GEODESICS IN A TENSOR BUNDLE

Abdullah Kopuzlu & A. A. Salimov

Abstract

The main purpose of the present paper is to study geodesics in a tensor bundle $T^p_q(M_n)$ with respect to the horizontal lift $H\nabla$ of an affine connection $\nabla$.

Key words and phrases: Tensor, Tensor Bundle, Connection, Horizontal Lift, Geodesic.*

1. Introduction

Let $M_n$ be an $n$-dimensional differentiable manifold of class $C^\infty$ and $T^p_q(Q)$ the vector space of tensors type $(p,q)$ at a point $Q$ of $M_n$ that is, the set of all tensors of type $(p,q)$, of $M_n$ at $Q$. Then the set

$$T^p_q(M_n) = \bigcup_{Q \in M_n} T^p_q(Q)$$

is, by definition, the tensor bundle over the manifold $M_n$. For any point $\tilde{Q}$ of $T^p_q(M_n)$ such that $\tilde{Q} \in T^p_q(Q)$, the correspondence $\tilde{Q} \rightarrow Q$ determines the bundle projection $\pi: T^p_q(M_n) \rightarrow M_n$.

Let $x^i$ be local coordinates in a neighborhood $U$ of $Q \in M_n$. Then a tensor $t$ of type $(p,q)$ at $Q$ which is an element of $T^p_q(M_n)$ is expressible in the form $(x^i, t^{i_1 \cdots i_p}_{j_1 \cdots j_q}) = (x^i, \tilde{x}^\tau) (x^\tau = t^{i_1 \cdots i_p}_{j_1 \cdots j_q}, \tau = h + 1, \ldots, h + n^p+q)$, where $t^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ are components of $t$ with respect to the natural frame $\frac{\partial}{\partial x^i}$. We may consider $(x^i, \tilde{x}^\tau)$ as local coordinates in a neighborhood $\pi^{-1}(U)$ of $T^p_q(M_n)$.

*AMS Subject classification number: Primary 53A45; secondary 53C55
To a transformation of local coordinates of \( M_n \); \( x' = x'(x^1, \ldots, x^n) \), there corresponds in \( T^p_q(M_n) \) the coordinates transformation

\[
x'(x^1, \ldots, x^n)
\]

\[
x' = x'(x^1, \ldots, x^n)
\]

\[
x^i = t^j_{i_1 \cdots i_q} A^i_{j_1} \cdots A^i_{j_p} X^j = A^i_{(j)} A^{(i)}_{(i')} x'^i
\]

where \( A^i_{(j)} = \frac{\partial x^i}{\partial x^j} \), \( A^{(i)}_{(i')} = \frac{\partial x^i}{\partial x^{i'}} \), \( A^{(j)}_{(i)} = A^i_{j} \cdots A^i_{j_p} A_j^{(i)} = A^i_{j} \cdots A^i_{j_q} \).

The Jacobian of (1) is given by the matrix

\[
\left( \frac{\partial x^{i'}}{\partial x^i} \right) = \begin{pmatrix}
A^i_{(j)} & A^{(i)}_{(i')}
\end{pmatrix}
\]

where \( I = (i, i'), I' = (j', i') \), \( t^{j_1 \cdots j_p}_{i_1 \cdots i_q} \) are components of \( \Gamma_{ij}^k \) in \( M_n \).

2. Horizontal Lifts of Affine Connection

We denote by \( T^p_q(M_n) \) the set of all tensor fields of class \( C^\infty \) and of type \( (p, q) \) in \( M_n \).

We now assume that \( M_n \) is a manifold with an affine connection \( \nabla \). Let \( X^h \) and \( \Gamma^h_{ji} \) be components of \( X \in T^1_0 \) and \( \nabla \), respectively, with respect to the local coordinates \( (x^h) \) in \( M_n \). Then the horizontal lift of \( X \) have components

\[
^H X = \begin{pmatrix}
^H X^i \\
^H X^i
\end{pmatrix} = \begin{pmatrix}
\sum_{\mu=1}^p \Gamma^m_{hj_k} X^h t^{j_1 \cdots j_p}_{i_1 \cdots i_q} = \sum_{\lambda=1}^p \Gamma^\lambda_{hk} X^h t^{j_1 \cdots j_p}_{i_1 \cdots i_q}
\end{pmatrix}
\]

with respect to the coordinates \( (x^i, x^j) \) in \( T^p_q(M_n) \) (see [1]).

Let \( A^{j_1 \cdots j_p}_{i_1 \cdots i_q} \) be components of \( A \in T^p_q(M_n) \). We can easily verify by means of (2) that the \( ^H A \) defined by

\[
^H A = 0, \quad A^{j_1 \cdots j_p}_{i_1 \cdots i_q}
\]

determine in \( T^p_q(M_n) \) a vector field. This vector field is called the vertical lift of the tensor field \( A \in T^p_q(M_n) \) to \( T^p_q(M_n) \) and denoted by \( ^V A \) (see [2]).

We shall now define the horizontal lift \( ^H \nabla \) of an affine connection \( \nabla \) in \( M_n \) to \( T^p_q(M_n) \) by the conditions

\[
^H (\nabla_X Y) = ^H \nabla_X ^H Y, \quad ^V (\nabla_X A) = ^H \nabla_X ^V A, \quad ^H \nabla_v ^A X = 0, \quad ^H \nabla_v ^A B = 0
\]
for any \( X, Y \in T^1_n(M_n) \), \( A, B \in T^q_n(M_n) \), from which we have (see [3])

\[
H \Gamma^{i}_{ms} = \Gamma^{i}_{ms}
\]

\[
H \Gamma^{j}_{ms} = \sum_{b=1}^{q} \sum_{m=1}^{m} \delta_{lb} \delta_{mb} \delta_{j+\delta_{i+1} \delta_{j+1}} \delta_{j+\delta_{i+1} \delta_{j+1}} \delta_{j+\delta_{i+1} \delta_{j+1}} \delta_{j+\delta_{i+1} \delta_{j+1}}
\]

\[
= \sum_{c=1}^{q} \sum_{m=1}^{m} \delta_{lc} \delta_{mc} \delta_{j+\delta_{i+1} \delta_{j+1}} \delta_{j+\delta_{i+1} \delta_{j+1}} \delta_{j+\delta_{i+1} \delta_{j+1}} \delta_{j+\delta_{i+1} \delta_{j+1}}
\]

\[
H \Gamma^{k}_{ms} = \sum_{b=1}^{p} \sum_{m=1}^{m} \delta_{lb} \delta_{mb} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}}
\]

\[
= \sum_{c=1}^{q} \sum_{m=1}^{m} \delta_{lc} \delta_{mc} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}}
\]

\[
(3)
\]

\[
H \Gamma^{l}_{ms} = \sum_{b=1}^{p} \sum_{m=1}^{m} (\partial m \Gamma^{j}_{aq} + \Gamma^{j}_{mr} \Gamma^{a}_{aq} - \Gamma^{j}_{ms} \Gamma^{a}_{mr} \Gamma^{j}_{q} + \delta_{j+\delta_{i+1} \delta_{j+1}} \delta_{j+\delta_{i+1} \delta_{j+1}})
\]

\[
+ \sum_{c=1}^{q} \sum_{m=1}^{m} \delta_{lc} \delta_{mc} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}}
\]

\[
= \sum_{b=1}^{p} \sum_{m=1}^{m} \delta_{lb} \delta_{mb} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}}
\]

\[
+ \frac{1}{2} \sum_{b=1}^{p} \sum_{m=1}^{m} \sum_{l=1}^{l} \delta_{lb} \delta_{mb} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}} \delta_{k+\delta_{i+1} \delta_{k+1}}
\]

where \( \delta^{j}_{i} \)-Kronecker delta, \( \sum_{m}^{m} = \sum_{m=1}^{m} \).
3. Geodesics (paths) In A Tensor Bundle of The Horizontal Lift $H\nabla$

Let $\tilde{C} : [0, 1] \to T^p_q(M_n)$ be a curve in $T^p_q(M_n)$ and suppose that $\tilde{C}$ is expressed locally by $x^A = x^A(t)$, i.e.,

$$
x^h = x^h(t)$$
$$x^\overline{h} = x^\overline{h}(t)
$$

with respect to the induced coordinates $(x^h, x^{\overline{h}})$ in $T^p_q(M_n)$, $t$ being a parameter. Then the curve $C = \pi \circ \tilde{C}$ in $M_n$ is called the projection of the curve $\tilde{C}$ and denoted by $\pi \tilde{C}$ which is expressed locally by $x^h = x^h(t)$.

A curve $\tilde{C}$ in $T^p_q(M_n)$ is a geodesic with respect to $H\nabla$ (a path of $H\nabla$), when it satisfies the differential equation

$$
\frac{d^2 x^I}{dt^2} + H\Gamma_{MS} \frac{dx^M}{dt} \frac{dx^S}{dt} = 0 \quad (4)
$$

Consider the case where $p = 1$, $q = 2$, for example. By means of (3), (4) reduces to

$$
\frac{d^2 x^i}{dt^2} + \Gamma_{ms}^i \frac{dx^m}{dt} \frac{dx^s}{dt} = 0
$$

$$
\frac{d^2 x^\overline{m}}{dt^2} + H\Gamma_{ms}^\overline{m} \frac{dx^m}{dt} \frac{dx^s}{dt} + H\Gamma_{ms}^\overline{r} \frac{dx^r}{dt} \frac{dx^s}{dt} + H\Gamma_{ms}^\overline{l} \frac{dx^l}{dt} \frac{dx^s}{dt} = \frac{d^2 \overline{t}^j_{i\overline{1}\overline{2}}}{dt^2}
$$

$$
+ \left[ (\partial_m \Gamma_{sa}^i + \Gamma_{mr}^i \Gamma_{sa}^2 - \Gamma_{ms}^2 \Gamma_{ra}^i) \overline{t}^a_{i\overline{1}\overline{2}} \right]
$$

$$
+ \sum_{c=1}^{2} \left( -\partial_m \Gamma_{s|i}^a + \Gamma_{ms}^2 \Gamma_{s|x}^a + \Gamma_{ms}^2 \Gamma_{s|r}^a \right) \overline{t}^a_{i\overline{1}\overline{2}}
$$

$$
- \sum_{c=1}^{2} t^r_{i\overline{1}} \left( \Gamma_{mr}^a \Gamma_{s|i}^a + \Gamma_{ms}^a \Gamma_{s|v}^a \right)
$$

$$
+ \sum_{b=1}^{2} \sum_{c=1}^{2} \overline{t}^a_{i\overline{1}} \left( \Gamma_{ms}^a \Gamma_{s|b}^a + \Gamma_{ms}^a \Gamma_{s|w}^a \right) \frac{dx^m}{dt} \frac{dx^s}{dt}
$$

$$
$$
+ [\Gamma^j_{i_1} \delta^{m_1}_{i_1} \delta^{m_2}_{i_2} - \sum_{c=1}^2 \Gamma^m_{m_{i_2}} \delta^{m_{c+1}}_{i_{c+1}} \delta^{m_{c}}_{i_{c}}] \frac{dt^{j_1}_{m_{i_2}m_2}}{dt} \frac{dx^{s}}{dt} \\
+ (\Gamma^j_{m_k} \delta^{s}_{i_1} \delta^{s}_{i_2} - \sum_{c=1}^2 \Gamma^r_{m_{i_2}} \delta^{r_{c+1}}_{i_{c+1}} \delta^{r_{c}}_{i_{c}}) \frac{dx^{m}}{dt} \frac{dt^{k_1}_{i_{1i_2}}}{dt} = 0,
\]

where \( x^i = t^{j_1}_{i_1i_2} \).

From first equation in (5), we have

\[ \Gamma^i_{i_1} \frac{d^2x^m}{dt^2} = \Gamma^i_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt}, \]

\[ -\Gamma^i_{i_1} \frac{d^2x^m}{dt^2} = \Gamma^i_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt}, \quad (6) \]

. By means of (6), the second equation in (5) is reduced to

\[ \frac{\delta^2 t^{j_1}_{i_{1i_2}}}{dt^2} = 0 \]

where the left-hand side is defined by

\[ \frac{\delta^2 t^{j_1}_{i_{1i_2}}}{dt^2} = \frac{d}{dt} \left( \frac{dt^{j_1}_{i_{1i_2}}}{dt} \right) + \Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt}, \]

\[ -\Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} = \Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt}, \]

\[ +\Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} + \Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} \]

\[ -\Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} = \Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt}, \]

\[ +\Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} + \Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} \]

\[ -\Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} = \Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt}, \]

\[ +\Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} + \Gamma^{i}_{i_1} \frac{dx^m}{dt} \cdot \frac{dx^m}{dt} \]
By similar devices, we can prove the formula (7) for general case. Thus we have from (5) and (7).

Theorem. A curve $\tilde{C}: x^i = x^i(t), \tilde{x^i} = \tilde{t}_{i_{1} \cdots i_{p}}^{j_{1} \cdots j_{q}} = t_{i_{1} \cdots i_{p}}^{j_{1} \cdots j_{q}}(t) = \tilde{x^i}(t)$, in $T^p(M_n)$ is a geodesic of the horizontal lift $\pi^r$ of an affine connection $\nabla$ given in $M_n$, if and only if the projection $\pi C$ is a geodesic of $\nabla$ in $M_n$ and the tensor field $t_{i_{1} \cdots i_{p}}^{j_{1} \cdots j_{q}}$ along $\pi \tilde{C}$ has vanishing second covariant derivative.

References