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DYNAMICAL SYSTEM TOPOLOGY PRESERVED IN THE PRESENCE OF NOISE

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Abstract

We first give a precise definition of the terms "topological horseshoe" and "generalized quadrilateral" and then examine the behavior of a homeomorphism F on a locally compact, separable, locally connected metric space X (X is usually a manifold in applications) such that F restricted to some generalized quadrilateral Q in X is a topological horseshoe map. For a set $Q \subset X$ we define and describe (1) the *permanent set* Z of Q to be $\{x \in X : F^n(x) \in Q$ for all integers $n\}$, and (2) the *entrainment set* of Q to be $E(Q) = \{x \in X : F^{-n}(x) \in Q$ for all sufficiently large $n\}$. We give conditions under which various closed sets of $\overline{E(Q)}$ are associated, in a strong way, with indecomposable, closed, connected spaces invariant under F. (A connected set A is *indecomposable* if it is not the union of two proper connected sets, each of which is closed relative to A.) Next we show that even when small amounts of noise are added to the dynamical system, there are associated indecomposable sets. These sets are not, in general, invariant sets for our process with noise, but they are the physically observable sets, while invariant Cantor sets are not, and they are the sets that can be measured.

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1. Introduction

Let $F : M \to M$ be a C^1 map on a manifold M. The study of a dynamical system

$$x_n \to x_{n+1} = F(x_n) \tag{1}$$

often becomes the study of sets such as fixed points, periodic orbits, stable and unstable manifolds, basins, attractors, and basic sets -in general, those sets that are invariant under F. A particularly interesting case is the invariant set of a horseshoe map.

Some physical processes are better modelled by including some representation of noise as follows: Let $\epsilon > 0$. We define a time-dependent process (n is time)

$$(n, x) \to (n + 1, F_n(x)) \tag{2}$$

where $F_n : M \to M$ is a C^1 map for each integer n such that

$$\| F_n - F \|_{C^1} \leq \epsilon.$$

Hence (2) is a time-dependent system that we could say is ϵ-close to the time-dependent system (1). The dependence of time means there are in general no sets that are invariant for (2), that is no nontrivial sets S such that $F_n(S) = S$ for all integers n.

Let $F : \mathbf{R^2} \to \mathbf{R^2}$ be a homeomorphism. Let Q be the quadrilateral pictured in Figure 1, and imagine that a computer prints this picture by plotting separately the images of the four sides of the quadrilateral. Let $\tilde{F}(S)$ be the computer image of S, where S is any of the sides of the quadrilateral. We can view $\tilde{F}(S)$ as a finite collection of dots or line segments. Assume furthermore that for some small ϵ (say
epsilon is less than 0.001 times the length of the sides of Q), and for each side S of the quadrilateral, $\tilde{F}(S)$ and $F(S)$ each lie in an ϵ-neighborhood of the other (so in the Hausdorff metric the distance between them is less than ϵ).

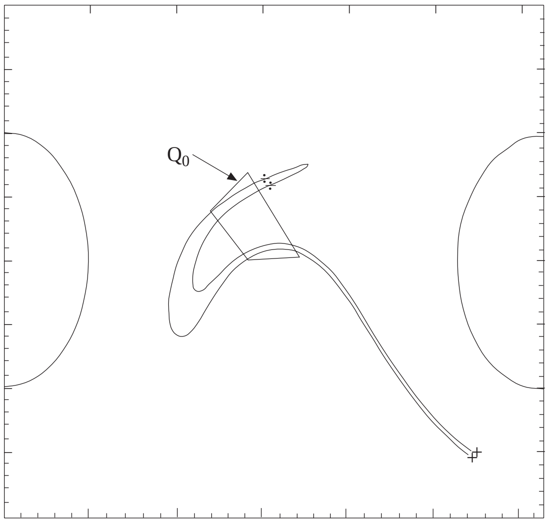

Figure 1. The figure is part of a computer investigation of a fluid flow model. (See [SKGY] and [KSYG] for detailed discussions of this work.) The image of the quadrilateral Q_0 is pictured (the curve crossing Q_0 twice), and was obtained by computing the image of Q_0 under the Poincaré return map F associated with the fluid flow model. Evidently, the action of F on Q_0 results in a "topological horseshoe". The crosses mark the images of the vertices of the quadrilateral Q_0 under F. In the figure, the upper right side of Q_0 (i.e., the upper of the two side which can be expressed as a linear function of the horizontal x-axis with negative slope) is the short portion of the image curve just outside the upper right side, while the image of the lower left side (the lower of the two sides which can be expressed as a linear function of the horizontal x-axis with negative slope) is the portion of the image curve with vertices marked at the lower right of the figure. The image of this side actually extends back up into the image, and is not just a short segment between the two crosses.

Such a picture is often called a topological horseshoe. We ask what can be rigorously concluded from this situation about the points $Z = \{p \in Q : F^n(p) \in Q$ for all integers $n\}$. In the literature it is usally assumed that F is a diffeomorphism and that the map is hyperbolic in Q (which we need not define here), and in this case we call the example a Smale horseshoe. It is often easy to verify that F is diffeomorphism,but it is far more difficult to verify hyperbolicity in a picture such as we have above, and indeed hyperbolicity may not hold. Our Theorems 3 and 4 will apply to Figure 1 assuming hypothesis Ω given below. We remark that the homeomorphic image of a side S can be quite complicated and still be within ϵ of $\tilde{F}(S)$ as shown, and we must draw our conclusions despite this difficulty.

We are interested in the entrainment sets (and the destination sets) associated with topological horseshoes for several reasons:

1. The dynamics on the permanent set in a generalized quadrilateral Q for a topological horseshoe are described "in the large " at least, by the dynamics of the shift on M symbols. In particular, although we know there are periodic *sets* of all periods in the permanent set, we don't know, without further information about the space and the homeomorphism involved, if there are any periodic orbits in those sets. In addition to the usual M-shift dynamics inside the set, interesting behavior and topology can happen *outside* Q in the entrainment set associated with Q as well. Compare, for example, the Smale horseshoe map where the crossing number M is 2 (Figure 3) with the fluid flow horseshoe pictured in Figure 1. (To be completely accurate, Figure 1 actually shows the action of the square root $\sqrt{F}$ of the fluid flow diffeomorphism, or the time-1/2 map, rather than the time-1 map studied in [SKGY] and [KSYG]. These papers study the dynamics of a periodically varying fluid flow past an array of cylinders.) For the Smale horseshoe map, the entrainment set consists entirely of points attracted to a fixed point outside the rectangle (see Figure 3), while the fluid flow diffeomorphism, on the other hand, has a much more complicated and interesting entrainment set, both topologically and dynamically. (See Figure 2.) These vastly different entrainment sets occur in spite of the fact that inside the respective quadrilaterals, the dynamics are exactly the same.

2. The entrainment sets for a topological horseshoe are physically observable in real experiments in the sense that they can be observed. (See [SKG].) Of course, no experiment can reveal the infinitely fine structure of an entrainment set. Nonetheless, the entrainment set can be thought of as the result of pouring dye into a quadriateral Q and then watching it evolve. The entrainment set is the limit as time goes to ∞ of the theoretical position of the dye. Thus, it may well be possible in experiments

to measure and compute the entrainment set's fractal dimension, Lyapunov exponents, Hausdorff dimension, etc. (See [HOY].) Cantor sets, or quotient Cantor sets, and periodic points, on the other hand, are nearly impossible to observe forming in a real, as opposed to simulated, flow.

3. Similar indecomposable sets often appear as the "strange" sets associated with nonlinear dynamics (e.g., strange attractors, fractal basin boundaries, and closures of stable and unstable manifolds of chaotic saddles, as well as entrainment sets), and, when present, they provide a usefull conceptual characterization of these phenomena. (See [SKOY].)

2. Background, Definitions and Notation

A *continuum* is a compact, connected metric space. A subset of a continuum which is itself a continuum is a *subcontinuum*. A continuum is *indecomposable* if it is not the union of two (necessarily overlapping) proper subcontinua. Equivalently, a continuum is indecomposable if and only if every proper subcontinuum has empty interior (relative to the continuum). If x is a point in the continuum X, then the *composant Com(x)* in X containing x is the set of all points y in X such that there is a proper subcontinuum in X that contains both x and y. The collection $C(X)$ of all composants of an indecomposable continuum X partitions X into $\mathbf{c}$ (the cardinality of the real numbers) many mutually disjoint, first category, connected F_σ-sets. (For more information and references concerning indecomposable continua, see [K].)

In this paper, we use the term "indecomposable" in its original, more general sense: If X is a metric space, then the connected subset A of X is *indecomposable* if it cannot be expressed as the union of two proper (necessarily overlapping) connected sets. When L. E. J. Brouwer [B] constructed the first indecomposable continuum, he was disproving a conjecture of Schoenflies that the common boundary between two simply connected open plane sets had to be *decomposable,* i.e., that such a boundary would be a connected, closed set which was itself the union of two proper, closed, connected sets. Although the boundary in Brouwer's example is compact, and failed to be decomposable so that it is an indecomposable continuum, for a number of years indecomposability was studied as a property of connected sets which were not necessarily compact. In recent years, indecomposability has been studied mostly in compact sets. In this paper, we return to the original focus of indecomposability as a (possible) property of connected sets, and one that often arises in the presence of chaotic dynamics.

If X is a locally compact, separable metric space, then it must still be the case that a connected subset A is indecomposable if and only if every proper, closed, connected subset of A has empty interior (relative to the subspace A). (See [Ku].) If x is a point in the set A, then the *composant Com(x)* in A containing x is the set of all points y in A such that there is a proper, connected subset of A that contains both x and y, and is closed in A. If A is a completely metrizable (i.e., the set A admits a metric d_A which is compatible with its topology and with respect to which A is complete, and it should be noted that the metric d_A may not extend to a metric d on X compatible with the topology on X), indecomposable subset of X, then the collection $\mathcal{C}(A)$ of all composants of the set A partitions it into uncountably many mutually disjoint, first category, connected F_σ-sets.

In this paper, the indecomposable sets we consider all lie in a larger spaces X that are connected, locally compact, locally connected, spearable metric spaces. There are connected, locally compact, locally connected, separable metric spaces which are one-dimensional, and have the property that no countable collection of disjoint arcs separates the space. Examples include the Sierpinski curve (or gasket), the Menger cube (or sponge), and higher dimensional analogs of these spaces. **Thus in this paper the statement that "X is a background space" means that "X is a locally compact, locally connected, separable metric space"**. Subspaces of X, however, need not be locally compact, connected, locally connected, or closed.

If X is a background space and A is a subset of X, then we use the notation $A^\circ, \overline{A}$, and ∂A to denote the interior, closure, and boundary of A in X, respectively. If Y is a subspace of X (with the inherited topology), $A \subseteq Y$, and we wish to discuss the interior, closure, or boundary of A in the subspace Y, we use the notation $Int_Y(A)$, $Cl_Y(A)$, and $Bdy_Y(A)$, respectively, to avoid confusion. The symbols $\mathbf{Z}, \mathbf{N}$, and $\tilde{\mathbf{N}}$ are used to denote the integers, the positive integers, and the nonnegative integers, respectively. We use d to denote a metric on X (which is, of course, compatible with its topology), unless this leads to confusion, in which case we differentiate metrics by using a variety of other letters and symbols. If $\epsilon > 0$, $x \in X$, let $D_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$. If $\epsilon > 0$, and $A, B \subset X$, let $D_\epsilon(A) = \{y \in X| \ d(x, y) < \epsilon$ for some $x \in A\}$, and let

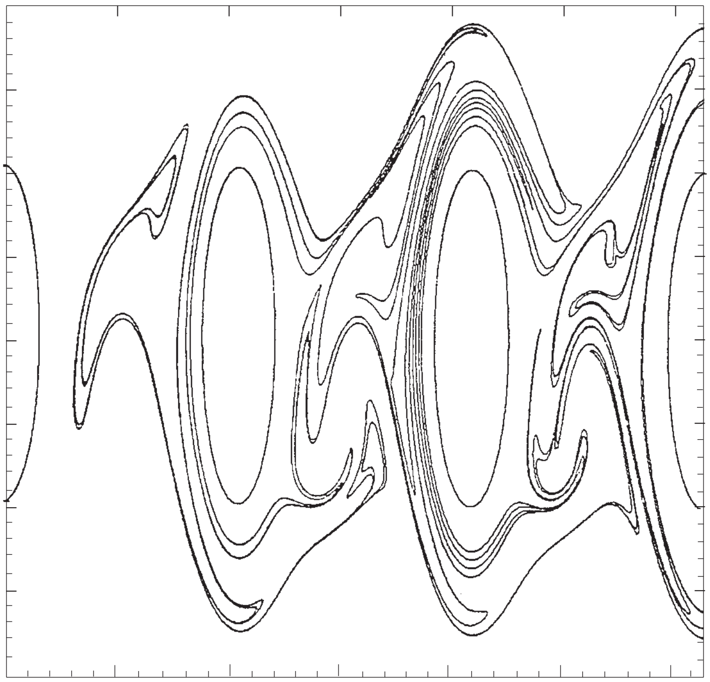

Figure 2. The figure is part of a computer investigation of a fluid flow model. (See [SKGY] and [KSYG] for detailed discussions of this work.)

$d(A, B) = inf\{d(x, y)|x \in A, y \in B\}$. **Since our background spaces X are locally connected we can make the additional assumption that for each $\epsilon > 0$, if x is a point in X, then $D_\epsilon(x)$ is connected**.

Hypothesis Ω_0: Suppose that X is a background space, and $F : X \to X$ is a homeomorphism. The set $Q \subset X$ is a *generalized quadrilateral* if Q is a compact, connected, locally connected neighborhood of X such that $(\overline{Q^\circ}) = Q$, and end_0 and end_1 are disjoint closed subsets of ∂Q each of which has nonempty interior in the relative topology on ∂Q, i.e., $Int_{\partial Q}(end_i) \neq \emptyset$ for $i = 1, 2$. We say that end_0 and end_1 are the *ends* of Q, and that $\overline{Q \backslash (end_0 \cup end_1)} = side$ is the "side" of Q, and define the map F

to be a *(topological) horseshoe map on* Q if there is a positive integer $M \geq 2$ such that

1. if K an arc in Q that intersects both end_0 and end_1, then $Q \cap F(K)$ contains at least M components each of which intersects both end_0 and end_1,

2. neither $F(end_0)$ nor $F(end_1)$ intersects Q, and

3. $F(side) \cap side = \emptyset$.

We say that F *satisfies* Hyothesis Ω_0 *on the generalized quadrilateral Q (with respect to* end_0, $end_1 = \overline{Q \backslash (end_0 \cup end_1)}$) if the definition above is satisfied. If M is a positive integer with the property that if K is an arc in Q that intersects both end_0 and end_1, then $Q \cap F(K)$ contains at least M components each of which intersects both end_0 and end_1, then M is unique, and we say that M is the *crossing number* of F on Q. If, in addition, X is a differentiable manifold and F is a diffeomorphism, then F is a *hyperbolic horseshoe map on* Q if it is a topological horseshoe map on Q (i.e., it satisfies the conditions above) and if Γ denotes the invariant set $\cap_{n \in \mathbf{Z}} F^n(Q)$, then F is hyperbolic on Γ. (See Figure 3 for illustrations of hyperbolic horseshoes with several crossing numbers, and Figure 4 for illustrations of topological horseshoes, and what is and is not allowed by this definitintion).

If $F : X \to X$ is a homeomorphism, then the closed set B satisfies the *lockout property* if when $q \in B$ and $F^k(q) \notin B$ for some $k > 0$, then further iterates of q remain outside B; i.e., $F^n(q) \notin B$ if $n \geq k$. For the noisy case we need a stronger version of this property: thus, the closed set B satisfies the *uniform lockout property* if there are a positive integer N_F, a closed neighborhood B^+ such that $B \subset (B^+)^\circ$, and a positive number ϵ such that if $q \in B$ and $F(q) \notin B$, then $d(F^n(q), B^+) > \epsilon$ if $n \geq N_F$.

Suppose that $\sum_M = \{(\ldots i_{-1\star} i_0 i_1 i_2 \ldots)$: for each integer $j, i_j \in \{1, 2, \ldots M\}\}$. Then $\sum_M$ is a Cantor set expressed as the product space $\{1, 2, \ldots M\}^{\mathbf{Z}}$, and the shift (on M symbols) $\sigma_M : \sum_M \to \sum_M$ defined by $\sigma_M((\ldots i_{-1\star} i_0 i_1 i_2 \ldots)) = (\ldots j_{-1\star} j_0 j_1 j_2 \ldots)$, where $j_k = i_{k+1}$ for each integer k, is a homeomorphism. Thus, $\sigma_M((\ldots i_{-1\star} i_0 i_1 i_2 \ldots)) = (\ldots i_{-1} i_{0\star} i_1 i_2 \ldots)$. The properties of $\sum_M$ and σ_M have been thoroughly studied: see [R], for example, for a discussion).

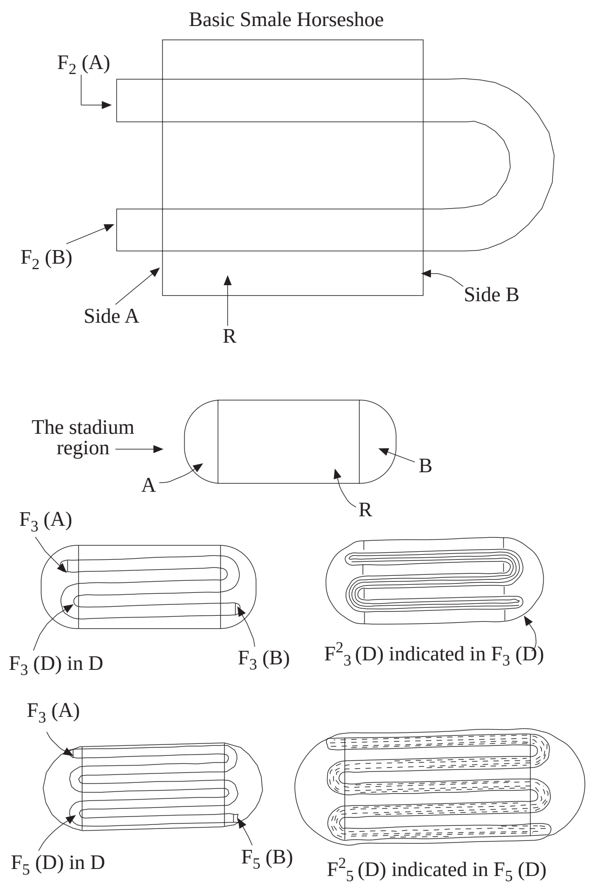

Figure 3. Smale horseshoes with several crossing numbers are pictured.

Now if $\overline{X}$ denotes a compactification of the background space X, which is itself a metric space (and since the one-point compactification is a metric space, one does exist), there is a metric $\overline{d}$ on $\overline{X}$ which is compatible with its topology. Since $\overline{X}$ is a compact metric space, so is the space $\mathcal{F}(\overline{X})$ consisting of all closed subsets of $\overline{X}$ with the topology induced by the Hausdorff metric ν (relative to the metric $\overline{d}$). Thus, if H and K are in $\mathcal{F}(\overline{X})$, then $\nu(H,K)$ is the inf $\{\epsilon > 0|$ each point of H is within ϵ (under the metric $\overline{d}$) of some point of K and each point of K is within ϵ (under the metric $\overline{d}$) of some point of $H\}$. A *Cantor set of continua* is a collection $\mathcal{C}$ of continua in $\overline{X}$ such that if $\tilde{\mathcal{C}}$ denotes the subset of $\mathcal{F}\overline{X}$ whose points are the components of $\mathcal{C}$, then $\tilde{\mathcal{C}}$ is a Cantor set in $\mathcal{F}(\overline{X})$. We can also talk directly about the Hausdorff metric on closed subsets $\mathcal{F}(X)$ of X, if we use the metric $\overline{d}$ inherited by X when it is considered as a subspace of $\overline{X}$.

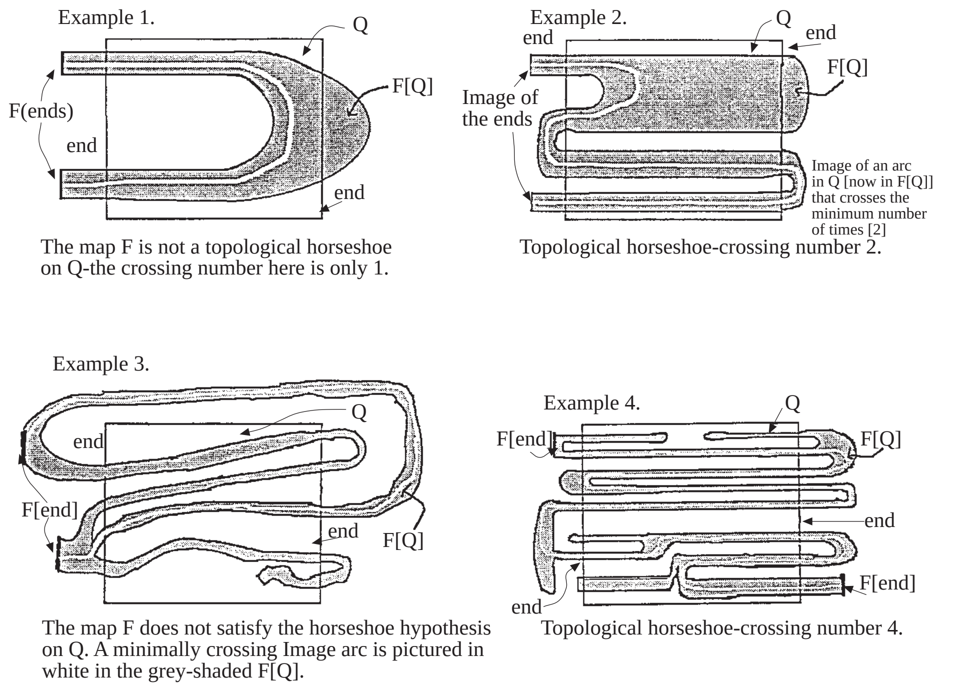

Figure 4. Examples 1 and 3 are not topological horseshoes, while Examples 2 and 4 are.

Another topology on collections of closed subsets of a compact subset D of X that we need is the quotient topology. Suppose that D is a compact subset of X, and $\mathcal{D}$ denotes a decomposition of D into disjoint closed sets which is upper semicontinuous. The collection $\mathcal{D}$, when endowed with the quotient topology, is a compact metric space, with the points of $\mathcal{D}$ (considered as space) being the sets in the collection $\mathcal{D}$ (considered as collection in X). Let $P : D \to \mathcal{D}$ denote the projection map associated with the decomposition. (Thus, for $x \in D$, $P(x) = D_x$, where $D_x \in \mathcal{D}$ and $x \in D_x$.) The map P is continuous and onto. We say that the set D is a *quotient Cantor set* (relative to the decomposition $\mathcal{D}$) if there is an upper semicontinuous decomposition $\mathcal{D}$ of D such that $\mathcal{D}$ endowed with the quotient topology is a Cantor set. Note that $\mathcal{D} \subset \mathcal{F}(\overline{X})$. Every set that is open in the quotient topology on $\mathcal{D}$ is also open in the topology induced by the Hausdorff metric on $\mathcal{D}$. If $D_0 \in \mathcal{D}$, then D_0 is a *point of continuity* of $\mathcal{D}$ if every sequence $D_1, D_2, \ldots$ in $\mathcal{D}$ which converges to D_0 in the quotient topology also converges to D_0 in the topology induced by the Hausdorff metric on $\mathcal{D}$. The points of continuity of an upper semicontinuous decomposition on a compact metric space contain a dense G_δ-subset of the decomposition space. (See [Ku] for more details.)

A finite collection $C = \{c_0, c_1, \ldots, c_n\}$ of subsets of a space X is a *chain* if $c_i \cap c_j \neq \emptyset$ if and only if $|i - j| \leq 1$. If the elements of C are open sets, the C is an *open chain.* However, in this manuscript our chains have elements consisting of closed neighborhoods intersecting only at their boundaries. Thus, we define a *tiling chain* $C = \{c_0, c_1, \ldots, c_n\}$ to be a chain whose elements are closed sets and such that $c_i^\circ \cap c_j^\circ = \emptyset$ for $i \neq j$. The *mesh* of a chain $C = \{c_0, c_1, \ldots, c_n\}$ is the positive number $\varpi = \sup\{diam(c_i)\}$, and each $c_i \in C$ is a *link* of the chain C.

For the noisy case, we consider a new assumption.

Adding noise. *Let* $\epsilon > 0$. *Instead of applying a homeomorphism* F *at each time* i*, we instead assume that for each* i*, a homeomorphism* F_i *which is close to* F *in the sense that* $d(F(q), F_i(q)) < \epsilon$ *for each* i *and* q*, is applied. We refer to* ϵ *as the "noise level".*

With this assumption we can still talk about the trajectory of a point if we replace $F(q_0)$ with $F_0(q_0), F^2(q_0)$ with $F_1 \circ F_0(q_0)$, and so forth. In general, the *trajectory* of q_0 is the bisequence $\ldots q_{-2}, q_{-1}, q_0, q_1, \ldots,$ where $q_1 = F_{i-1} \circ F_{i-2} \circ \ldots \circ F_0(q_0)$ and $q_{-i} = (F_{-1} \circ F_{-2} \circ \ldots \circ F_{-i})^{-1}(q_0) = F_{-i}^{-1} \circ \ldots \circ F_{-2}^{-1} F_{-1}^{-1}(q_0)$ for $i > 0$. It no longer makes sense to talk about *invariant* Cantor sets, *invariant* points, or *invariant* continua. Define then the *permanent set* Z_0 to be the *set of all points* x_0 *whose entire trajectory (under a sequence of noisy maps) is inside the generalized quadrilateral* B*,* the *entrainment set* $E(Z_0)$ *to be the set of points* x_0 *whose backward trajectory (under a sequence of noisy maps) is eventually inside* B*, and the destination set* $D(Z_0)$ *to be the set of points* x_0 *whose forward trajectory (under a sequence of noisy maps) is eventually inside* B*.*

For notational convenience, we make the following definition: for $m \leq n$, and $\{F_n, F_{n-1}, \ldots, F_m\}$ a collection of $n-m+1$ homeomorphisms on X, $F_n \circ F_{n-1} \circ \ldots \circ F_m = F_{n,m}$. Even though the maps F_j are chosen randomly, we are assuming that once the jth map F_j is chosen, it is the map that is applied at integer time j. Thus, we have a family of maps $\{F_j\}_{j=-\infty}^{\infty}$ which also defines a map $\mathbf{F} : \mathbf{Z} \times X \to X$, where $\mathbf{F}(j, x) = \mathbf{F}_j(x)$ for

$x \in X$, just as a flow on the space X is a map $\varphi : \mathbf{R} \times X \to X$, such that for each time $t, \varphi_t : X \to X$ is the homeomorphism defined by $\varphi_t(x) = \varphi(t,x)$ for $x \in X$. However, there is one important difference: the group operation on $\mathbf{Z}$ is not preserved by $\mathbf{F}$, while that on $\mathbf{R}$ is preserved by φ. Thus, although it is the case that $\varphi(t+t',x) = \varphi(t,\varphi(t',x))$, or $\varphi_{t+t'}(x) = \varphi_t(\varphi_{t'}(x))$, it is not generally the case that $\mathbf{F}(j+j',x) = \mathbf{F}_{j+j'}(x)$ is the same as $\mathbf{F}(j,\mathbf{F}(j',x)) = \mathbf{F}_j(\mathbf{F}_{j'}(x))$ since $F_j \circ F_{j'} \neq F_{j'} \circ F_j$ (for most cases). We say that $\mathbf{F} : \mathbf{Z} \times X \to X$ is a *floating dynamical system*, or just a *floating system.*

Since we can talk about the trajectory of a point x in X under the system $\mathbf{F}$ of maps, we can talk about the trajectory of the permanent set Z_0, if we define $Z_n = \tilde{F}_{n,0}(Z_0)$ for $n \geq 0$ and $Z_{-n} = \tilde{F}^{-1}_{-1,-n}(Z_0)$ for $n < 0$. Note that each Z_n is contained in B. Because composition is not commutative for maps in the family $\{F_j\}_{j=-\infty}^{\infty}$, in order to keep track of our trajectories, we need to designate a "relative center" for each trajectory to know where it is in relation to the family $\{F_j\}_{j=-\infty}^{\infty}$ and the sets $\{Z_j\}_{j=-\infty}^{\infty}$: for each integer n, let the *trajectory* of q_n be the centered bisequence $\ldots q_{n-2}, q_{n-1} \star q_n \star q_{n+1}, \ldots$ Note that if the bisequence that denotes the *trajectory* of q_0 is $\ldots q_{-2}, q_{-1}, q_0, q_1, \ldots,$ then $\ldots q_{-2}, q_{-1} \star q_0 \star q_1, \ldots, = \ldots q_{-2}, q_{-1}, q_0, q_1, \ldots,$ but the symbol $\star$ puts the *center* for this trajectory at q_0, and we know that the next map to be applied to q_0 is F_0, and the last map already applied to obtain q_0 is F_{-1}. Similarly, $\ldots q_{-2}, q_{-1}, q_0, q_1, \ldots = \ldots q_{n-2}, q_{n-1}, q_n, q_{n+1}, \ldots = \ldots q_{n-2}, q_{n-1} \star q_n \star q_{n+1}, \ldots,$ but the next map to be applied to q_n is F_n, and $F^{-1}_{n-1}(q_n) = q_{n-1}$. Also, for $i \geq 0$, $q_{n+i+1} = \tilde{F}_{n,n+i}(q_n)$, and for $i > 0$, $q_{n-i} = \tilde{F}^{-1}_{n-1,n-i}(q_n)$.

Further, if there is an ordered collection $\mathcal{Z} = \{Z_n\}_{n \in \mathbf{Z}}$ of subsets of X, then the floating system $\mathbf{F} : \mathbf{Z} \times X \to X$ *preserves* $\mathcal{Z}$ if for each integer $n, F_n(Z_n) = Z_{n+1}$ (where $F_n = \mathbf{F}|\{n\} \times Z_n$). We use the notation $\mathbf{F}|\mathcal{Z}$ to denote $\{F_n|Z_n\}_{n \in \mathbf{Z}}$. We say that $\mathbf{F}|\mathcal{Z}$ is *conjugate* to the map $\phi : Y \to Y$ if there is a collection $\mathbf{H} = \{h_n\}_{n \in \mathbf{Z}}$ of homeomorphisms such that for each n, h_n is a homeomorphism from Z_n onto Y, and $\phi \circ h_n = h_{n+1} \circ F_n|Z_n$. More generally, if $\mathcal{Y} = \{Y_n\}_{n \in \mathbf{Z}}$ is an ordered collection of subsets of the space X, or is an ordered collection of spaces, and $\sqcup\mathcal{Y}$ denotes the disjoint union of the collection $\mathcal{Y}$, then if $\mathbf{F} : \sqcup\mathcal{Y} \to \cup\mathcal{Y}$ is one-to-one and onto, we say that $\mathbf{F}$ *preserves* the collection $\mathcal{Y}$ if for each n and each $y \in Y_n \in \mathcal{Y}$, $\mathbf{F}|Y_n : Y_n \to Y_{n+1}$ is a homeomorphism. The statement that $\mathbf{F}$ is *conjugate* to the map $\phi : Y \to Y$ means that there is a collection $\mathcal{H} = \{\theta_n\}_{n \in \mathbf{Z}}$

of homeomorphisms such that for each n, θ_n is a homeomorphism from Y_n onto Y, and $\phi \circ \theta_n = \theta_{n+1} \circ \mathbf{F}|Y_n$.

3. The Results

Theorem 2.1. *Suppose that X is a background space, A is an indecomposable subset of X such that A is nowhere dense in X, and $\overline{X}$ is a metric compactification of X. Let $\overline{A}$ denote the closure of A in $\overline{X}$. Then*

1. *if some composant Cps of A is has the property that if o is an open set in X such that $\overline{o}$ is compact, and any component of $\overline{o} \cap Cps$ is also a component of $\overline{o} \cap \overline{A}$, then $\overline{A}$ is an indecomposable continuum in $\overline{X}$ if and only if A is an indecomposable set in X; and*
2. *if A is a closed subset of X, then $\overline{A}$ is an indecomposable continuum in $\overline{X}$ if and only if A is an indecomposable set in X.*

Proof. We prove the first statement. The second then follows immediately. That when $\overline{A}$ is an indecomposable continuum in $\overline{X}$, A is an indecomposable set in X follows from the fact that a dense connected subset of an indecomposable set is indecomposable. (See [Ku], p.208.)

Suppose then that A is an indecomposable set in X. If $\overline{A} = A \subset X$, then A is an indecomposable continuum, so there is nothing to prove. Then suppose that $\overline{A} \backslash A \neq \emptyset$ (which means that either $\overline{A} \cap (\overline{X} \backslash X) \neq \emptyset$ or $\overline{A} \cap X \neq A$). If $\overline{A}$ is decomposable, then there is some proper subcontinuum H of $\overline{A}$ that has nonempty interior relative to $\overline{A}$. Then H must intersect $\overline{A}/A$. Let Cps denote a composant of A that has the property that if o is an open set in X such that $\overline{o}$ is compact, then any component of $\overline{o} \cap Cps$ is also a component of $\overline{o} \cap \overline{A}$. Note that if x and y are points in Cps, then there is a continuum C_{xy} which is contained in Cps, in nowhere dense in A, and contains both x and y.

Consider the subspace $A' = \overline{A} \cap X$ of X. Choose a point x_0 from $Cps \cap (\overline{A} \backslash H)$, and a nonempty open subset o such that $\overline{o} \subset Int_{\overline{A}}(H) \cap X$. Then x_0 is not in $\overline{o}$, and x_0 is in some component C_0 of $A \backslash o$, C_0 is a nowhere dense continuum in Cps and in

A. Now choose an open set u from $A\backslash H$ such that $\overline{u}$ does not intersect C_0 and $\overline{u} \subset X$. The continuum C_0 is properly contained in some component C_1 of $A\backslash u$, and C_1 is a nowhere dense continuum in Cps and in A.

There is $\epsilon > 0$ such that $D_\epsilon(C_1)$ does not contain o. For each z in $C_1\backslash(\{x_0\} \cup \overline{u})$, there is $\epsilon/2 > \epsilon_z > 0$ such that $D_{2\epsilon_z}(z)$ does not intersect $\{x_0\} \cup \overline{u}$ and $D_{2\epsilon_z}(z) \subset X$. Let $C_2 = Cl_X(\cup\{D_{\epsilon_z}(Z) : z \in C_1\backslash(\{x_0\} \cup \overline{u})\})$. Without loss of generality, we can assume that $C_2 \subset X$. Suppose that $\mathcal{D}$ denotes the upper semicontinuous decomposition of C_2 into its components. That is, we are considering the space $\mathcal{D}$ whose points are the components of C_2 endowed with the quotient topology. Then C_1 is a component of C_2 (and, in particular, C_1 is not properly contained in any component of C_2), since each component of $C_2 \cap A$ is a component of $C_2 \cap \overline{A}$, and $\mathcal{D}$ itself is a totally disconnected, compact metric space. Let $P : C_2 \to \mathcal{D}$ denote the projection map associated with the decomposition. The map P is continuous and onto. Note that C_2 does not contain $\overline{o}$, but $C_2 \cap o \neq \emptyset$ since $C_1 \cap \overline{o} \neq \emptyset$.

Since A is indecomposable, $\overline{A}$ is nowhere dense in $\overline{X}$, and each point x of $C_2 \cap A$ is in $Cl_{\overline{X}}\{y \in C_2 \cap A : y$ is not in C_x, the component of $C_2 \cap A$ that contains $x\}$, $\mathcal{D}$ is a Cantor set. Since $C_2 \cap Cps$ is dense in $C_2, P(C_2 \cap Cps)$ is dense in $\mathcal{D}$. The set $\mathcal{C} = \{D_x \in \mathcal{D} : H \cap \partial C_2 \cap D_x = \emptyset\}$ is open in $\mathcal{D}$, and $C_1 \in \mathcal{C}$. Thus, there is a subset $\mathcal{O}$ of $\mathcal{C}$ that contains C_1, and is both open and closed in $\mathcal{D}$. Then $P^{-1}(\mathcal{O})$ is both closed and open relative to C_2.

But we have a contradiction: then $P^{-1}(\mathcal{O}) \cap Int_{\overline{A}}(H) \neq \emptyset$, and $P^{-1}(\mathcal{O})$ does not contain $Int_{\overline{A}}(H)$. It follows that H is not connected, since $P^{-1}(\mathcal{O}) \cap H \cap \partial C_2 = \emptyset$, and $P^{-1}(\mathcal{O}) \cap H$ is both closed and open in H. Thus, $\overline{A}$ is an indecomposable continuum. $\square$

Remark *In the theorem above, the assumption that X be locally connected is not needed. Also, it is clear that the indecomposable continua in the compactifications of* $\mathbf{R^2}$ *considered in [KY], [KSYG] and [SKGY] are also closed indecomposable sets when restricted to* $\mathbf{R^2}$ *(even without the compactifications considered in those papers), and this result could be used to simplify the results in those papers concerning indecomposable continua in the compactified spaces.*

Lemma 2. *Suppose that $\tilde{F} : X \to X$ is a homoemorphism, B is a generalized quadrilateral in X with ends end_0 and end_1, $\tilde{F}$ is a horseshoe map on B, and M is the crossing number of $\tilde{F}$ on Q. There is $\epsilon > 0$ such that if $F : X \to X$ is a homeomorphism such that for each $q \in X, |\tilde{F}(q) - F(q)| < \epsilon$, then*

1. *F is a horseshoe map on B,*
2. *$(F(B) \cap B)\backslash\overline{D_\epsilon(end_0 \cup end_1)}$ contains only finitely many components,*
3. *$(F(B) \cap B)\backslash\overline{D_\epsilon(end_0 \cup end_1)}$ is a subset of the union of M multually diswoint, closed sets (which are not necessarily components) $\mathcal{C} = \{C_1, C_2, \ldots, C_M\}$, and $\cup\mathcal{C}$ is contained in $F(B) \cap B$, and*
4. *there is a tiling chain $\mathcal{T} = \{T_0, T_1, \ldots, T_{2M}\}$ such that $\cup\mathcal{T} = F(B)$; $F(end_0) \subset T_0$, $F(end_1) \subset T_{2M}$, for $1 \le i \le M$, $T_{2i-1} \in \{C_1, C_2, \ldots, C_M\}$ and for $0 \le i \le M$, $T_{2i} \subset D_\epsilon(F(B)\backslash B)$.*

Proof. For each $\epsilon' > 0$, $\left(\tilde{F}(B) \cap B\right)\backslash D_{\epsilon'}(end_0 \cup end_1)$ contains only finitely many components (Otherwise $\tilde{F}(B)$ is not locally connected.) Choose $\epsilon_1 > 0$ such that

e1. $\tilde{F}(end_0 \cup end_1) \cap \overline{D_{4_{\epsilon_1}}(end_0 \cup end_1)} = \emptyset$,

e2. $\tilde{F}^{-2}(B) \cap \overline{D_{4_{\epsilon_1}}(end_0 \cup end_1)} = \emptyset$, and

e3. $D_{4_{\epsilon_1}}(\tilde{F}(side)) \cap D_{4_{\epsilon_1}}(side) = \emptyset$, and $D_{4_{\epsilon_1}}(end_0) \cap D_{4_{\epsilon_1}}(end_1) = \emptyset$,

There is $\epsilon_1/2 > \epsilon_2 > 0$ such that (i) if C and C' are distinct components of $\tilde{F}(B) \cap B$ that intersect $\left(\tilde{F}(B) \cup B\right)\backslash D_{\epsilon'}(end_0 \cup end_1)$, or (ii) if C and C' are distinct components of $\overline{D_{\epsilon'}(\tilde{F}(B)\backslash B)}$ that do not intersect the interior of any component C'' of $\tilde{F}(B) \cap B$ that intersects $\left(\tilde{F}(B) \cap B\right)\backslash D_{\epsilon'}(end_0 \cup end_1)$, then $d(C, C') > \epsilon_2$. Suppose $0 < \epsilon < \epsilon_2/2$, and F is a homeomorphism on X such that $\left|\tilde{F}(q) - F(q)\right| < \epsilon$ for each $q \in X$. There is an arc K in B that intersects both end_0 and end_1, both $F(K) \cap end_0$ and $F(K) \cap end_1$ consist of exactly one point, and $B \cap \tilde{F}(K)$ contains exactly M components each of which intersects both end_0 and end_1, and for no arc K' in B that intersects both end_0 and end_1 is it the case that $B \cap \tilde{F}(K')$ has fewer than M components each of which intersects both end_0 and end_1. Then $B \cap F(K)$ contains exactly M components each of which intersects both end_0 and end_1, and for no arc K' in B that intersects both end_0 and end_1 with both $F(K') \cap end_0$ and $F(K') \cap end_1$ consisting of exactly one point is it the case that $B \cap F(K)$ contains fewer than M components each of which intersects both end_0 and end_1. Let $\mathcal{K} = \{K' : K'$ is an arc in B that intersects both end_0 and end_1, and both $F(K') \cap end_0$ and $F(K') \cap end_1$ consist of exactly one point$\}$. For $K' \in \mathcal{K}$, let K'_B denote the collection of all components of $F(K') \cap B$ which intersect both end_0 and end_1. This set is finite, but it may well contain more than M components. List the $\tilde{M}_{K'}$ components in K'_B in the order that they occur in $F(K')$, beginning with the component of K'_B that intersects the closure of that component of $K' \backslash \cup \{K'_B\}$ that contains the degenerate set $F(K') \cap end_0$: $K'_B = \{K'_{B1}, K'_{B2}, \ldots, K'_{B\tilde{M}_{K'}}\}$. The set $F(K') \cap B$ may also have components that are not contained in $D_\epsilon(F(B)\backslash B)$ but do intersect both end_0 and end_1: list these components, if there are any, in the order in which they occur in K', that order being the one established above in the listing of the members of K'_B, giving a (possibly empty) collection $K'_{BO} = \{K'_{BO1}, K'_{BO2}, \ldots, K'_{BO\tilde{M}_{OK'}}\}$. Then let $K'_T = \{K'_{T1}, K'_{T2}, \ldots, K'_{T\underline{M}_{K'}}\}$ denote a listing, in the order in which the intervals occur in K', of the members of $K'_B \cup K'_{BO}$, with $\underline{M}_{K'} = \tilde{M}_{K'} + \tilde{M}_{OK'}$. For each K'_{Ti}, let $Com(K'_{Ti})$ denote that component of $F(B) \cap B$ that contains K'_{Ti}, and let $Com = \{Com(K'_{Ti}) : K' \in \mathcal{K}, 1 \leq i \leq \underline{M}_{K'}\}$, a finite collection. Let $\mathcal{N}$ denote the collection of components of $[D_\epsilon(F(B)\backslash B)\backslash(\cup Com)] \cap F(B)$. Then $F(B) = (\cup Com) \cup (\cup \mathcal{N})$. Let $\mathcal{N}' = Com U \mathcal{N}$.

Form the collection $\mathcal{T}' = \{T'_0, T'_1, \ldots T'_{2\underline{M}_{K'}}\}$ as follows: Let $\mathcal{T}'_0$ denote the collection of all members of $\mathcal{N}$ that contain a point of $F(end_0)$, and let $\mathcal{T}'_{2\underline{M}_{K'}}$ denote the collection of all members of $\mathcal{N}$ that contain a point of $F(end_1)$. Then $T'_0 \cap \mathcal{T}'_{2\underline{M}_{K'}} = \emptyset$, because otherwise there is an arc K in B that intersects both end_0 and end_1, and $F(K) \cap B \subset D_\epsilon(F(B)\backslash B)$. Let $T'_0 = \overline{U\mathcal{T}}'_0$ and $T'_{2\underline{M}_{K'}} = U\mathcal{T}_{2\underline{M}_{K'}}$. Inductively, let $\mathcal{T}'_1$ denote the collection of all members of Com that contain a point of T'_0 and let $T'_1 = \cup \mathcal{T}'_1$. Let $\mathcal{T}'_2$ denote the collection of all members of $\mathcal{N}$ that contain a point of T'_1, but are not contained in T'_0, and let $T'_2 = \overline{\cup \mathcal{T}}'_2$. Let $\mathcal{T}'_3$ denote the collection of all members of Com that contain a point of T'_2, but are not contained in T'_1. Continue this finite "out-in-out-in-out" process, until finally, $T'_{2\underline{M}_{K'}-2}$ denotes the collection of all members of $\mathcal{N}$ that contain a point of $T'_{2\underline{M}_{K'}-3}$ but are not contained in $\cup_{\leq j \leq \underline{M}_{K'}-2} T'_{2j-i}$; and let $\mathcal{T}'_{2\underline{M}_{K'}-1}$ denote the collection of all members of Com that contain a point of $T'_{2\underline{M}_{K'}-2}$, but are not contained in $\cup_{1\leq j\leq \underline{M}_{K'}-1} T'_{2j-1}$, with $T'_{2\underline{M}_{K'}-1} = \cup \mathcal{T}'_{2\underline{M}_{K'-1}}$. Note that $\mathcal{T}' = \{T'_0, T'_1, \ldots, T'_{2\underline{M}_{K'}}\}$ is a tiling chain such that $\cup\mathcal{T}' = F(B)$, for otherwise there is some arc $\tilde{K} \in \mathcal{K}$ such that $B \cap F(K)$ contains fewer than $\underline{M}_{K'}$ components each of which intersects both end_0 and end_1. However, $\mathcal{T}'$ may have more than $2M$ members. To remedy this, note that exactly M members of $\{T'_{2j-1} : 1 \leq j \leq \underline{M}_{K'}\}$ contain a component that intersects both end_0 and end_1. Suppose that $\{j_1, j_2, \ldots, j_M\}$ is that subsequence of $\{1, 3, \ldots, 2\underline{M}_{K'} - 1\}$ such that $\{T'_{j_i} : 1 \leq j_i \leq M\}$ is the collection of all links in $\mathcal{T}'$ that contain a component that intersects both end_0 and end_1. Finally, define $T_0 = \cup\{T'_{2j} : 2j < j_1\}$, $T_1 = \cup\{T'_{2j-1} : 2j-1 \leq j_1\}$. Inductively, for $1 < k < M$, define $T_{2k} = \cup\{T'_{2j} : j_{k-1} < 2j < j_k\}$, $T_{2k_{-1}} = \cup\{T'_{2j-1} : j_{k-1} < 2j-1 \leq j_k\}$; and define the two links, T_{2M-1} and T_{2M}, by $T_{2M-1} = \cup\{T'_{2j-1} : j_{M-1} < 2j-1 \leq M_{K'}\}$, $T_{2M} = \cup\{T'_{2j} : j_{M-1} < 2j \leq \underline{M}_{K'}\}$. Hence, $\mathcal{T} = \{T_0, T_1, \ldots, T_{2M}\}$ is a tiling chain with the properties desired. □

The following, especially part one, is in the spirit of a folk theorem, though to our knowledge, no one has published a precise definition of a topological horseshoe.

Theorem 3. Topological Horseshoe Theorem-Cantor Set Part. *Suppose that $F : X \to X$ is a homeomorphism, B is a generalized quadrilateral in X with ends end_0 and end_1, and F is a horseshoe map on B with crossing number M.*

1. *Then the permanent set $C_B = \{x : F^n(x) \in B$ for all integers $n\}$ has an upper semicontinuous decomposition $\mathcal{Q}$ which is a Cantor set in the quotient topology, and $C_B = \cup\mathcal{Q}$ is a quotient Cantor set contained in the interior of B. Furthermore, F preserves the members of $\mathcal{Q}$, and thus, if we define $\mathcal{F} : \mathcal{Q} \to \mathcal{Q}$ by $\mathcal{F}(Q) = F(Q)$, then $\mathcal{F}$ is a homeomorphism on $\mathcal{Q}$, $\mathbf{F}$ is conjugate to the shift $\sigma_M : \sum_M \to \sum_M$, and $F|C_B$ factors over $\sigma_M : \sum_M \to \sum_M$.*

2. *There is a dense G_δ-subset $\mathcal{Q}'$ of $\mathcal{Q}$ such that each member of $\mathcal{Q}'$ is a point of continuity of the decomposition $\mathcal{Q}$.*

3. *The set* $C'_B = \overline{\cup \mathcal{Q}'}$ *is an invariant quotient Cantor set with respect to the upper semicontinuous decomposition* $\mathcal{Q}'' = \{Q \cap \overline{\cup \mathcal{Q}'} : Q \in \mathcal{Q}\}$, *and for each* $Q'' \in \mathcal{Q}''$, *if* $x \in Q''$, *then* $x \in \overline{C'_B \backslash Q''}$. *(The set* C'_B *is just the set* C_B *with all "dynamically inert" sets, such as interiors of members of* $\mathcal{Q}$ *or isolated points of* $\mathcal{Q}$, *removed.) Furthermore,* F *preserves the members of* $\mathcal{Q}''$, *and thus, if we define* $\mathcal{F}'' : \mathcal{Q}'' \to \mathcal{Q}''$ *by* $\mathcal{F}''(Q'') = F(Q'')$ *for* $Q'' \in \mathcal{Q}''$, *then* $\mathcal{F}''$ *is a homeomorphism on* $\mathcal{Q}'', Q''$ *which is conjugate to the shift* $\sigma_M : \sum_M \to \sum_M$, *and* $F|C'_B$ *factors over* $\sigma_M : \sum_M \to \sum_M$.

4. *If* $x \in Q' \in \mathcal{Q}'$, *then* $\mathcal{Q}_x = \{Q \cap \overline{O(x)} : Q \in \mathcal{Q}\}$ *is an upper semicontinuous decomposition of* $\overline{O(x)}$, *and* $\overline{O(x)}$ *is a quotient Cantor set with respect to the decomposition* $\mathcal{Q}_x$. *Again,* F *preserves the members of* $\mathcal{Q}_x$, *and thus, if we define* $\mathcal{F}_x : \mathcal{Q}_x \to \mathcal{Q}_x$ *by* $\mathcal{F}(Q_x) = F(Q_x)$ *for* $Q_x \in \mathcal{Q}_x$, *then* $\mathcal{F}_x$ *is a homeomorphism on* $\mathcal{Q}_x$, *and* $\mathcal{F}_x$ *is conjugate to the shift* $\sigma_M : \sum_M \to \sum_M$.

5. *Let* $\mathcal{C}$ *denote the collection of all continua contained in* $C_B = \{x : F^n(x) \in B$ *for all integers* $n\}$. *Then* $\mathcal{C}$ *is an upper semicontinuous decomposition which is a Cantor set in the quotient topology, and* $C_B = \cup \mathcal{C}$ *is a quotient Cantor set relative to the decomposition* $\mathcal{C}$. *Furthermore,* F *preserves the members of* $\mathcal{C}$, *and thus, if we define* $\mathcal{F}_{con} : \mathcal{C} \to \mathcal{C}$ *by* $\mathcal{F}(C) = F(C)$ *for* $C \in \mathcal{C}$, *then* $\mathcal{F}_{con}$ *is a homeomorphism on* $\mathcal{C}$, *and* $\mathcal{F}$ *factors over the shift* $\sigma_M : \sum_M \to \sum_M$.

6. *If* $x \in Q' \in \mathcal{Q}'$, *and* $\mathcal{C}_x$ *denotes the collection of all continua contained in* $\overline{O(x)}$, *then* $\mathcal{C}_x$ *is an upper semicontinuous decomposition of* $\overline{O(x)}$, *and* $\overline{O(x)}$ *is a quotient Cantor set with respect to the decomposition* $\mathcal{C}_x$. *Again,* F *preserves the members of* $\mathcal{C}_x$, *and thus, if we define* $\mathcal{F}_{con,x} : \mathcal{C}_x \to \mathcal{C}_x$ *by* $\mathcal{F}_{con,x}(C_x) = F(C_x)$ *for* $C_x \in \mathcal{C}_x$, *then* $\mathcal{F}_{con,x}$ *is a homeomorphism on* $\mathcal{C}_x, \mathcal{F}_{con,x}$ *factors over the shift* $\sigma_M : \sum_M \to \sum_M$, *and the orbit of* x *is dense in* $\overline{O(x)} = \cup \mathcal{C}_x$.

Proof. Suppose B has ends end_0 and end_1, and side $side$. Consider $C_B = \{q \in B | F^n(q) \in B$ for all $n \in \mathbf{Z}\}$. Applying the previous lemma, there are a positive number ϵ, and a tiling chain $\mathcal{S}_1 = \{S_{1,0}, S_{1,1}, \ldots, S_{1,2M}\}$ such that

1. $\cup \mathcal{S}_1 = F(B), S_{1,0} \supset F(end_0)$, and $S_{1,2M} \supset F(end_1)$;

2. for $0 \leq j \leq M, S_{1,2j} \subset \overline{D_\epsilon(F(B)\backslash B)}$; $\bigcup_{1\leq j\leq M} S_{1,2j-1}$
3. $(B \cap F(B))\backslash D_\epsilon(end_0 \cup end_1) \subseteq B \cap F(B)$; and
4. $\overline{D_\epsilon(F(B)\backslash B)} \cap \overline{D_\epsilon(F^{-1}(B) \cap B)} = \emptyset$.

For each $0 \leq i \leq 2M$, let $S_{0,i} = S_{-1,i} = F^{-1}(S_{1,i})$, so that $\mathcal{S}_0 = \{S_{0,0}, S_{0,1}, \ldots, S_{0,2M}\} = \mathcal{S}_{-1} = \{S_{-1,0}, S_{-1,1}, \ldots S_{-1,2M}\}$ is a tiling chain, $\cup\mathcal{S}_0 = B$, and for $1 \leq i \leq 2M-1$, $S_{0,i}$ separates B.

Let $N_1 = 2M$. Now consider $F(\mathcal{S}_1) = \{F(S_{1,0}), F(S_{1,1}), \ldots, F(S_{1,N_1})\}$ and construct the tiling chain $\mathcal{S}_2 = \{S_{2,0}, S_{2,2}, \ldots, S_{2,N_2}\}$ that covers $F^2(B)$ as follows: Let $S_{2,0} = F(S_{1,0})$. The set $F(S_{1,1})$ intersects both $F(end_0)$ and $F(end_1)$, and this forces $F(S_{1,1}) \cap S_{1,j}$, for $0 \leq j \leq N_1$, to be a nonempty set. List these sets $S_{2,1}, \ldots, S_{2,N_1+1}$ so that $S_{2,0}, S_{2,1}, \ldots, S_{2,N_1+1}$ is a tiling chain that covers $F(S_{1,0}) \cup F(S_{1,1})$.

Let $S_{2,N_1+2} = F(S_{1,2})$. The set $F(S_{1,3})$ intersects both $F(end_0)$ and $F(end_1)$, and this forces $F(S_{1,3}) \cap S_{1,j}$, for $0 \leq j \leq N_1$, to be a nonempty set. List these sets $S_{2,N_1+3}, \ldots, S_{2,2N_1+3}$ so that $S_{2,0}, S_{2,1}, \ldots S_{2,N_1}, S_{2,N_1+1}, S_{2,N_1+2}, \ldots, S_{2,2N_1+3}$ is a tiling chain that covers $\cup_{j=0}^{3} F(S_{1,j})$. Continue this process, finally letting $S_{2,N_2} = F(S_{1,N_1})$.

Having the tiling chain $\mathcal{S}_2 = \{S_{2,0}, S_{2,2}, \ldots, S_{2,N_2}\}$, construct the tiling chain $\mathcal{S}_3 = \{S_{3,0}, S_{3,3}, \ldots, S_{3,N_3}\}$ that covers $F^3(B)$ as follows: Let $S_{3,0} = F(S_{2,0})$. The set $F(S_{2,1})$ intersects both $F^2(end_0)$ and $F^2(end_2)$ and has at least one component that intersects both $F^2(end_0)$ and $F^2(end_2)$, and this forces $F(S_{2,1}) \cap S_{2,j}$, for $0 \leq j \leq N_2$, to be a nonempty set. List these sets $S_{3,1}, \ldots, S_{3,N_2+1}$ so that $S_{3,0}, S_{3,1}, \ldots, S_{3,N_2+1}$ is a tiling chain that covers $F(S_{2,0}) \cup F(S_{2,1})$. Let $S_{3N_2+2} = F(S_{2,2})$. The set $F(S_{1,3})$ intersects both $F^2(end_0)$ and $F^2(end_2)$ and has at least one component that intersects both $F^2(end_0)$ and $F^2(end_2)$, and this forces $F(S_{2,3}) \cap S_{2,j}$ for $0 \leq j \leq N_2$, to be a nonempty set. List these sets $S_{3,N_2+3}, \ldots, S_{3,3N_2+3}$ so that $S_{3,0}, S_{3,1}, \ldots, S_{3,N_2+1}, S_{3,N_2+3}, \ldots, S_{3,3N_2+3}$ is a tiling chain that covers $\cup_{j=0}^{3} F(S_{2,j})$. Continue this process, finally letting $S_{3,N_3} = F(S_{2,N_2})$. Continue the inductive process begun here: For each positive intger k, having the tiling chain $\mathcal{S}_k = \{S_{k,0} S_{k,1}, \ldots, S_{k,N_k}\}$ that covers $F^k(B)$, construct the tiling chain $\mathcal{S}_{k+1} = \{S_{k+1,0}, S_{k+1,1}, \ldots, S_{k+1,N_{k+1}}\}$ that covers $F^{k+1}(B)$ in the manner described.

The for each $k, F^{-k}(\mathcal{S}_k) = \{F^{-k}(S_{k,0}), F^{-k}(S_{k,1}), \ldots, F^{-k}(S_{k,N_k})\}$ is a tiling chain cover of $B, F^{-k-1}(\mathcal{S}_{k+1})$ refines $F^{-k}(\mathcal{S}_k)$, and for $1 \leq l \leq N_k - 1$, $F^{-k}(S_{k,l})$ separates B. For $0 \leq j \leq N_k$, let $S_{-kj} = F^{-k}(S_{k,j})$, and $\mathcal{S}_{-k} = \{S_{-k,0}, S_{-k,1} \ldots, S_{-k,N_k}\}$. For each positive integer k, there is a finite subsequence $(i_{k,1}, i_{k,2}, \ldots, i_{k,M}) = (i_{-k,1}, i_{-k,2}, \ldots, i_{-k,M})$ of $(0, 1, \ldots, N_k)$ such that $F^j(S_{-k,i_{k,\alpha}}) \subset B$ and $F^{-j}(S_{k,i_{k,\alpha}}) \subset B$ for each $0 \leq j \leq k$, $i_{k,\alpha} \in \{i_{k,1}, i_{k,2}, \ldots, i_{k,M}\}$, $F^k(S_{-k,i_{k\alpha}})$ contains a component that intersects both end_0 and end_1, and $S_{k,i_{k,\alpha}} \subset F^k(B)$ contains a component that intersects both $F^k(end_0)$ and $F^k(end_1)$.

If x is in the permanent set C_B, then $F^n(x) \in B$ for each $n \in \mathbf{Z}$. Thus, there is a unique bisequence $(\ldots, p_{x,1}, p_{x,0}, p_{x,1}, \ldots)$ contained in $\{1, \ldots M\}^{\mathbf{Z}}$ such that for each $n, x \in S_{n,i_{n,p_{x,n}}}$. Further, by construction, if $(\ldots, p_{-1}, p_0, p_1, \ldots)$ is a bisequence contained in $\{1, \ldots M\}^{\mathbf{Z}}$, the $Q_{(\ldots,p-1,p0,p1,\ldots)} = \cap_{n \in \mathbf{Z}} S_{n,i_{n,pn}}$ is a nonempty closed subset of B°. If $(\ldots, p_{-1}p_0, p_1, \ldots)$ and $(\ldots, p'_{-1}, p'_0 p'_1, \ldots)$ are distinct bisequences, then $Q_{(\ldots,p_{-1}p_0,p_1,\ldots)} \cap\ Q_{(\ldots,p'_{-1},p'_0p'_1,\ldots)} = \emptyset$, $\mathcal{Q} = \{Q_{(\ldots,p_{-1}p_0,p_1,\ldots)} : (\ldots, p_{-1}, p_0, p_1, \ldots)$ is a bisequence contained in $\{1, \ldots M\}^{\mathbf{Z}}\}$ is a collection of disjoint closed sets in B, and $\mathcal{Q}$ is an upper semicontinuous decomposition of $C_B = \cup \mathcal{Q} = \{x \in B : F^n(x) \in B$ for each integer $n\}$. From standard arguments, if follows that $\mathcal{Q}$ is a Cantor et in the quotient topology, and C_B is a quotient Cantor set in the interior of B.

For each bisequence $(\ldots, j_{-1\star} j_0, j_1, \ldots) \in \{1, \ldots M\}^{\mathbf{Z}}$, it is also the case that $\cap_{k=-\infty}^{\infty} F^k(S_{0,2_{jk-1}}) \neq \emptyset$; and, in fact, since for each $n > 0$, $\cap_{k=0}^{n} F^k(S_{0,2_{jk-1}}) = S_{n,i_{n,qn}}$ for some $i_{n,qn} \in \{i_{n,1} i_{n,2} \ldots i_{n,M}\}$, and $\cap_{k=-n}^{-1} f^k(S_{0,2_{jk-1}}) = S_{-n,i_{-n,q_{-n}}}$ for some $i_{-n,q-n} \in \{i_{-n,1}, i_{-n,2,\ldots,} i_{-n,M}\}$, $\cap_{k=-\infty}^{\infty} F^k(S_{0,2_{jk-1}}) = \cap_{k=-\infty}^{\infty} S_{k,i_{k,q_k}} \in \mathcal{Q}$. Since $F(\cap_{k=-\infty}^{\infty} F^k(S_{0,2_{jk-1}})) = \cap_{k=-\infty}^{\infty} F^{k+1}(S_{0,2_{jk-1}}) \in \mathcal{Q}$, it follows that if $Q \in \mathcal{Q}$, then the set $F(Q)$ is also in $\mathcal{Q}$. Thus, F preserves the decomposition $\mathcal{Q}$, and if we define $\mathcal{F} : \mathcal{Q} \to \mathcal{Q}$ by $\mathcal{F}(Q) = F(Q)$, then $\mathcal{F}$ is a homeomorphism on $\mathcal{Q}$. because for each $Q \in \mathcal{Q}$, there is a unique bisequence $(\ldots, j_{-1\star} j_0, j_1, \ldots) \in \{1, \ldots M\}^{\mathbf{Z}}$ such that $Q = \cap_{k=-\infty}^{\infty} F^k(S_{0,2_{jk-1}})$, and $F(Q) = F(\cap_{k=-\infty}^{\infty} F^k(S_{0,2_{jk-1}})) = \cap_{k=-\infty}^{\infty} F^{k+1}(S_{0,2_{jk-1}}) \in \mathcal{Q}, \mathcal{F}$ is conjugate to the shift $\sigma_M : \sum_M \to \sum_M$.

Since is $\mathcal{F}$ conjugate to σ_M, there must be a dense G_δ-subset $\mathcal{Q}'$ of $\mathcal{Q}$ such that (i) the orbit $O(Q) = \{\mathcal{F}^n(Q) : n \in \mathbf{Z}\}$ of each member Q of $\mathcal{Q}'$ is dense in $\mathcal{Q}$, and (ii) each Q in $\mathcal{Q}'$ is a point of continuity of $\mathcal{Q}$, because there are dense G_δ-subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{Q}$ such that the orbit of each point Q in $\mathcal{A}$ is dense in $\mathcal{Q}$, and such that each point Q of $\mathcal{B}$ is a point of continuity of $\mathcal{Q}$, and $\mathcal{Q}' = \mathcal{A} \cap \mathcal{B}$ is also a dense G_δ-subset of $\mathcal{Q}$. Then

$\overline{\cup \mathcal{Q}'} = C'_B$ is an invariant quotient Cantor set with respect to the upper semicontinuous decomposition $\mathcal{Q}'' = \{Q \cap \overline{\cup \mathcal{Q}'} : Q \in \mathcal{Q}\}$. If $x \in Q' \in \mathcal{Q}'$, then $\overline{O(x)} \cap Q / \neq \emptyset$ for any $Q \in \mathcal{Q}$. It follows that $\mathcal{Q}_x = \{Q \cap \overline{O(x)} : Q \in \mathcal{Q}\}$ is an upper semicontinuous decomposition of $\overline{O(x)}$, and $\overline{O(x)}$ is a quotient Cantor set with respect to the decomposition $\mathcal{Q}_x$. The rest follows easily. □

Theorem 4. Noisy Topological Horseshoe Theorem-Cantor Set Part. *Suppose that $F : X \to X$ is a homeomorphism, B is a genarlized quadrilateral in X with ends end_0 and end_1, and F is a horseshoe map on B with crossing number M. There is $\delta > 0$ such that if for each integer $j, F_j : X \to X$ is a homeomorphism with $d(F(x), F_j(x)) < \delta$ for each $x \in X$, then the following hold:*

1. *If for each integer η, $Z_\eta = \{x_\eta \in B :$ the trajectory $\{\ldots x_{\eta-2}, x_{\eta-1} \star x_\eta \star x_{\eta+1}, \ldots\} \subset B\}$, then there is an upper semicontinuous decomposition $\mathcal{Q}_\eta$ of Z_η which is a Cantor set in the quotient topology, and $Z_\eta = \cup \mathcal{Q}_\eta$ is a quotient Cantor set contained in the interior of B. Furthermore, the system $\mathbf{F} = \{F_j\}_{j \in \mathbf{Z}}$ preserves the members of $\cup_{\eta \in \mathbf{Z}} \mathcal{Q}_\eta$, in the sense that for $Q \in \mathcal{Q}_\eta$, $F_\eta(Q) \in \mathcal{Q}_{\eta+1}$, and thus, if we define $\mathcal{F}_n : \mathcal{Q}_\eta \to \mathcal{Q}_{\eta+1}$ by $\mathcal{F}_\eta(Q) = F_\eta(Q)$, then $\mathcal{F}_\eta$ is a homeomorphism from $\mathcal{Q}_\eta$ onto $\mathcal{Q}_{\eta+1}$. There is a collection $\Gamma = \{\gamma_\eta : \mathcal{Q}_\eta \to \sum_M\}_{\eta \in \mathbf{Z}}$ of homeomorphisms such that $\sigma_M \circ \gamma_\eta = \gamma_{\eta+1} \circ \mathcal{F}_\eta$ for each η. Hence, if $\mathcal{Z} = \{Z_\eta\}_{\eta \in \mathbf{Z}}$, then the floating system $\mathbf{F}|\mathcal{Z}$ factors over the shift $\sigma_M : \sum_M \to \sum_M$.*

2. *For each η, there is a dense G_δ-subset $\mathcal{Q}'_\eta$ of $\mathcal{Q}_\eta$ such that (i) each member Q of $\mathcal{Q}'_\eta$ is a point of continuity of the decomposition $\mathcal{Q}'_\eta$, and (ii) for each member Q of $\mathcal{Q}'_\eta, \gamma_\eta(Q)$ has a dense orbit in $\sum_M$ (under the action of σ). If $Z'_\eta = \overline{\cup \mathcal{Q}'_\eta}$, then the set $Z'_\eta = \overline{\cup \mathcal{Q}'_\eta}$ is a quotient Cantor set with respect to the upper semicontinuous decomposition $\mathcal{Q}''_\eta = \{Q \cap \overline{\cup \mathcal{Q}'} : Q \in \mathcal{Q}_\eta\}$ and if $x \in Q \in \mathcal{Q}''_\eta$, then $x \in \overline{\mathcal{Q}''_\eta \backslash \{Q\}}$. (Here, as before, the set Z'_η is just the set Z_η with all "dynamically inert" sets (relative to the system $\mathbf{F}$), such as interiors of members of $\mathcal{Q}_\eta$ or isolated points of $\mathcal{Q}_\eta$, removed.) Furthermore, the system $\mathbf{F}$ preserves the members of $\cup_{\eta \in \mathbf{Z}} \mathcal{Q}''_\eta$, in the sense that for $Q \in \mathcal{Q}''_\eta$, $F_\eta(Q) \in \mathcal{Q}''_{\eta+1}$ and thus, if we define*

$\mathcal{F}''_\eta : \mathcal{Q}''_\eta \to \mathcal{Q}''_{\eta+1}$ by $\mathcal{F}''_\eta(Q) = F_\eta(Q)$, then $\mathcal{F}''_\eta$ is a homeomorphism from $\mathcal{Q}''_\eta$ onto $\mathcal{Q}''_{\eta+1}$. There is a collection $\Gamma'' = \{\gamma''_\eta : \mathcal{Q}''_\eta \to \sum_M\}_{\eta\in\mathbf{Z}}$ of homeomorphisms such that $\sigma_M \circ \gamma''_\eta = \gamma''_{\eta+1} \circ \mathcal{F}''_\eta$ for each η. Hence, if $\mathcal{Z}'' = \{Z''_\eta\}_{\eta\in\mathbf{Z}}$, then the system $\mathbf{F}|\mathcal{Z}''$ factors over the shift $\sigma_M \to \sum_M$.

3. *If, for each $\eta, \mathcal{C}_\eta$ denotes the collection of all continua contained in Z_η, then $\mathcal{C}_\eta$ is an upper semicontinuous decomposition of Z_η which is a quotient Cantor set relative to the decomposition $\mathcal{C}_\eta$. Furthermore, the system $\mathbf{F}$ preserves the members of $\cup_{\eta\in\mathbf{Z}}\mathcal{C}_\eta$, in the sense that for $C \in \mathcal{C}_\eta$, $F_\eta(C) \in \mathcal{C}_{\eta+1}$, and thus, if we define $\mathcal{F}_{\eta,con} : \mathcal{C}_\eta \to \mathcal{C}_\eta$ by $\mathcal{F}_{\eta,con}(C) = F(C)$ for $C \in \mathcal{C}_\eta$, then $\mathcal{F}_{\eta,con}$ is a homeomorhism on $\mathcal{C}_\eta$. There is a collection $\Gamma_{con} = \{\gamma_{con,\eta} : \mathcal{C}_\eta \to \sum_M\}_{\eta\in\mathbf{Z}}$ of continuous surjections such that $\sigma_M \circ \gamma_{con,\eta} = \gamma_{\eta+1} \circ \mathcal{F}_{\eta,con}$ for each η.*

Proof. Suppose B has side *side*. There is $\epsilon > 0$ such that if $F' : X \to X$ is a homeomorphism with $d(F(x), F'(x)) < \epsilon$ for each $x \in X$, then

- e1) $F'(end_0 \cap end_1) \cap \overline{D_{4\epsilon}(end_0 \cup end_1)} = \emptyset$,
- e2) $(end_0 \cup end_1) \cap \overline{D_{4\epsilon}F'^{-1}(end_0 \cup end_1)} = \emptyset$,
- e3) $D\epsilon(end_0) \cap D_{4\epsilon}(end_1) = \emptyset$
- e4) $D_{4\epsilon}(F'(side)) \cap D_{4\epsilon}(side) = \emptyset$ and
- e5) each F' is a horseshoe map on the generalized quadrilateral B (with ends end_0 and end_1) with the same crossing number M as F.

Suppose that $\mathcal{T} = \{T_0, T_1, \ldots T_{2M}\}$ is the tiling chain obtained by applying Lemma 2 to F on the generalized quadrilateral B. Thus, $\cup\mathcal{T} = F(B)$; $F(end_0) \subset T_0, F(end_1) \subset T_{2M}, \cup_{1\le i\le M} T_{2i-1} \subset B$, and $\cup_{0\le i\le M} T_{2i-1} \subset D_\epsilon(F(B)\backslash B)$. There is some $\delta > 0$ such that $\delta < \epsilon/2$, and

- e6) if C and C' are different components of $B \cap F'(B)$ that intersect $B\backslash D_\epsilon(end_0 \cup end_1)$, then $d(C, C') > 4\delta$, and
- e7) if C and C' are different components of $\overline{D_\epsilon(F'(B)\backslash)}$, then $d(C, C') > 4\delta$, and

e8) $d(T_{2i-1},T_{2j-1}) > 4\delta$ if $i \neq j, i,j \in \{1,2,\ldots M\}$.

For each integer j, suppose that $F_j : X \to X$ is a homeomorphism with $d(F(x), F_j(x)) < \delta$ for each $x \in X$. Note that for each $j, \overline{D_\epsilon(end_0 \cup end_1)} \cap F_j^{-1}(B) = \emptyset$. Let $\mathbf{F}$ denote the floating system $\{F_j\}_{j=-\infty}^{\infty}$, and fix the integer η. For this fixed η, consider the floating permanent set $Z_\eta = \{x_\eta \in B :$ the trajectory $\{\ldots x_{\eta-2}, x_{\eta-1} \star x_\eta \star x_{\eta+1}, \ldots\} \subset B\}$. Let $\varepsilon_j = \{C : C$ is a component of $F_j(B) \cap B$ and C intersects $B \backslash D_\epsilon(end_0 \cup end_1)\}$ and let $\varepsilon_j' = \{C : C$ is a component $\overline{F_j(B) \backslash (\cup \varepsilon_j)}\}$. Hence, both $\cup\varepsilon$ and $\cup\varepsilon_j'$ are closed, and $\cup\varepsilon_j \subset B$, while $\cup\varepsilon_j' \subset \overline{D_\epsilon(F(B) \backslash B)}$.

Let $N_1 = 2M$. Now consider a positive integer n, and the composition $\tilde{F}_{\eta+n,\eta-n}$. First, consider $S_n = \{S_{n,0}, S_{n,1}, \ldots, S_{n,N_1}\}$ where S_n is the tilin chain constructed in Lemma 2 for $F_{\eta+n}$, and let $\mathcal{T}_{n,n} = \{F_{\eta+n}^{-1}(S_{n,0}), \ldots, F_{\eta+n}^{-1}(S_{n,N_1})\} = \{T_{n,n,0}, T_{n,n,1}, \ldots, T_{n,n,N_1}\}$, which is a tiling chain cover of B. Let $\underline{\alpha_{n,n}} = \{\alpha_{n,n,i}\}_{i=1}^{M}$, where $\alpha_{n,n,i} = 2i-1$. Then consider the tiling chain $\mathcal{S}_{n-1} = \{S_{n-1,0}, S_{n-1,1}, \ldots, S_{n-1,N_1}\}$, the Lemma 2 tiling chain cover of $F_{\eta+n-1}(B)$ that covers $F_{\eta+n-1}(B)$, and modify it to produce the tiling chain cover $\mathcal{D}_{n,n-1}$ as follows: Let $\mathcal{D}_{n,n-1} = \{S_{n-1,2i} : 0 \leq i \leq M\} \cup \{S_{n-1,2i-1} \cap T_{n,n,j} : 1 \leq i \leq M$ and $0 \leq j \leq N_1\}$. The collection $\mathcal{D}_{n,n-1}$ is a tiling chain cover of $F_{\eta+n-1}(B)$ that refines S_{n-1}, and we can list the links of $\mathcal{D}_{n,n-1}$ so that they reflect this chain structure as $\mathcal{D}_{n,n-1} = \{D_{n,n-1,0}, D_{n,n-1,1}, \ldots, D_{n,n-1,N_2}\}$, with $F_{\eta+n-1}(end_0) \subset D_{n,n-1,0}$ and $F_{\eta+n-1}(end_1) \subset D_{n,n-1,N_2}$. Let $\underline{\alpha_{n,n-1}} = \{\alpha_{n,n-1,i}\}_{i=1}^{M^2}$ be that subsequence of $\{i\}_{i=0}^{N_2}$ such that $\underline{\alpha_{n,n-1}} = \{j : D_{n,n-1,j} = S_{n-1,2i-1} \cap T_{n,n,\alpha_{n,n,k}} : 1 \leq i \leq M$ and $1 \leq k \leq M\}$. Let $\mathcal{T}_{n,n-1} = \{F_{\eta+n-1}^{-1}(D_{n,n-1,1}), \ldots, F_{\eta+n-1}^{-1}(D_{n,n-1,N_2})\} = \{T_{n,n-1,0}, T_{n,n-1,1}, \ldots, T_{n,n-1,N_2}\}$, which is a tiling chain cover of B. Note that (1) $T_{n,n-1,0}$ contains end_0, $T_{n,n-1,N_2}$ contains end_1, and each $T_{n,n-1,i}$, for $1 \leq i \leq N_2 - 1$, separates B; and (2) for $1 \leq i \leq M^2$, the set $\tilde{F}_{\eta+n,\eta+n-1}(T_{n,n-1,\alpha_{n,n-1,i}})$ is contained in B and contains a component that intersects both end_0 and end_1. Continue with this constraction: For $-n < m < n$, having constructed the tiling chain cover $\mathcal{T}_{n,m} = \{F_{\eta+m}^{-1}(D_{n,m,1}), \ldots, F_{\eta+m}^{-1}(D_{n,m,N_{n-m+1}})\} = \{T_{n,m,0}, T_{n,m,1}, \ldots, T_{n,m,N_{n-m+1}}\}$, a tiling chain cover of B such that (1) $T_{n,m,0}$ contains end_0, $T_{n,m,N_{n-m+1}}$ contains end_1, and each $T_{n,m,i}$ for $1 \leq i \leq N_{n-m+1} - 1$, separates B; and (2) $\underline{\alpha_{n,m}} = \{\alpha_{n,m,i}\}_{i=1}^{M^{M-n+1}}$ is a subsequence of $\{i\}_{i=0}^{N_{n-m+1}}$ such that for $1 \leq i \leq M^{n-m+1}$, the set $\tilde{F}_{\eta+n,\eta+m}(T_{n,m,\alpha_{n,m,i}})$ is contained in B and contains a component that intersects both end_0 and end_1, let

$\mathcal{D}_{n,m-1} = \{S_{m-1,2i} : 0 \leq i \leq M\} \cup \{S_{m-1,2i-1} \cap\ T_{n,m,j} : 1 \leq i \leq M$ and $0 \leq j \leq N_{n-m+1}\}$. The collection $\mathcal{D}_{n,m-1}$ is a tiling chain cover of $F_{\eta+m-1}(B)$ that refines S_{m-1}, and we can list the links of $\mathcal{D}_{n,m-1}$ so that they reflect this chain structure and $\mathcal{D}_{n,m-1} = \{D_{n,m-1,0}, D_{n,m-1,1}, \ldots, D_{n,m-1,N_{n-m+2}}\}$, with $F_{\eta+m-1}(end_0) \subset D_{n,m-1,0}$ and $F_{\eta+m-1}(end_1) \subset D_{n,m-1,N_{n-m+2}}$. Let $\underline{\alpha_{n,m-1}} = \{\alpha_{n,m-1,i}\}_{i=1}^{M^{n-m+2}}$ be that subsequence of $\{i\}_{i=0}^{N_{n-m+2}}$ such that $\underline{\alpha_{n,m-2}} = \{j : D_{n,m-1,j} = S_{m-1,2i-1} \cap T_{n,m,\alpha_{m,k}}$ for some $1 \leq i \leq M$ and $1 \leq k \leq M^{n-m+1}\}$. Let $\mathcal{T}_{n,m-1} = \{F^{-1}_{\eta+m-1}(D_{n,m-1,1}), \ldots, F^{-1}_{\eta+m-1}(D_{n,m-1,N_{n-m+2}})\} = \{T_{n,m-1,0}, T_{n,m-1,1}, \ldots, T_{n,m-1,N_{n-m+2}}\}$, which is a tiling chain cover of B. As before, (1) $T_{n,m-1,0}$ contains end_0 contains end_0, $T_{n,m-1,N_{n-m+2}}$ contains end_1, and each $T_{n,m-1,i}$, for $1 \leq i \leq N_{m-n+2} - 1$, separates B; and (2) for $1 \leq i \leq M^{n-m+2}$, the set $\tilde{F}_{\eta+n,\eta+m-1}(T_{n,m-1\alpha_{n,m-2,i}})$ is contained in B and contains a component that intersects both end_0 and end_1.

Now back up a bit: Let $\mathcal{L}_{-n} = \tilde{F}_{\eta-1,\eta-n}(\mathcal{T}_{n,-n}) = \{\tilde{F}_{\eta-1,\eta-n}\ (T_{n,-n,0}),\ \tilde{F}_{\eta-1,\eta-n}\ (T_{n,-n,1}) \ldots,\ \tilde{F}_{\eta-1,\eta-n}\ (T_{n,-n,N_{2n+1}})\} = \{L_{-n,0}, L_{-n,1},\ \ldots L_{-n,N_{2n+1}}\}$, so that $\mathcal{L}_{-n}$ is a tiling chain cover of $\tilde{F}_{\eta-1,\eta-n}(B)$. Consider the subsequence $\underline{\alpha_{n,-n}} = \{\alpha_{n,-n,i}\}_{i=1}^{M^2n+1}$ of $\{i\}_{i=0}^{N_{2n+1}}$. For $1 \leq j \leq M^{2n+1}$, $F_{\eta,n}(T_{n,-n,\alpha_{n,-n,j}}) = D_{n,-n,\alpha_{-n,j}} = S_{-n,2k_n-1} \cap T_{n,-n+1,\alpha_{n,-n+1,i_n}}$ for some appropriate k_n, i_n and is contained in B. Then $\tilde{F}_{\eta-n+1,\eta-n}$ $(T_{n,-n,\alpha_{n,-n,j}}) = F_{\eta-n+1}(S_{-n,2k_n-1} \cap\ T_{n,-n+1,\alpha_{n,-n+1,i_n}}) \subset S_{-n+1,2k_{n-1}-1} \cap$ $T_{n,-n+2,\alpha_{n,-n+2,i_{n-1}}}$ for some appropriate k_{n-1}, i_{n-1}, and is contained in B, and so on, until finally, we conclude that $\tilde{F}_{\eta-1,\eta-n}(T_{n,-n,\alpha_{n,-n,j}}) \subset \tilde{F}_{\eta-1,\eta-n+1}(S_{-n,2k_{n-1}} \cap T_{n,-n+1,\alpha_{n,-n+1,i_n}}) \subset \tilde{F}_{\eta-1,\eta-n+2}(S_{-n+1,2k_{n-1}-1} \cap T_{n,-n+2,\alpha_{n,-n+2,i_{n-1}}}) \subset \ldots \subset S_{-1,2k_1-1} \cap$ $T_{n,0,\alpha_{n,-1,i_1}}$ for some appropiriate sequences $\{k_{n-\xi}\}_{\xi=0}^{n-1}$, $\{i_{\eta-\xi}\}_{\xi=0}^{n-1}$, and is contained in B. Note that the subsequence $\overline{\alpha_{n,0}} = \{\alpha_{n,-1,i}\}_{i=1}^{M^{n+1}}$ of $\{i\}_{i=0}^{N_{n+1}}$ has M^{n+1} members, and each $T_{n,0,\alpha_{n,-1,k}}$ separates B and contains M^n members of $\{T_{n,-n,\alpha_{n,-n,j}} : \alpha_{n,-n,j} \in \overline{\alpha_{n,-n}}\}$. Further, $F_{\eta-n}(T_{n,-n,\alpha_{n,-n,j}}) \subset S_{-n,2k_{n-1}}; \tilde{F}_{\eta-n+1,\eta-n}(T_{n,-n,\alpha_{n,-n,j}}) \subset F_{\eta-n+1}$ $(S_{-n,2k_n-1}) \cap S_{-n+1,2k_{n-1}-1}$; and so on, until we conclude that $\tilde{F}_{\eta-1,\eta-n}(T_{n,-n,\alpha_{n,-n,j}}) \subset \tilde{F}_{\eta-1,\eta-n+1}(S_{-n,2k_n-1}) \cap \tilde{F}_{\eta-1,\eta-n+2}(S_{-n+1,2k_{n-1}-1}) \cap \ldots \cap S_{-1,2k_1-1}$.

For each sequence $\{k'_n, k'_{n-1}, \ldots, k'_1\}$ in $\{1, 2, \ldots, M\}^n$, let $\tilde{S}(k'_n, k'_{n-1}, \ldots, k'_1) = \tilde{F}_{\eta-1,\eta-n+1}(S_{-n,2k'_n-1}) \cap \tilde{F}_{\eta-1,\eta-n+2}(S_{-n+1,2k'_{n-1}-1}) \cap \ldots \cap S_{1,2k'_1-1}$. Each $\tilde{S}(k'_n, k'_{n-1}, \ldots, k'_1)$ contains at least one component that intersects both end_0 and end_1. It follows that each $T_{n,0,\alpha_{n-1,i}} \cap \tilde{S}(k'_n, k'_{n-1}, \ldots, k'_1) \neq \emptyset$ for $\alpha_{n,-1,j} \in \underline{\alpha_{n,-1}}, \{k'_n, k'_{n-1}, \ldots, k'_1\} \in$

$\{1, 2, \ldots, M\}^n$, and, in fact, $T_{n,0,\alpha_{-1,i}} \cap \tilde{S}\{k'_n, k'_{n-1}, \ldots, k'_1\} = \tilde{F}_{\eta-1,\eta-n}(T_{n,-n,\alpha_{n,-n,i}})$ for some i.

Now the ηth floating permanent set $Z_\eta = \{x_\eta \in B$: the trajectory $\{\ldots x_{\eta-2}, x_{\eta-1} \star x_\eta \star x_{\eta+1}, \ldots\} \subset B\} = (\cap_{j=0}^{\infty} \tilde{F}_{\eta+j,\eta}^{-1}(B)) \cap B \cap (\cap_{j=1}^{\infty} \tilde{F}_{\eta-1,\eta-j}(B))$. For each centered bisequence $\{\ldots, i_{\eta-1} \star i_\eta \star i_{\eta+1}, \ldots\}$, let $Q^\eta(\{\ldots, i_{\eta-1} \star i_\eta \star i_{n+1}, \ldots\}) = (\cap_{j=0}^{\infty} \tilde{F}_{\eta+j,\eta}^{-1}(S_{j,2i_{\eta+j}-1}))$ $\cap S_{-1,2i_\eta-1} \cap (\cap_{j=1}^{\infty} \tilde{F}_{\eta-1,\eta-j}(S_{-j,2i_{\eta+j}-1}))$. Also, for each centered bisequence $\{\ldots, i_{\eta-1} \star i_\eta \star i_{\eta+1}, \ldots, \} \in \{1, 2, \ldots, M\}^{\mathbf{Z}}$, there is a unique sequence $\{\hat{i}_j\}_{j=1}^{\infty}$ such that (1) the sequence $\tilde{F}_{-1,-1}(T_{1,-1,\alpha_{1,-1,\hat{i}_1}}) \supset \tilde{F}_{-1,-2}(T_{2,-2,\alpha_{2,-2,\hat{i}_2}}) \supset \ldots$ is nested, and (2) $Q^\eta(\{\ldots, i_{\eta-1} \star i_\eta \star i_{\eta+1}, \ldots\}) = \cap_{j=1}^{\infty} \tilde{F}_{\eta-1,\eta-j}(T_{n,n,\alpha_{n,-n,\hat{i}_j}})$. Let $\mathcal{Q}_\eta = \{Q^\eta(\{\ldots, i_{\eta-1} \star i_\eta \star i_{\eta+1}, \ldots\}) : \{\ldots, i_{\eta-1} \star i_\eta \star i_{\eta+1}, \ldots\} \in \{1, 2, \ldots, M\}^{\mathbf{Z}}\}$. Then, by construction, $Z_\eta = \cup \mathcal{Q}_\eta$, and thus, Z_η is a quotient Cantor set with upper semicontinuous decomposition $\mathcal{Q}_\eta$. Define $\gamma_n : \mathcal{Q}_\eta \to \sum_M$ by $\gamma_n(Q^\eta(\{\ldots, i_{\eta-1} \star i_\eta \star i_{\eta+1}, \ldots\})) = \{\ldots, i_{\eta-1 \star} i_\eta i_{\eta+1} \ldots\}$. It is easy to check that $\gamma_{\eta+1} \circ F_\eta = \sigma_M \circ \gamma_\eta$. Thus, the first statement of the theorem is proved. The rest is similar to the corresponding theorem in the previous section, and so we omit those parts. □

Theorem 5. Topological Horseshoe Theorem-Indecomposable Set Part. *Suppose that $F : X \to X$ is a homeomorphism, B is a generalized quadrilateral in X with ends end_0, F is a horseshoe map on B with crossing number M, and F has the lockout property on B. In the Hausdorff metric, the sequence $B, F(B), F^2(B), \ldots$ of continua in X has a unique limit point $\tilde{B}$, and $\tilde{B}$ is a closed, invariant set which contains the entrainment set $E(B) = \{x \in X | F^{-n}(x) \in B$ for all sufficiently large $n\}$. Furthermore,*

1. *$\tilde{B}$ contains an invariant indecomposable continuum $\tilde{I}$ which for some $z \in C_B$, contains the invariant quotient Cantor set $\overline{O(z)} = \cap \mathcal{Q}_z \subseteq C'_B \subseteq C_B$ contained in interior of B and discussed in Theorem 3, and*
2. *there is an upper semicontinuous decomposition $\mathcal{G}$ of $\tilde{B}$ such that the quotient space $\mathcal{G}$ is an indecomposable, locally compact, separable metric space, each composant of which is an arc-component.*

Proof. For each integer k, let $\mathcal{S}_k = \{S_{k,0}, S_{k,1}, \ldots, S_{k,N_k}\}$ denote the tiling chain constructed in the proof of Theorem 3. Thus, for $k > 0$, $\cup\mathcal{S}_k = F^k(B)$, $F^k(end_0) \subset S_{k,0}$, $F^k(end_1) \subset S_{k,2M}$, and each $S_{k,i}$, for $1 \leq i \leq N_{k-1}$, separates $F^k(B)$. If $k < 0$, then $S_{-k} = \{S_{-k,0}, S_{-k,1}, \ldots, S_{-k,N_k}\} = F^{-k}(\mathcal{S}_k) = \{F^{-k}(S_{k,0}), F^{-k}(S_{k,1}), \ldots, F^{-k}(S_{k,N_k})\}$, and $\mathcal{S}_0 = \{S_{0,0}, S_{0,1}, \ldots, S_{0,N_0}\} = \mathcal{S}_{-1} = \{S_{-1,0}, S_{-1,1}, \ldots, S_{-1,N_1}\}$. If $(i_0, i_1 \ldots)$ is a sequence contained in $\{1, \ldots M\}^{\tilde{\mathbf{N}}}$, then $Q^+_{(i_0,i_1,\ldots)} = \cap_{n\in\tilde{\mathbf{N}}} F^n(S_{1,2i_n-1})$ is a nonempty closed subset of B and some component of $Q^+_{(i_0,i_1,\ldots)}$ intersects both end_0 and end_1. If $(p_0, p_i, \ldots)$ and $(p'_0, p'_1, \ldots)$ are distinct sequences, then $Q^+_{(i_0,i_1,\ldots)} \cap Q^+_{(i'_0,i'_1,\ldots)} = \emptyset$. Let $\mathcal{W}_0 = \{Q^+_{(i_0,i_1,\ldots)} : (i_0, i_1, \ldots)$ is a sequence contained in $\{1, \ldots M\}^{\tilde{\mathbf{N}}}\}$. Thus, $\mathcal{W}$ is a collection of disjoint closed sets in B, and $\mathcal{W}_0$ is an upper semicontinuous decomposition of $\tilde{B}_0 = \cap\mathcal{W}_0 = \cap_{i\leq 0} F^i(B)$. Arguing as we did in the proof of Theorem 3, $\tilde{B}_0$ is a quotient Cantor set in B, with each member of $\mathcal{W}_0$ containing a continuum that intersects both end_0 and end_1. At most countably many of the members of $\mathcal{W}_0$ have interior because there are at most countably many disjoint open sets in X. For each $n \leq 0$, $\tilde{B}_n = \cap_{i\leq n} F^i(B) = F^n(\tilde{B}_0)$ is a quotient Cantor set with respect to the upper semicontinuous decomposition $F^n(\mathcal{W}_0) = \mathcal{W}_n = \{Q^+_{(p_n,p_{n+1},\ldots)} : (p_n, p_{n+1}, \ldots)$ is a sequence contained in $\{1, \ldots M\}^{\tilde{\mathbf{N}}\setminus\{0,1,\ldots,n-1\}}$, and each member of $\mathcal{W}_n$ contains a component that intersects both ends $F^n(end_0)$ and $F^n(end_1)$ of $F^n(B)$. For each $n, \tilde{B}_n \subset \tilde{B}_{n+1}$ Consider $\overline{\cup_{n\leq 0}\tilde{B}_n} = \tilde{B}_\infty$. (Since X may not be compact, it is possible that $\tilde{B}_\infty\setminus(\cup_{n\leq 0}\tilde{B}_\infty)$ is empty, or, even if it is not empty, it may be disconnected.) Since $\tilde{B}_\infty \subset \tilde{B}_{n+1}$ for each $n, \tilde{B}_\infty$ is the Hausdorff limit of the sequence $\tilde{B}_0, \tilde{B}_1, \tilde{B}_2, \ldots$ Since $F(\tilde{B}_n) = \tilde{B}_{n+1}, F(\tilde{B}_\infty) = \tilde{B}_\infty$.

We can partition each $\tilde{B}_n$ into its components: denote this collection of components as $\mathcal{B}_n$. For each point x in $\tilde{B}_n, x$ is contained in some component $R_{x,n}$ of $\tilde{B}_n$. Then $R_{x,n} \subseteq R_{x,n+1}$ for each x, each n, and $F(R_{x,n}) = R_{F(x),n+1}$. Let $R_x = \cup_{n\leq 0} R_{x,n}$, so that R_x is a connected set in X. Note that for each positive integer n, $R_{x,n} \subset \tilde{B}_n \subset F^n(B)$. Choose $z \in C'_B$ such that (i) if Q''_z denotes the member of Q'' (from Theorem 3) that contains z, then (i) both $O^-(Q''_z) = \{\mathcal{F}^{-n}(Q''_z) : n \in \tilde{\mathbf{N}}\}$ and $O^+(Q''_z) = \{\mathcal{F}^{+n}(Q''_z) : n \in \tilde{\mathbf{N}}\}$ are dense in Q''; and (ii) R_z has empty interior. Since $\mathcal{B}_0$ is an upper semiccontinuous decomposition of $\tilde{B}_0$, and $\mathcal{B}'_0 = \{Q \in \mathcal{B}_0 : Q \cap C'_B \neq \emptyset\}$ is a closed subset of $\mathcal{B}_0$ that contains $\overline{O(z)}$, $\mathcal{B}'_0$ is an upper semicontinuous decomposition of $\tilde{B}'_0 = \cup\mathcal{B}'_0$. The points of continuity of $\mathcal{B}'_0$ form a dense G_δ-subset of $\mathcal{B}'_0$. Let

$\hat{B}_0$ denote such a dense G_δ-subset of $\mathcal{B}_0'$, and, without loss of generality, assume that $O(z) \subset \hat{\mathcal{B}}_0$, that no member of $\hat{\mathcal{B}}_0$ has interior, and that $F(\cup\hat{\mathcal{B}}_0) = \cap\hat{\mathcal{B}}_0$. Since each $R_{z,n}$ has empty interior, R_z is first category in $\tilde{B}$ and connected. For each $n > 1$, each $R_{z,0}$ is in the interior (relative to the subspace B) of some component $E_{z,0,n}$ of $\cap_{0\leq l\leq n} F^l(B)$, and moreover, the collection $\{E_{z,0,n}\}_{n=1}^{\infty}$ forms a neighborhood base in B for the component $R_{z,0}$. Thus, for each $m \geq 0$ and each $n > m$, each $R_{z,m}$ is in the interior (relative to the subspace $F^m(B)$) of some component $E_{z,m,n}$ of $\cap_{m\leq l\leq n} F^l(B)$, and the collection $\{E_{z,m,n}\}_{n=1}^{\infty}$ forms a neighborhood base for the component $R_{z,m}$ of $\tilde{B}_m$. Let $\tilde{J} = \{R_{F^n(z)} : n \in \mathbf{Z}\}$, and let $\tilde{I} = \bar{\tilde{J}} \subset X$.

If $n \in \mathbf{Z}$, there is a sequence $m_1, m_2, \ldots$ of integers, all different from n, such taht $\gamma(F^{m_1}(z)), \gamma(F^{m_2}(z)), \ldots$ converges to $\gamma(F^n(z))$. Hence, each limit point of $\gamma(F^{m_1}(z)$, $\gamma(F^{m_2}(z), \ldots$ is contained in $R_{F^n(z)}$. Since $R_{F^n(z),0}$ is a point of continuity of $\mathcal{B}_0$, $R_{F^{m_1}(z),0}, R_{F^{m_2}(z),0,\ldots}$ converges to $R_{F^n(z),0}$ (in the Hausdorff metric), and for each integer $k, R_{F^{m_1}(z),k}, R_{F^{m_1}(z),k,\ldots}$ converges to $R_{F^n(z),k}$ (in the Hausdorff metric). Thus, each $R_{F^n(z)}$ is dense in $\tilde{J}$. Then, applying Theorem 1, $\tilde{I}$ is indecomposable. Thus, the first part of the theorem is proved.

Now consider the set C_B. Define the upper semicontinuous decomposition $\mathcal{G}$ of $\tilde{B}_\infty$ as follows: If $(\ldots, i_{-1}, i_0, i_1, \ldots)$ is a bisequence contained in $\{1, \ldots M\}^{\mathbf{Z}}$, then $Q_{(\ldots,i_{-1},i_0,i_1,\ldots)} = \cap_{n\in\mathbf{Z}} F^n(S_{1,2i_n-1})$ is a nonempty closed subset of $B, \mathcal{Q} = \{Q_{(\ldots,i_{-1},i_0,i_1,\ldots)} :$ $(\ldots, i_{-1}, i_0, i_1, \ldots)$ is a bisequence contained in $\{1, \ldots M\}^{\mathbf{Z}}\}$ is a collection of disjoint closed sets in B that is an upper semicontinuous decomposition of $C_B = \{x \in B :$ $F^n(x) \in B$ for each integer $n\}$, and $\mathcal{W}_0 = \{Q^+_{(i_0,i_1,\ldots)} : (i_0, i_1, \ldots)$ is a sequence contained in $\{1, \ldots M\}^{\tilde{\mathbf{N}}}\}$ is a collection of disjoint closed sets in B whose union contains C_B. Since for $0 \leq j \leq N_k, S_{-k,j} = F^{-k}(S_{k,j})$, and $\mathcal{S}_{-k} = \{S_{-k,0}, S_{-k,1}, \ldots, S_{-k,N_k}\}$, where $\mathcal{S}_k = \{S_{k,0}, S_{k,1}, \ldots, S_{k,N_k}\}$ is the tiling chain cover of $F^k(B)$ in Theorem 3, $\mathcal{S}_{-k}$ is a tiling chain cover of B, and each link $S_{-k,j}, 1 \leq j < N_k$, separates B. Then since each $Q^+_{(i_0,i_1,\ldots)} \in \mathcal{W}_0$ intersects both end_0 and end_1, and each $Q^+_{(i_0,i_1,\ldots)}$ is separated by each link $S_{-k,j}, 1 \leq j < N_k, Q^+_{(i_0,i_1,\ldots)} \backslash C_B$ is a countable union of disjoint open sets relative to the subspace $Q^+_{(i_0,i_1,\ldots)}$.

For each positive integer n, use Urysohn's Lemma to construct inductively the Urysohn functions $f_n : B \to [0, 1/2^n]$ as follows:

1) There is a continuous function $f_1 : B \to [0, 1/2]$ such that

a) $f_1(\cup_{1\le i\le M} S_{-1,2i-1}) = 1/2$,

b) $f_1(end_0 \cup end_1) = 0$, and

c) for $x \in B\backslash((end_0 \cup end_1) \cup (\cup_{1\le i\le M} S_{-1,2i-1})), 0 < f_1(x) < 1/2$.

2) Having chosen f_{n-1}, there is a continuous function $f_n : B \to [0, 1/2^n]$ such that

a) for $x \in F^{-n+1}(B) \cap B, f_n(x) = f_{n-1}(F^{n-1}(x))/2$, and

b) for $x \in B\backslash F^{-n+1}(B), f_n(x) = 0$.

Then define $f(x) = \sum_{n=1}^{\infty} f_n(x)$ so that $f : B \to [0, 1]$, $f(x) = 1$ if and only if $F^n(x) \in B$ for each $n \ge 0$, and $f(end_0 \cup end_1) = 0$. Next let $g_0 = f$ and $\Xi_0 = \{g_0^{-1}(t) : t \in [0, 1]\}$. Since f is continuous, Ξ_0 is an upper semicontinuous decomposition of B. Then, making use of the way f and F are related, define $g_n : F^n(B) \to [0, 1]$ by $g_n = f \circ F^{-n} = g_0 \circ F^{-n}$, and $\Xi_n = \{g_n^{-1}(t) : t \in [0, 1]\}$ so that Ξ_n is an upper semicontinuous decomposition of $F^n(B)$. Note that if $Y_n \in \Xi_n$ such that $Y_n \subset B$, then Y_n is contained in some unique member Y_0 of Ξ_0. In fact, if $Y_n \in \Xi_n, Y_n \subset F^m(B)$ for $0 \le m < n$, then Y_n is contained in some unique member Y_m of Ξ_m.

For each $Q^+_{(p_0,p_1,\ldots)} \in \mathcal{W}_0$, let $\mathcal{G}_{p_0,p_1,\ldots)} = \{Q^+_{(p_0,p_1,\ldots)} \cap Y : Y \in \Xi_0\}$, and for $n > 0$, for each $Q^+_{(p_n,p_{n+1},\ldots)} \in \mathcal{W}_n$, let $\mathcal{G}_{(p_n,p_{n+1},\ldots)} = \{(Q^+_{(p_n,p_{n+1},\ldots)} \backslash (\cup_{0\le m<n} Q^+_{(p_m,p_{m+1},\ldots)})) \cap Y : Y \in \Xi_n\}$. Again, because of the way f and F are related, the lockout property of F on B with its consequence that $\tilde{Q}_{(p_0,p_1,\ldots)} = \cup_{n\ge 0} Q^+_{(p_n,p_{n+1},\ldots)}$ is a countable union of nested compact sets, and because each $F^n(B)$ is compact, $\tilde{G}_{(p_0,p_1,\ldots)} = \cup_{n\ge} \mathcal{G}_{(p_n,p_{n+1},\ldots)}$ is an upper semicontinuous decomposition of $\tilde{Q}_{(p_0,p_1,\ldots)}$. If $(p_0, p_1, \ldots)$ and $(p'_0, p'_1 \ldots)$ are sequences contained in $\{1, \ldots M\}^{\tilde{\mathbf{N}}}$, then if $\tilde{Q}_{(p_0,p_1,\ldots)} \cap \tilde{Q}_{(p'_0,p'_1,\ldots)} \ne \emptyset$, $\tilde{Q}_{(p_0,p_1,\ldots)} = \tilde{Q}_{(p'_0,p'_1,\ldots)}$. Hence, $\mathcal{G} = \cup\{\tilde{\mathcal{G}}_{(p_0,p_1,\ldots)} : (p_0, p_1, \ldots)$ is a sequence each member of which is an element of $\{1, \ldots, M\}\}$ is an upper semicontinuous decomposition of $\cap_{n\ge 0} \tilde{B}_n$. Because of the lockout property of F on B, $(\cup\mathcal{G}')\cap$ $(\tilde{B}_\infty \backslash \cup_{n\ge 0} \tilde{B}) = \emptyset$, and thus $\mathcal{H} = \mathcal{G}' \cup \{\tilde{B} \backslash \cup_{n\ge 0} \tilde{B}_n\}$ is an upper semicontinuous decomposition of $\tilde{B}_\infty$. (Note that $\cup_{n\ge 0} \tilde{B}_n$ is open in $\tilde{B}_\infty$). Since each $\tilde{Q}_{(p_0,p_1,\ldots)}$ is dense and connected in $\tilde{B}$, so is the subspace $\tilde{\mathcal{G}}_{(p_0,p_1,\ldots)}$ of the space $\mathcal{H}$ (which we have endowed with the quotient topology). By construction, each set $\tilde{\mathcal{G}}_{(p_0,p_1,\ldots)}$ is dense in $\tilde{\mathcal{G}}'$. Suppose that $P : \tilde{B}_\infty \to \mathcal{H}$ denotes the projection associated with the given upper semicontinuous decomposition. For each $Q^+_{(p_n,p_{n+1}\ldots)}, P(Q^+_{(p_n,p_{n+1}\ldots)})$ is an arc in $\mathcal{H}$, and $P(Q^+_{(p_{n+1},p_{n+2}\ldots)}) \supset P(Q^+_{(p_n,p_{n+1}\ldots)})$.

Specifically, each $P(Q^+_{(p_{n+1},p_{n+2}\ldots)})$ contains exactly those M members of $\mathcal{W}_n$ in the set $\{Q^+_{(i,p_{n+1}\ldots)} : i \in \{1,2,\ldots M\}\}$, and the collection $\{\tilde{\mathcal{G}}_{(p_0,p_1,\ldots)} : (p_0,p_1,\ldots)$ is a sequence each member of which is an element of $\{1,\ldots,M\}\}$ is uncountable. Thus, each $\tilde{\mathcal{G}}_{(p_0,p_1,\ldots)}$ is a "folded" ray or line, and each $\tilde{\mathcal{G}}_{(p_0,p_1,\ldots)}$ is a composant of $\mathcal{G}'$. Then $\mathcal{G}'$ is indecomposable, and applying Theorem 1, so is $\mathcal{H}$. Furthermore, each composant of $\mathcal{H}$ is an arc-component, and $\mathcal{H}$ is a locally compact, separable, indecomposable metric space.

Finally, for each nonnegative integer n, let $D_n = \overline{\cup_{m\geq n}F^m(B)}$. Then $\tilde{B} = \cap_{n\geq 0}D_n$ is a closed, connected invariant subset of X which contains $\tilde{B}_\infty$. In fact, if $G = \cup_{n\geq 0}(F^n(B))^\circ$, then $\tilde{B}_\infty \cap G = \tilde{B}\cap G$. Thus, $\mathcal{G} = \mathcal{G}' \cup \{\tilde{B}\setminus \cup_{n\geq 0}\tilde{B}_n\}$ is an upper semicontinuous decomposition of $\tilde{B}$, and, when endowed with the quotient topology, $\mathcal{G}$ is a locally compact, separable indecomposable metric space which is homeomorphic to $\mathcal{H}$, and each composant of $\mathcal{H}$ is an arc-component. □

Theorem 6. Noisy Topological Horseshoe Theorem-Indecomposable Set Part. *Suppose that $F : X \to X$ is a homeomorphism, B is a generalized quadrilateral in X with ends end_0 and end_1, F is a horseshoe map on B, and F satisfies the uniform lockout property on B. There is $\delta > 0$ such that if for each integer $j, F_j : X \to X$ is a homeomorphism with $d(F(x),F_j(x)) < \delta$ for each $x \in X$, then the following hold:*

1. *In the Hausdorff metric, the sequence $B, F_{-1}(B), \tilde{F}_{-1,-2}(B),\ldots$ of continua in X has a unique limit point $\tilde{B}$, and $\tilde{B}$ is a closed, connected, invariant set which contains the entrainment set $E(B) = \{x_0 \in X | (\tilde{F}_{-1,-n})^{-1}(x_0) \in B$ for all sufficiently large $n\}$, and there is an upper semicontinuous decomposition $\mathcal{G}$ of $\tilde{B}$ such that the quotient space $\mathcal{G}$ is an arc-component.*

2. *The system $\mathbf{F} : \mathbf{Z}\times X \to X$ defined by $F(n,x) = F_n(x)$ preserves*

(a) *the collection $\tilde{\mathcal{B}} = \{\tilde{B}(\eta)\}_{\eta\in\mathbf{Z}}$, where $\tilde{B}(\eta+1) = \tilde{F}_{0,\eta}(\tilde{B})$ for $\eta \geq 0$, $\tilde{B}(\eta) = \tilde{F}^{-1}_{-1,\eta}(\tilde{B})$ for $\eta < 0$, and $\tilde{B}(0) = \tilde{B}$; and*

(b) *if we define for $G \in \mathcal{G}$, $G(\eta+1) = \tilde{F}_{0,\eta}(G)$ for $\eta \geq 0$, $G(\eta) = \tilde{F}^{-1}_{-1,\eta}(G)$ for $\eta < 0$, and $G(0) = G$; and for $\eta \in \mathbf{Z}$, $\mathcal{G}(\eta) = \{G(\eta) : G \in \mathcal{G}\}$, then each $\mathcal{G}(\eta)$ is an upper* *semicontinuous decomposition of* $\tilde{B}(\eta)$ *, and as space, is an indecomposable, locally compact, separable metric space, when* $\mathcal{G}(\eta)$ *is endowed with the quotient topology;*

(c) the system $\mathbf{F}$ *preserves the ordered collection* $\mathbf{G} = \{\mathcal{G}(\eta)\}_{\eta\in\mathbf{Z}}$ *since for* $G \in \mathcal{G}(\eta)$ *,* $\mathbf{F}(\eta, \mathbf{G}) = \mathbf{F}_\eta(\mathbf{G})$ *, and if we define* $\Im : \sqcup\mathbf{G} \to \sqcup\mathbf{G}$ *by* $Im(G_\eta) = F_\eta(G_\eta)$ *for each* $G_\eta \in \mathcal{G}(\eta)$ *, then each* $\Im|\mathcal{G}(\eta) : \mathcal{G}(\eta) \to \mathcal{G}(\eta+1)$ *is a homeomorphism, and* $\Im$ *preserves the collection* $\mathbf{G}$ *; and*

(d) the system $\Im : \sqcup\mathbf{G} \to \sqcup\mathbf{G}$ *is conjugate to the homeomorphism* $\mathcal{F} : \mathcal{G} \to \mathcal{G}$ *, where we define* $\mathcal{F}(G) = F(G)$ *for* $G \in \mathcal{G}$ *(from the previous theorem), with the conjugating homeomorphism* $\gamma_\eta : \mathcal{G}(\eta) \to \mathcal{G}$ *being the one induced naturally by the respective constructions of* $\mathcal{G}(\eta)$ *and* $\mathcal{G}$ *.*

Proof. Suppose B has ends end_0 and end_1 ; side; the lockout set for F is B^+ , a closed neighborhood that contains B in its interior and has the property that for some $\epsilon > 0$, if $x \in \partial B^+$, then $d(F(x), B^+) > \epsilon$; and the lockout number for F is $N_F = N$. There is $\epsilon/2 > \epsilon_1 > 0$ such that

e1) if $F' : X \to X$ is a homeomorphism with $d(F(x), F'(x)) < \epsilon_1$ for each $x \in X$, then $\frac{F'(end_0 \cup end_1)\cap}{D_{4\epsilon_1}(end_0\cup end_1)} = \emptyset$, $(F')^{-1}(end_0 \cup end_1) \cap \overline{D_{4_{\epsilon_1}}(end_0 \cup end_1)} = \emptyset$, and $D_{4_{\epsilon_1}}(F'(side)) \cap D_{4_{\epsilon_1}}(side) = \emptyset$,

e2) each F' is a horseshoe map on the generalized quadrilateral B with ends end_0 and end_1 and the same crossing number M as F; and

e3) if $F'_1, \ldots, F'_N$ is a collection of N homeomorphisms on X such that for each $q \in X$, $|F(q) - F'_i(q)| < \epsilon_1$, and if $F_{i_1} \circ \ldots \circ F_{i_N}$ has the property that if $q \in D_{\epsilon_1}(F(B) - B)$, then $F_{i_1} \circ \ldots \circ F_{i_N}(q)$ is in $X \backslash B^+$ (in other words, each F'_i is chosen so close to F that the resulting composition of N homeomorphisms satisfies an appropriately modified version of the strong lockout property).

Next there is some $\delta > 0$ such that $\delta < \epsilon_1/2$, and if

e4) if C and C' are different components of $B \cap F'(B)$ that intersect $B \backslash D_{\epsilon_1}(end_0 \cup end_1)$, then the Hausdorff distance from C and C' is greater that 4δ, and e5) if C and C' are different components of $\overline{D_{\epsilon_1}(F'(B)\backslash B)}$, then the Hausdorff distance from C and C' is greater than 4δ.

For each integer j, suppose that $F_j : X \to X$ is a homeomorphism with $d(F(x), F_j(x)) < \delta$ for each $x \in X$. Note that for each $j, F_j(end_0 \cup end_1) \cap \overline{D_{\epsilon_1}(end_0 \cup end_1)} = \emptyset$, and $F_j^1(end_0 \cup end_1) \cap \overline{D_{\epsilon_1}(end_0 \cup end_1)} = \emptyset$.

For each $\eta \in \mathbf{Z}$, the permanent set $Z_\eta = \{x_\eta \in B\colon$ the trajectory $\{\ldots x_{\eta-2}, x_{\eta-1} \star x_\eta \star x_{\eta+1}, \ldots\} \subset B\} = (\cap_{j=0}^{\infty} \tilde{F}_{\eta+j,\eta}^{-1}(B)) \cap B \cap (\cap_{j=1}^{\infty} \tilde{F}_{\eta-1,\eta-j}(B))$. For each centered bisequence $\{\ldots, i_{\eta-1} \star i_\eta \star i_{\eta+1}, \ldots\}$, there corresponds exactly on member $Q^\eta(\{\ldots, i_{\eta-1} \star i_\eta \star i_{\eta+1}, \ldots\})$ of the upper semicontinuous decomposition $\mathcal{Q}_\eta$ of Z_η, and $Q^\eta(\{\ldots, i_{\eta-1} \star i_\eta \star i_{\eta+1}, \ldots\}) = (\cap_{j=0}^{\infty} \tilde{F}_{\eta+j,\eta}^{-1}(S_{j,2i_{\eta+j}-1})) \cap S_{-1,2i_\eta-1} \cap (\cap_{j=1}^{\infty} \tilde{F}_{\eta-1,\eta-j}^{-1}(S_{-j,2i_{\eta+j}-1}))$. We are adopting the notation of Theorem 4.

Let $\eta = 0$. If $(p_1, p_2, \ldots)$ is a sequence contained in $\{1, \ldots M\}^{\mathbf{N}}$, then $Q_{(p_1,p_2,\ldots)}^+ = S_{-1,2p_1-1} \cap (\cap_{j=2}^{\infty} \tilde{F}_{-1,-j}(S_{-j,2p_j-1}))$ is a nonempty closed subset of B and some campponent of $Q_{(p_1,p_2,\ldots)}^+$ intersects both end_0 and end_1. If $(p_1, p_2, \ldots)$ and $(p_1', p_2', \ldots)$ are distinct sequences, then $Q_{(p_1,p_2,\ldots)}^+ \cap Q_{(p_1',p_2',\ldots)}^+ = \emptyset$, $\mathcal{W}_0 = \{Q_{(p_1,p_2,\ldots)}^+ : (p_1, p_2, \ldots)$ is a sequence contained in $\{1, \ldots M\}^{\mathbf{N}}\}$ is an upper semicontinuous decomposition of $\tilde{B}_0 = B \cap (\cap_{i \geq 1} \tilde{F}_{-1,-i}(B))$, $Z_0 \subset \cup \mathcal{W}_0$, and $\tilde{B}_0$ is a quotient Cantor set in B. For each $n \geq 1$, $\tilde{B}_n = \cap_{i \geq n} \tilde{F}_{-1,-i}(B) = \tilde{F}_{-1,-n}(\tilde{B}_0)$ is a quotient Cantor set with respect to the upper semicontinuous decomposition $\tilde{F}_{-1,-n}(\mathcal{W}_0) = \mathcal{W}_n = \{Q_{p_{n+1},p_{n+2},\ldots)}^+ : (p_{n+1}, p_{n+2}, \ldots)$ is a sequence contained in $\{1, \ldots M\}^{\mathbf{N} \backslash \{\mathbf{1},\ldots,\mathbf{n}\}}\}$, and each member of $\mathcal{W}_n$ contains a component that intersects both ends $\tilde{F}_{-1,-n}(end_0)$ and $\tilde{F}_{-1,-n}(end_1)$ of $\tilde{F}_{-1,-n}(B)$. For each n, $\tilde{B}_n \subset \tilde{B}_{n+1}$. Consider $\cup_{n \geq 0} \tilde{B}_n = \tilde{B}_\infty$. (As before, it is possible that $\tilde{B}_\infty \backslash (\cup_{n \geq 0} \tilde{B}_n)$ may be empty or disconnected.) Since $\tilde{B}_n \subset \tilde{B}_{n+1}$ for each n, the sequence $\tilde{B}_0, B_1, B_2, \ldots$ has the Hausdorff limit $\tilde{B}_\infty$. Arguing as we did in the previous proof, the entrainment set $E(B) = \{x \in X :$ for some positive integer N_x, if $n \geq N_x$, $\tilde{F}_{-1,-n}^{-1}(x) \in B\} \subseteq \tilde{B}_\infty$.

For each negative integer n, let $D_n = \overline{\cup_{m \geq n} \tilde{F}_{-1,-m}(B)}$. Then $\tilde{B} = \cap_{n \geq 0} D_n$ is a closed, connected invariant subset of X which contains $\tilde{B}_\infty$. In fact, if $G = B^\circ \cup (\cup_{n \geq 0} (\tilde{F}_{-1,-n}(B))^\circ)$, then $\tilde{B}_\infty \cap G = \tilde{B} \cap G$, and the Hausdorff limit of the sequence $\tilde{F}_{-1,-1}(B), \tilde{F}_{-1,-2}(B), \tilde{F}_{-1,-3}(B), \ldots$ is $\tilde{B}$. Since for each positive integer k, each $Q_{(p_1,p_2,\ldots)}^+$ is separated by each link $S_{-k,j}, 1 \leq j < N_k$ (from the proof of Theorem 4), $Q_{(p_1,p_2,\ldots)}^+ \backslash Z_0$ is a countable union of disjoint open sets (relative to the subspace

$Q^+_{(p_1,p_2,\ldots)})$.

Adopting the notation of Theorem 4, for each $n \geq 1$, let $\mathcal{T}_{-1,-n} = \{T_{-1,n,0},\ T_{-1,-n,1},\ \ldots,\ T_{-1,-n,N_{-n}}\}$ denote the tiling chain cover of B obtained by applying Lemma 2 to $\tilde{F}_{-1,-n}$ and then $\tilde{F}^{-1}_{-1,-n}$ on B. Thus, for each $n \geq 1$,

1. $T_{-1,-n,0}$ contains end_0 and $T_{-1,n,N_{-n}}$ contains end_1,
2. $\mathcal{T}_{-1,-n-1}$ refines $\mathcal{T}_{-1,-n}$, and
3. $\tilde{F}_{-1,-n}(\mathcal{T}_{-1,-n}) = \{\tilde{F}_{-1,-n}(T_{-1,-n,0}), \tilde{F}_{-1,-n}(T_{-1,-n,1}), \ldots, \tilde{F}_{-1,-n}(T_{-1,-n,N_{-n}})\}$ is a tiling chain cover of $\tilde{F}_{-1,-n}(B)$.

For each nonnegative integer n, use Urysohn's Lemma to construct inductively the Urysohn functions $f_n : B \to [0, 1/2^n]$ as follows:

1) There is a continuous function $f_0 : B \to [0, 1/2]$ such that

a) $f_0(B \cap F_0^{-1}(B)) = 1/2$,

b) $f_0(end_0 \cup end_1) = 0$, and

c) for $x \in B \backslash ((end_0 \cup end_1) \cup (B \cap F_0^{-1}(B))$, $0 < f_0(x) < 1/2$.

2) Having chosen f_{n-1}, there is a continuous function $f_n : B \to [0, 1/2^{n+1}]$ such that

a) for $x \in \tilde{F}^{-1}_{n-1,0}(B) \cap B$, $f_n(x) = f_{n-1}(\tilde{F}_{n-1,0}(x))/2$, and

b) for $x \in B \backslash \tilde{F}^{-1}_{n-1,0}(B)$, $f_n(x) = 0$.

Then define $f(x) = \sum_{n=1}^{\infty} f_n(x)$, so that $f : B \to [0,1], f(x) = 1$ if and only if $\tilde{F}_{-1,-n}(x) \in B$ for each $n \geq 0$, and $f(end_0 \cup end_1) = 0$. Further, note that f has been constructed carefully so that f "preserves the system $\mathbf{F}|\mathbf{N} \times X$ in that preimages of points in [0,1] are carefully chosen to coordinate with the F_n'$s(n \geq 0)$ in the appropriate order. Next let $g_0 = f$ and $\Xi_0 = \{g_0^{-1}(t) : t \in [0,1]\}$. Since f is continuous, Ξ_0 is an upper semicontinuous decomposition of B. Then, making use of the way f and F are related, define $g_n : \tilde{F}_{-1,-n}(B) \to [0,1]$ by $g_n = f \circ \tilde{F}^{-1}_{-1,-n} = g_0 \circ F^{-1}_{-1,-n}$, and $\Xi_n = \{g_n^{-1}(t) : t \in [0,1]\}$ so that Ξ_n is an upper semicontinuous decomposition of $\tilde{F}_{-1,-n}(B)$. Note that for $Y_n \in \Xi_n$, $Y_n \subset B$, and Y_n is contained in some unique member Y_m of Ξ_m.

For each $Q^+_{(p_0,p_1,\ldots)} \in \mathcal{W}$, let $\mathcal{G}_{(p_0,p_1,\ldots)} = \{Q^+_{(p_0,p_1,\ldots)} \cap Y : Y \in \Xi_0\}$, and for $n > 0$, for each $Q^+_{(p_n,p_{n+1},\ldots)} \in \mathcal{W}, \mathcal{G}_{(p_n,p_{n+1},\ldots)} = \{(Q^+_{(p_n,p_{n+1},\ldots)} \backslash (\cup_{0 \leq m < n} Q^+_{(p_m,p_{m+1},\ldots)})) \cap Y :$

$Y \in \Xi_n\}$. Note that $Z_0 \subset \cup\mathcal{W}_0$. For $n > 0$, for each $Q^+_{(p_n,p_{n+1},\ldots)} \in \mathcal{W}_n$, let $\mathcal{G}_{(p_n,p_{n+1},\ldots)} = \{(Q^+_{(p_n,p_{n+1},\ldots)} \backslash \tilde{F}_{-1,-n+1}(B)) \cap Y : Y \in \Xi_n\} = \{\tilde{F}_{-1,-n}(Q^+_{(p_0,p_1,\ldots)}) \backslash \tilde{F}_{-1,-n+1}(B)) \cap Y : Y \in \Xi_n\}$. Again, for each sequence $(p_0, p_1, \ldots)$ contained in $\{1, \ldots M\}^{\tilde{\mathbf{N}}}$, let $\tilde{Q}^+_{(p_0,p_1,\ldots)} = \cup_{n\geq 0} Q^+_{(p_n,p_{n+1},\ldots)}$ so that $\tilde{\mathcal{G}}_{(p_0,p_1,\ldots)} = \cup_{n\geq 0}\tilde{\mathcal{G}}_{(p_n,p_{n+1},\ldots)}$ is an upper semicontinuous decomposition of $\tilde{Q}_{(p_0,p_1,\ldots)}$. Further, $\mathcal{G}' = \cup\{\tilde{\mathcal{G}}_{(p_0,p_1,\ldots)} : (p_0, p_1, \ldots)$ is a sequence each member of which is an element of $\{1, \ldots, M\}\}$ is an upper semicontinuous decomposition of $\cup_{n\geq 0}\tilde{B}_n$, and, as in the previous proof, $\mathcal{G} = \mathcal{G}' \cup \{\tilde{B} \backslash U_{n\geq 0}\tilde{B}_n\}$ is an upper semicontinuous decomposition of $\tilde{B}$, each $\tilde{Q}_{(p_0,p_1,\ldots)}$ is dense, first category, and connected in $\tilde{B}$, and so is the subspace $\tilde{\mathcal{G}}_{(p_0,p_1,\ldots)}$ of the space $\mathcal{G}$ (which we have endowed with the quotient topology). If $(p_0, p_1, \ldots)$ and $(p'_0, p'_1, \ldots)$ are sequences contained in $\{1, \ldots M\}^{\tilde{\mathbf{N}}}$, then if $\tilde{Q}_{(p_0,p_1,\ldots)} \cap \tilde{Q}_{(p'_0,p'_1,\ldots)} \neq \emptyset$, $\tilde{Q}_{(p_0,p_1,\ldots)} = \tilde{Q}_{(p'_0,p'_1,\ldots)}$. Since $\{\tilde{Q}^+_{(p_0,p_1,\ldots)} : (p_0, p_1, \ldots)$ is a sequence each member of which is an element of $\{1, \ldots M\}\}$ is uncountable and for each $(p_0, p_1, \ldots) \in \{1, \ldots, M\}^{\tilde{\mathbf{N}}}, Q^+_{(p_1,p_2,\ldots)}$ contains exactly M members of $\mathcal{G}', \mathcal{G}'$ has uncountably many distinct composants and is indecomposable, and applying Theorem 1, so is $\mathcal{G}$. Because of the uniform lockout property of F on B, with its result that for each $n \geq 0$, $Q^+_{(p_n,p_{n+1},\ldots)} \cap (\cup_{n\geq 1}\tilde{F}_{-1,-n}(B)) = Q_{(p_0,p_1,\ldots)} \cap (\cup_{n\geq 1}\tilde{F}_{-1,-n}(B))$, and $Q^+_{(p_0,p_1,\ldots)} \subset Q^+_{(p_1,p_2,\ldots)} \subset \ldots, \mathcal{G}$ is an indecomposable, locally compact, separable, closed metric space. The rest then follows easily. □

Remark *Results similar to those of the last two theorems concerning indecomposability associated with entrainment sets hold, of course, for destination sets as well. The entrainment set and destination set for a topological horseshoe need not be homeomorphic, however.*

4. Associated and Future Work

Since doing this research, we have learned that Konstantin Mischaikow, M. Carbinatto, J. Kwapisz, and M. Mrozek have been studying topological horseshoes for the last several yeras. (See [CKM], [CM], [MM] and [S].) However, those results approach the subject from a different, more algebraic perspective namely that of the Conley index.

We are currently writing a second paper generalizing the results in this paper by (1) eliminating the requirement that F be a homeomorphism (F needs only be continuous); (2) eliminating the requirement that a generalized quadrilateral Q be connected (Q needs only to be a closed neighborhood); and (3) eliminating the requirement that the background space X be locally connected (X needs only to be a locally compact, separable, connected metric space). Our results hold even with these more general hypotheses. We also prove more facts about topological horseshoes and their entrainment sets, and we hope to be able to relate our work to that of Mischaikow, Carbinatto, Kwapisz, and Mrozek.

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