (z \in U; p \in N). \tag{22}
\]

**Corollary 2.** If \( f(z) \in T(p) \) satisfies the inequality
\[
\text{Re} \left\{ 1 + z \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right\} < \frac{1}{2}, \quad (z \in U; p \in N), \tag{23}
\]
then \( f(z) \in TS(p) \) and
\[
\left| \frac{zf'(z)}{f(z)} - p \right| < p, \quad (z \in U; p \in N). \tag{24}
\]

By taking \( q = 1 \) in Theorems 1 and 2, we have the following.
Corollary 3. If $f(z) \in T(p)$ satisfies the inequality

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} - p \right\} < \frac{1}{2}, \quad (z \in U; p \in N),$$

then $f(z) \in TK(p)$ and

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p, \quad (z \in U; p \in N).$$

Corollary 4. If $f(z) \in T(p)$ satisfies the inequality

$$\text{Re} \left\{ 1 + z \left( \frac{f'''(z)}{f''(z)} - \frac{f''(z)}{f'(z)} \right) \right\} < \frac{1}{2}, \quad (z \in U; p \in N),$$

then $f(z) \in TC(p)$ and

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| < p - 1, \quad (z \in U; p \in N - \{1\}).$$

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