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ON 3 DIMENSIONAL ISOTROPIC SUBMANIOLDS OF A SPACE FORM

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Abstract

We study 3-dimensional isotropic submanifolds of a space form with low-dimensional first normal space

1. Introduction

B. O'Neill [3] introduced first the notion of isotropic submanifold of a Riemannian manifold. Many differential-geometrs have studied isotropic submanifolds of spheres. In particular, L. Vrancken [10] proved recently the following results.

Proposition 1. *Let M be a 3-dimensional constant isotropic submanifold in an n-dimensional unit sphere $S^n(1)$. If the dimension of the first normal space of M is $\leqq 3$ at every point, then one of the following holds.*

(1) M is totally geodesic in $S^n(1)$.

(2) There exists a totally geodesic $S^4(1)$ in $S^n(1)$ such that the image of M is (a part of) a small hypersphere of $S^4(1)$.

(3) There exists a totally geodesic $S^7(1)$ in $S^n(1)$ such that the image of M is congruent to (a part of) $R \times S^2\left(\frac{3}{2}\right)$ in $S^7(1)$.

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Proposition 2. *A 3-dimensional minimal isotropic submanifold in S^n is of constant curvature.*

In the present paper, we will study a 3-dimensional isotropic submanifolds in an n-dimensional space form $\tilde{M}^n(c)$ of constant curvature c and at first prove the following.

Theorem 1. *Let M be a 3-dimensional isotropic submanifold in an n-dimensional space form $\tilde{M}(c)$. If the dimension of the first normal space of M is $\leqq 3$ at every point, then M is constant isotropic.*

By Theorem 1, we have the following result which can be considered as a hyperbolic version of Proposition 1.

Theorem 2. *Let M be a 3-dimensional isotropic submanifold in an n-dimensional hyperblic space $\mathbf{H}^n$. If the dimension of the first normal space of M is $\leqq 3$ at every point, then one of the following holds.*

(1) M is totally geodesic in $\mathbf{H}^n$,

(2) There exists a totally geodesic $\mathbf{H}^4$ in $\mathbf{H}^n$ such that M is a geodesic sphere, a horosphere or a hypersphere in $\mathbf{H}^4$.

Moreover, we have the following generalization of Proposition 2.

Theorem 3. *A 3-dimensional minimal isotropic submanifold in a space form is of constant curvature.*

2. Preliminaries

Let $\tilde{M}(c)$ be an n-dimensional space form of constant curvature c, that is, an n-dimensional Riemannian manifold of cosntant curvature c. Let M be a 3-dimensional submanifold in $\tilde{M}(c)$. We denote by g (resp. $\tilde{g}$) the Riemannian metric of M (resp. $\tilde{M}^n(c)$). Let $T_p(M)$ be the tangent space of M at $p \in M$ and $\nu_p(M)$ be the normal space to M at $p \in M$. We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation on M (resp. $\tilde{M}^n(c)$) and $\nabla^\perp$ the covariant differentiation on the normal bumdle $\nu(M)$. Then, for vector field X, Y tangent to M and a vector field ξ normal to M, the formulas of Gauss and Weingarten are

$$\begin{cases} \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \\ \tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi, \end{cases} \tag{2.1}$$

where σ is the *second fundamental form* and A is the *shape operator* which are related by $\sigma(X,Y) = g(AX,Y)$. We define the covariant derivative $\nabla\sigma$ of σ by

$$(\nabla_X \sigma)(Y,Z) = \nabla_X^\perp(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

Since the ambient space is of constant curvature c, the equations of Gauss, Codazzi and Ricci are given respectively by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\} + A_{\sigma(Y,Z)}X - A_{\sigma(X,Z)}Y, \tag{2.2}$$

$$(\nabla_X \sigma)(Y,Z) = (\nabla_Y \sigma)(X,Z), \tag{2.3}$$

$$\tilde{g}(R^\perp(X,Y)\xi, \eta) = g([A_\xi, A_\eta]X, Y), \tag{2.4}$$

for tangent (rep. normal) vector fields X, Y and Z (resp. ξ and η), where R (resp. $R^\perp$) denotes the Riemannian (resp. normal) curvature tensor of M.

We choose a local field of orthonormal frames $e_1, e_2, e_3, e_4, \ldots, e_n$ in $\tilde{M}(c)$ in such a way that, restricted to M, e_1, e_2, e_3 are tangent to M and consequently, the remaining vectors are normal to M. Let $\tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3, \tilde{\omega}^4, \ldots, \tilde{\omega}^n$ be the field of duat frames. We use the following convention on the range of indices unless otherwise stated: $A, B, C, \ldots = 1, 2, \ldots, n; i, j, k, \ldots = 1, 2, 3; \alpha, \beta, \gamma, \ldots = 4, 5, \ldots, n$. We agree that repeated indices under a summation sign without indication are summed over the respective range. Then the structure equations of $\tilde{M}(c)$ are given by

$$\begin{cases} d\tilde{\omega}^A = -\sum \tilde{\omega}_B^A \wedge \tilde{\omega}^B, \quad \tilde{\omega}_B^A + \tilde{\omega}_A^B = 0, \\ d\tilde{\omega}_B^A = -\sum \tilde{\omega}_C^A \wedge \tilde{\omega}_B^C + c\tilde{\omega}^A \wedge \tilde{\omega}^B. \end{cases} \tag{2.5}$$

Restricting these forms to M, we have the structure equations of M:

$$\begin{cases} \omega^\alpha &= 0,\ \omega_i^\alpha = \sum h_{ij}^\alpha \omega^j,\ h_{ij}^\alpha = h_{ji}^\alpha, \\ d\omega^i &= -\sum \omega_j^i \wedge \omega^j,\ \omega_j^i + \omega_i^j = 0, \\ d\omega_j^i &= -\sum \omega_k^i \wedge \omega_j^k + \Omega_j^i, \quad \Omega_j^i = \frac{1}{2}\sum R_{jkl}^i \omega^k \wedge \omega^l, \\ R_{jkl}^i &= c\left(\delta_k^i \delta_{jl} - \delta_l^i \delta_{jk}\right) + \sum \left(h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha\right). \end{cases} \tag{2.6}$$

The last equation of (2.6) is nothing but the Gauss equation (2.2).

$$\begin{cases} d\omega_\beta^\alpha &= -\sum \omega_\gamma^\alpha \wedge \omega_\beta^\gamma + \Omega_\beta^\alpha, \quad \Omega_\beta^\alpha = \frac{1}{2}\sum R_{\beta ij}^\alpha \omega^i \wedge \omega^j, \\ R_{\beta ij}^\alpha &= \sum \left(h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta\right). \end{cases} \tag{2.7}$$

Then the second fundamental form σ may be expressed by

$$\sigma(X,Y) = \sum h_{ij}^\alpha \omega^i(X)\omega^j(Y)e_\alpha,$$

and the last equation of (2.7) is nothing but the Ricci equation (2.4). Define $h_{ijk}^\alpha (i,j,k = 1,2,3)$ by

$$\sum h_{ijk}^\alpha \omega^k = dh_{ij}^\alpha - \sum h_{kj}^\alpha \omega_i^k - \sum h_{ik}^\alpha \omega_j^k + \sum h_{ij}^\beta \omega_\alpha^\beta.$$

Then we have $(\nabla_X \sigma)(Y,Z) = \sum h_{ijk}^\alpha \omega^i(Y)\omega^j(Z)\omega^k(X)$ and $h_{ijk}^\alpha = h_{ikj}^\alpha, i,j,k = 1,2,3$, which is nothing but the Codazzi equation (2.3).

At a point $p \in M$, let ν_p^1 be the space spanned by all vectors $\sigma(u,v), u,v \in T_p(M)$, which is called the *first normal space* of M at p.

The vector $\sigma(X,X)$ is called the *normal curvature vector* in the direction of $X \in T_p(M)$. M is said to be *isotropic* at $p \in M$ if $\| \sigma(X,X) \| / \| X \|^2$ is independent of the choice of $X \in T_p(M)$ and, in particular, λ-*isotropic* at $p \in M$ if $\| \sigma(X,X) \| / \| X \|^2 = \lambda$ for all $X \in T_p(M)$. M is said to be *isotropic* if M is isotropic at every point. In such a case, λ is considered as a differentiable function on M and M is said to be *constant isotropic* if λ is constant on M. In particular, M is 0-isotropic if and only if it is totally geodesic.

If M is λ-isotropic, then we have the following equations ([9]):

$$\tilde{g}(\sigma(X,X), \sigma(X,Y)) = 0, \tag{2.9}$$

$$\lambda^2 - \tilde{g}(\sigma(X,X),\sigma(Y,Y)) - 2\tilde{g}(\sigma(X,Y),\sigma(X,Y)) = 0, \tag{2.10}$$

$$\tilde{g}(\sigma(X,X),\sigma(Y,Z) + 2\tilde{g}(\sigma(X,Y),\sigma(X,Z) = 0, \tag{2.11}$$

$$\tilde{g}(\sigma(X,Y),\sigma(Z,W) + \tilde{g}(\sigma(X,Z),\sigma(W,Y) + \tilde{g}(\sigma(X,W),\sigma(Y,Z) = 0, \tag{2.12}$$

for orthnormal X, Y, Z, W.

3. Proof of Theorems.

Let M be a 3-dimensional λ-isotropic submanifold in a space form $\tilde{M}^n(c)$.

Lemma 3.1. *If dim $\nu_p^1 \leqq 3$ at a pont $p \in M$, then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $T_p(M)$ with respect to which one of the following holds:*

$$\begin{cases} \sigma(e_1,e_1) &= \sigma(e_2,e_2) = \sigma(e_3,e_3) = 0, \\ \sigma(e_1,e_2) &= \sigma(e_1,e_3) = \sigma(e_2,e_3) = 0, \end{cases} \tag{3.1}$$

$$\begin{cases} \sigma(e_1,e_1) &= \sigma(e_2,e_2) = \sigma(e_3,e_3) = \lambda e_4, \\ \sigma(e_1,e_2) &= \sigma(e_1,e_3) = \sigma(e_2,e_3) = 0, \end{cases} \tag{3.2}$$

$$\begin{cases} \sigma(e_1,e_1) &= -\sigma(e_2,e_2) = \sigma(e_3,e_3) = \lambda e_4, \\ \sigma(e_1,e_2) &= \lambda e_5, \\ \sigma(e_1,e_3) &= 0, \\ \sigma(e_2,e_3) &= \lambda e_6, \end{cases} \tag{3.3}$$

where e_4, e_5, e_6 are orthonormal normal vectors at p and $\lambda \neq 0$.

Proof. In the case $\dim \nu_p^1 = 0$ M is geodesic at p, hence (3.1) holds for an arbitrary $\{e_1, e_2, e_3\}$.

We next consider the case where $\dim \nu_p^1 = 1$. Since p is not a geodesic point, $\lambda(p) \neq 0$. For an arbitrary orthonormal basis $\{e_1, e_2, e_3\}$ of $T_p(M)$, (2.9) implies that $\sigma(e_1, e_2)$ is orthogonal to $\sigma(e_1, e_1)$ so that it follows from $\dim \nu_p^1 = 1$ and $\lambda(p) \neq 0$ that $\sigma(e_1, e_2) = 0$. We similarly have $\sigma(e_1, e_3) = \sigma(e_2, e_3) = 0$. Then from (2.10) we have

$\lambda^2 = \tilde{g}(\sigma(e_1, e_1), \sigma(e_2, e_2))$, which, together with the Cauchy-Schwarz inequlity, implies $\sigma(e_1, e_1) = \sigma(e_2, e_2)$. By the same way, we have $\sigma(e_1, e_1) = \sigma(e_3, e_3)$. Then we have (3.2).

Let $S_p = \{(u, v) | u, v \in T_p(M),\ g(u, v) = 0,\ \| u \| = \| v \| = 1\}$ and consider a function f on S_p defined by

$$f(u, v) = \| \sigma(u, v) \|^2 .$$

Since S is compact, we can choose $(e_1, e_2) \in S_p$ at which f takes its maximum. We choose furthermore $e_3 \in T_p(M)$ in such a way that e_1, e_2, e_3 are orthonormal. Since f takes its maximum at (e_1, e_2), we have

$$\frac{d}{d\theta} f(e_1, \cos\theta e_2 + \sin\theta e_3) = \frac{d}{d\theta} f(\cos\theta e_1 + \sin\theta e_3, e_2) = 0$$

at $\theta = 0$ so that we get

$$\begin{cases} \tilde{g}(\sigma(e_1, e_2), \sigma(e_1, e_3)) = 0 \\ \tilde{g}(\sigma(e_1, e_2), \sigma(e_2, e_3)) = 0. \end{cases} \tag{3.4}$$

We consider the case where $\dim v_p^1 = 2$. If $f = 0$ holds identically, then we easily see that (3.2) holds so that $\dim v_p^1 \leqq 1$. This contradicts the assumption that $\dim v_p^1 = 2$. Therefore f is not identically zero so that $\| \sigma(e_1, e_2) \| \neq 0$. Then $\sigma(e_1, e_1)$ and $\sigma(e_1, e_2)$ span ν_p^1. On the other hand, it follows from (2.9), (2.11) and (3.4) that $\sigma(e_1, e_3)$ and $\sigma(e_2, e_3)$ are orthogonal to $\sigma(e_1, e_1)$. Since $\dim v_p^1 = 2$, we get $\sigma(e_1, e_3) = \sigma(e_2, e_3) = 0$. This, together with (2.10) and the Cauchy-Schawarz inequality, implies $\sigma(e_1, e_1) = \sigma(e_2, e_2) = \sigma(e_3, e_3)$. Thus, using (2.10), we get $\| \sigma(e_1, e_2) \| = 0$. This is a contradiction so that this case does not occur.

Finnally, we consider the case where $\dim v_p^1 = 3$. It is clear that f is not identically zero so that $\| \sigma(e_1, e_2) \| \neq 0$. It follows from (2.11) and (3.4) that

$$\tilde{g}(\sigma(e_1, e_3), \sigma(e_2, e_2)) = -2\tilde{g}(\sigma(e_1, e_2), \sigma(e_2, e_3)) = 0,$$

which, together with (2.9), (2.11) and (3.4), implies that $\sigma(e_1, e_3)$ and $\sigma(e_2, e_3)$ are orthogonal to $\sigma(e_1, e_1)$, $\sigma(e_2, e_2)$ and $\sigma(e_1, e_2)$. Suppose that $\sigma(e_1, e_1)$, $\sigma(e_2, e_2)$ and

$\sigma(e_1, e_2)$ span ν_p^1. Then $\sigma(e_1, e_3) = \sigma(e_2, e_3) = 0$. Using (2.10) and the Cuchy-Schwarz inequality, we have

$$\begin{aligned} \lambda^2 &= \tilde{g}(\sigma(e_i, e_i), \sigma(e_3, e_3)) + 2\tilde{g}(\sigma(e_i, e_3), \sigma(e_i, e_3)) \\ &= \tilde{g}(\sigma(e_i, e_i), \sigma(e_3, e_3)) \leqq \lambda^2, \qquad (i = 1, 2), \end{aligned}$$

so that $\sigma(e_1, e_1), \sigma(e_2, e_2)$ and $\sigma(e_3, e_3)$ are proportional. This contradicts the assumption that $\sigma(e_1, e_1), \sigma(e_2, e_2)$ and $\sigma(e_1, e_2)$ span 3-dimensional space ν_p^1. Therefore $\sigma(e_1, e_1), \sigma(e_2, e_2)$ and $\sigma(e_1, e_2)$ must be linearly dependent. Since $\sigma(e_1, e_2)$ is orthogonal to $\sigma(e_1, e_1)$ and $\sigma(e_2, e_2)$, it follows from (2.9) and (2.10) that $\sigma(e_1, e_1) = -\sigma(e_2, e_2)$ and $\| \sigma(e_1, e_2) \| = \lambda$. Moreover, since $\dim \nu_p^1 = 3$, it follows from (3.4) that there exist orthonormal normal vectors ξ_1, ξ_2, ξ_3 satisfying

$$\begin{aligned} \sigma(e_1, e_1) &= \lambda\xi_1, \quad \sigma(e_2, e_2) = -\lambda\xi_1, \quad \sigma(e_1, e_2) = \lambda\xi_2, \\ \sigma(e_1, e_1) &= \mu\xi_3, \quad \sigma(e_2, e_3) = \mu\xi_3, \quad \sigma(e_3, e_3) = \alpha\xi_1 + \beta\xi_2, \end{aligned}$$

for constants μ_1, μ_2, α and β. It follows from (2.9) $\sim$ (2.11) that

$$\beta\lambda + 2\mu_1\mu_2 = 0, \quad 2\mu_1^2 = \lambda^2 - \alpha\lambda, \quad 2\mu_2^2 = \lambda^2 + \alpha\lambda.$$

From the last two equations, we have $\mu_1^2 + \mu_2^2 = \lambda^2$. We may put $\mu_1 = \lambda \sin\theta$ and $\mu_2 = \lambda\cos\theta$ so that we have $\alpha = \lambda\cos 2\theta$ and $\beta = -\lambda\sin 2\theta$. Put $\tilde{e}_1 = (\cos\theta)e_1 - (\sin\theta)e_2$, $\tilde{e}_2 = (\sin\theta)e_1 + (\cos\theta)e_2$, $e_4 = (\cos 2\theta)\xi_1 - (\sin 2\theta)\xi_2$, $e_5 = (\sin 2\theta)\xi_1 + (\cos 2\theta)\xi_2$, $e_6 = \xi_3$. Then $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5$ and $\tilde{e}_6$ satisfy (3.3). □

We see in the proof of Lemma 3.1 that if $\dim \nu_p^1 \leqq 3$, then $\dim \nu_p^1 = 0, 1$ or 3. Let K denote the sectional curvature of M. Then we have

Lemma 3.2. *(1) If* $\dim \nu_p^1 = 0$, *then* $K \equiv c$.
(2) If $\dim \nu_p^1 = 1$, *then* $K \equiv c + \lambda^2$.
(3) If $\dim \nu_p^1 = 3$, *then* $c - 2\lambda^2 \leqq K \leqq c + \lambda^2$.

Proof. (1) is clear.

If $\dim \nu_p^1 = 1$, then it follows from the equation of Gauss and (3.2) that

$$g(R(X,Y)Y,X) = c + \lambda^2$$

for an arbitrary orthnormal X and Y in $T_p(M)$.

If $\dim \nu_p^1 = 3$, then it follows from the equation of gauss and (3.3) that, for an arbitrary orthonormal $X = \sum_i^3 x_i e_i$ and $Y = \sum_i^3 y_i e_i$,

$$g(R(X,Y)Y,X) = c - 2\lambda^2 + 3\lambda^2(x_1 y_3 - x_3 y_1)^2.$$

Since $0 \leqq (x_1 y_3 - x_3 y_1)^2 \leqq 1$, we have

$$c - 2\lambda^2 \leqq g(R(X,Y)Y,X) \leqq c + \lambda^2.$$

Proof of Theorem 1. Let $M_k = \{p \in M \mid \dim \nu_p^1 = k\}$. Then Lemma 3.1 implies that $k = 0, 1$ or 3. It is clear that M_3 is an open subset of M.

We first consider the case $M_3 \neq \phi$. There exists a neighborhood U of a point $p \in M_3$ such that $U \subset M_3$ and we can take a local field of orthonormal frames $\{e_1, e_2, e_3, e_4, e_5, e_6, \ldots, e_n\}$ on U satisfing (3.3) in Lemma 3.1. With respect to such a frame field, we have

$$\begin{cases} h_{11}^4 &= -h_{22}^4 = h_{33}^4 = \lambda, \quad h_{ij}^4 = 0 \ (i \neq j), \\ h_{12}^5 &= \lambda, \quad h_{ij}^5 = 0 \ (\{i,j\} \neq \{1,2\}), \\ h_{23}^6 &= \lambda, \quad h_{ij}^6 = 0 \ (\{i,j\} \neq \{2,3\}), \\ h_{ij}^\alpha &= 0 \ (\alpha \geqq 7; i,j = 1,2,3) \end{cases} \tag{3.5}$$

or equivalently

$$\begin{cases} \omega_1^4 &= \lambda\omega^1, \ \omega_2^4 = -\lambda\omega^2, \ \omega_3^4 = \lambda\omega^3, \\ \omega_1^5 &= \lambda\omega^2, \ \omega_2^5 = \lambda\omega^1, \ \omega_3^5 = 0, \\ \omega_1^6 &= 0, \ \omega_2^6 = \lambda\omega^3, \ \omega_3^6 = \lambda\omega^2, \\ \omega_1^\alpha &= \omega_2^\alpha = \omega_3^\alpha = 0 \ (\alpha \geqq 7). \end{cases} \tag{3.5$'$}$$

It follows from (2.8) and (3.5)' that

$$\omega_5^4 = \omega_2^1,\ \omega_6^4 = -\omega_3^2,\ \omega_6^5 = \omega_3^1. \tag{3.6}$$

It follows from (3.5)' that, for $\alpha \geqq 7$,

$$\begin{cases} 0 &= d\omega_1^\alpha = -\lambda(\omega_4^\alpha \wedge \omega^1 + \omega_5^\alpha \wedge \omega^2) \\ 0 &= \alpha\omega_2^\alpha = \lambda(\omega_4^\alpha \wedge \omega^2 - \omega_5^\alpha \wedge \omega^1 - \omega_6^\alpha \wedge \omega^3) \\ 0 &= d\omega_3^\alpha = -\lambda(\omega_4^\alpha \wedge \omega^3 + \omega_6^\alpha \wedge \omega^2), \end{cases}$$

which implies,

$$\omega_4^\alpha = f_\alpha \omega^2,\ \omega_5^\alpha = f_\alpha \omega^1,\ \omega_6^\alpha = f_\alpha \omega^3, \tag{3.7}$$

$f_\alpha (\alpha = 7, 8, \ldots, n)$ are differentiable functions on U.

Using (2.6), (2.7), (3.5), (3.6) and (3.7), we have

$$\begin{cases} \sum f_\alpha^2 = c \\ \sum f_\alpha^2 + c - 4\lambda^2 = 0. \end{cases} \tag{3.8}$$

Therefore we have

$$2\lambda^2 = c, \tag{3.9}$$

which implies that $\lambda = \sqrt{c/2}$ on U. Since M is connected, $\lambda = \sqrt{c/2}$ on M and $M_3 = M$. We have proved that if $M_3 \neq \phi$, then $\dim \nu_p^1 = 3$ every where on M and M is constant isotropic.

We must now remark the following.

Remark. *The case $M_3 \neq \phi$ does not occur when $c < 0$ by (3.9).*

We next consider the case where $M_3 = \phi$ and $M_1 \neq \phi$. Since $M_2 = \phi, M_1$ is open in M. (3.2) of Lemma 3.1 implies that M is umbilic on M_1 so that $M_1 = M$ by the connectedness of M, that is, M is a totally umbilic submanifold of $\tilde{M}^n(c)$, and hence M is constant isotropic.

We finally consider the case where $M_1 = M_3 = \phi$ and $M_0 \neq \phi$, that is, $M_0 = M$. If this is the case, M is totally geodesic in $M^n(c)$ so that M is clearly constant isotropic.

Thus we have proved Theorem 1. □

Now we review a hyperbolic space $\mathbf{H}^n$ and totally umbilic hypersurfaces of $\mathbf{H}^n$. An n-dimensional hyperbolic space $\mathbf{H}^n$ is an n-dimensional complete, connected and simply connected Riemannian manifold of constant curvature -1. A model space of $\mathbf{H}^n$ is the half-space of an R^n given by $\mathbf{H}^n = \{x_1, x_2, \ldots, x_n) \in R^n | x_n > 0\}$ with metric $\tilde{g} = \sum_{i=1}^n dx_i^2/x_n^2$.

Let $(R^n, \bar{g})$ be an n-dimensional Euclidean space with the Euclidean metric $\bar{g}$ and its Riemannian connection $\bar{\nabla}$. A hypersurface M in $(R^n, \bar{g})$ is said to be *umbilic* if, at each point $p \in M$,

$$\bar{g}(\bar{\nabla}_x \xi, Y) = \kappa \bar{g}(X, Y)$$

holds for all $X, Y \in T_p(M)$ and a unit normal vector field ξ where κ is a constant on M.

Consider a conformal change $\tilde{g} = \mu \bar{g}$ of metric and denote the Riemannian connection of $\tilde{g}$ by $\tilde{\nabla}$. Then we have

$$\tilde{\nabla}_{\bar{X}} \bar{Y} = \bar{\nabla}_{\bar{X}} \bar{Y} + S(\bar{X}, \bar{Y}) \tag{3.10}$$

for al $\bar{X}$ and $\bar{Y}$, where $S(\bar{X}, \bar{Y}) = \frac{1}{2\mu}\{(\bar{X}\mu)\bar{Y} + (\bar{Y}\mu)\bar{X} - \bar{g}(\bar{X}, \bar{Y}) \text{ grad } \mu\}$ and grad μ is calcuated with respect to the metric $\bar{g}$, that is, $\bar{X}(\mu) = \bar{g}(\bar{X}, \text{grad } \mu)$. If M is umbilic in $(R^n, \bar{g})$, that is, $\bar{g}(\bar{\nabla}_X \xi, Y) = \kappa \bar{g}(X, Y)$, using (3.10), then at each point $p \in M$ we have

$$\tilde{g}(\tilde{\nabla}_X (\frac{\xi}{\sqrt{\mu}}), Y) = \frac{2\kappa\mu + \xi(\mu)}{2\mu\sqrt{\mu}} \tilde{g}(X, Y), \text{ for all } X, Y \in T_p(M),$$

which implies that M is also umbilic in $(R^n, \tilde{g})$.

The hyperbolic space $\mathbf{H}^n$ is considered an open submanifold of R^n with the metric $\tilde{g}$ of R^n.

Since umbilic hypersurface in $(R^n, \bar{g})$ are $(n-1)$-planes or $(n-1)$-spheres, umbilic hypersurfaces of $\mathbf{H}^n$ are therefore the intersections with $\mathbf{H}^n$ of $(n-1)$-planes or $(n-1)$-

spheres of R^n, and so *totally umbilic hypersurfaces of* $\mathbf{H}^n$ *are the geodesic spheres, the horospheres and the hyperspheres*.

Proof of Theorem 2. Since $\mathbf{H}^n$ is of negative curvature -1, as stated in Remark above, $M_3 = \phi$ so that the dimension of the first normal space of M is everywhere 0 or 1. Since M is constant isotropic by Theorem 1, $M_0 = M$ or $M_1 = M$.

If $M_0 = M$ is the case, then M is totally geodesic in $\mathbf{H}^n$.

We consider next the case $M_1 = M$, as stated in the proof of Theorem 1, M is totally umbilic in $\mathbf{H}^n$, and hence M is a totally umbilic hypersurface in a 4-dimensional hyperbolic space $\mathbf{H}^4$, which is totally geodesic in $\mathbf{H}^n$. Therefore, as stated above, M is a geodesic sphere, a horosphere or a hypersphere of $\mathbf{H}^4$.

Proof of Theorem 3. We may assume that M has no geodesic points. It follows from Lemma 3.1 and the minimality of M that the dimension of the first normal space of M is 4 or 5.

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $T_p(M)$ which satisfies (3.4). Since $\sigma(e_1, e_3)$ is orthogonal to $\sigma(e_1, e_1,)$ and $\sigma(e_3, e_3)$ from (2.9), $\sigma(e_1, e_3)$ is also orthogonal to $\sigma(e_2, e_2)$ by the minimality of M. By (3.4), furthermore, $\sigma(e_1, e_3)$ is orthogonal to $\sigma(e_1, e_2)$, too. By the same reason as above, $\sigma(e_2, e_3)$ is orthogonal to $\sigma(e_1, e_1)$, $\sigma(e_2, e_1)$, $\sigma(e_1, e_2)$ and $\sigma(e_3, e_3)$. It follows from (2.9), (2.11) and the minimality that

$$\begin{aligned} 2\tilde{g}(\sigma(e_1, e_3), \sigma(e_2, e_3)) &= -\tilde{g}(\sigma(e_1, e_2), \sigma(e_3, e_3)) \\ &= \tilde{g}(\sigma(e_1, e_2), \sigma(e_1, e_1) + \sigma(e_2, e_2)) \\ &= 0. \end{aligned}$$

On the other hand, we see from (2.10) and the minimality of M that $\sigma(e_1, e_3) \neq 0$ and $\sigma(e_2, e_3) \neq 0$.

Therefore we have orthonormal normal vector fields e_4, e_5, e_6, e_7, e_8 satisfing

$$\begin{cases} \sigma(e_1, e_1) &=& \lambda e_4, \\ \sigma(e_1, e_2) &=& \mu_1 e_5, \\ \sigma(e_1, e_3) &=& \mu_2 e_6, \\ \sigma(e_2, e_3) &=& \mu_3 e_7, \\ \sigma(e_2, e_2) &=& \mu_4 e_4 + \mu_5 e_8, \end{cases} \tag{3.11}$$

Then we have

$$\mu_4^2 + \mu_5^2 = \lambda^2. \tag{3.12}$$

Moreover it follows from the minimality that $\sigma(e_3, e_3) = -(\lambda + \mu_4)e_4 - \mu_5 e_8$ which implies

$$2\lambda\mu_4 + \mu_4^2 + \mu_5^2 = 0. \tag{3.13}$$

On the other hand, we see from (2.10) and (3.11) that

$$\lambda^2 - \lambda\mu_4 - 2\mu_1^2 = 0, \tag{3.14}$$

$$2\lambda^2 + \lambda\mu_4 - 2\mu_2^2 = 0, \tag{3.15}$$

$$\lambda^2 + \lambda\mu_4 + \mu_4^2 + \mu_5^2 - 2\mu_3^2 = 0. \tag{3.16}$$

It follows from (3.12), (3.13), (3.14), (3.15) and (3.16) that

$$\mu_4 = -\frac{\lambda}{2} \tag{3.17}$$

and

$$\mu_1^2 = \mu_2^2 = \mu_3^2 = \mu_5^2 = \frac{3}{4}\lambda^2.$$

We may assume without loss of generality that

$$\mu_1 = \mu_2 = \mu_3 = \mu_5 = \frac{\sqrt{3}}{2}\lambda. \tag{3.18}$$

Using (2.2), (3.11), (3.17) and (3.18), we have

$$g(R(e_1, e_2)e_2, e_1) = g(R(e_1, e_3)e_3, e_1) = g(R(e_2, e_3)e_3, e_2) = c - \frac{5}{4}\lambda^2,$$

which implies that M is of constant curvature.

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