ON THE DIFFERENTIAL PRIME RADICAL OF A
DIFFERENTIAL RING

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Abstract

In this paper we have obtained the following results for a differential ring (associative or nonassociative):

1. For a differential ring (\( D \)-ring) we introduce definitions of a \( D \)-prime \( D \)-ideal, \( D \)-semiprime \( D \)-ideal and a strongly \( D \)-nilpotent element. We define the \( D \)-prime radical as the intersection of all \( D \)-prime \( D \)-ideals. For any \( D \)-ring the \( D \)-prime radical, the intersection of all \( D \)-semiprime \( D \)-semiprime \( D \)-ideals and the set of all strongly \( D \)-nilpotent elements are equal.

2. For a \( D \)-ring we introduce a definition of an s-nilpotent \( D \)-ideal. If a \( D \)-ring satisfies the ascending chain condition for \( D \)-ideals then its \( D \)-prime radical is s-nilpotent.

3. Let \( Q \) be a field of rational numbers. If \( \delta \) is a differentiation of a \( Q \)-algebra \( R \) with 1 then \( \delta(Pr.rad(R)) \subseteq Pr.rad(R) \).

4. Let \( K \) be a differential ring. Then every radical \( D \)-ideal of \( K \) is an intersection of \( D \)-prime \( D \)-ideals.

1. The differential prime radical

This paper is a continuation of our papers [1-3]. Further, we use notions and notations of books [4-6].

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Definition ([7], p.556). A differential ring (D-ring) is a system $(K, \mathcal{D})$, where $K$ is a ring (associative or nonassociative) and $\mathcal{D}$ is a set of differentiations of $K$. A $\mathcal{D}$-subgroup $H$ is an additive subgroup of the ring $K$ such that $dh \in H$ for all $h \in H, d \in \mathcal{D}$.

Let $K$ be a $\mathcal{D}$-ring. Denote by $\text{Ad}(K)$ the set of all $\mathcal{D}$-subgroups of $K$. $\text{Ad}(K)$ is a complete lattice with respect to the inclusion relation. Introduce a multiplication operation on it by the following manner ([5], p.12): For $A, B \in \text{Ad}(K)$, $A \cdot B$ consists of all finite sums $\sum_{i=1}^{n} a_i b_i$, where $a_i \in A, b_i \in B$.

Definition ([7], p.556). A differential ideal (D-ideal) of $K$ is an ideal $H$ of $K$ such that $dh \in H$ for all $h \in H, d \in \mathcal{D}$.

Denote by $\text{Id}(K)$ the set of all $\mathcal{D}$-ideals of $K$. $\text{Id}(K)$ is a complete lattice with respect to the inclusion relation. If $K$ is associative then $A \cdot B \in \text{Id}(K)$ for all $A, B \in \text{Id}(K)$.

But there is nonassociative $\mathcal{D}$-ring $K$ and $A, B \in \text{Id}(K)$ such that $A \cdot B \notin \text{Id}(K)$.

Therefore for any $\mathcal{D}$-ring we define a multiplication operation of $\mathcal{D}$-ideals in the following manner. For $A, B \in \text{Id}(K)$ denote by $A \cdot B$ the intersection of all $\mathcal{D}$-ideals of $K$ containing the set $\{x \in K : x = a \cdot b, a \in A, b \in B\}$.

Proposition 1.1. For any $\mathcal{D}$-ring $K$ the lattice $\text{Id}(K)$ with above multiplication operation is a complete l-groupoid.

Proof. Let $A, B_t \in \text{Id}(K), t \in T$. The inequality

$$A \cdot (\vee_{t \in T} B_t) \geq \vee_{t \in T} (A \cdot B_t)$$

is obvious. We now prove the inverse inequality. The $\mathcal{D}$-ideal $A \cdot (\vee_{t \in T} B_t)$ is the smallest $\mathcal{D}$-ideal containing all elements $a \cdot (b_1 + b_2 + \ldots + b_k)$, where $a \in A$, $b_i \in B_t$. From the equality $a \cdot (b_1 + b_2 + \ldots + b_k) = ab_1 + \ldots + ab_k$ we obtain that $a \cdot (b_1 + b_2 + \ldots + b_k) \in \vee_{t \in T}(A \cdot B_t)$. Therefore

$$A \cdot (\vee_{t \in T} B_t) \leq \vee_{t \in T} (A \cdot B_t).$$

A proof of the equality
\((\forall t \in T B_t) \cdot A = \forall t \in T (B_t \cdot A)\)

is similar. \(\Box\)

**Definition** A \(D\)-ideal \(P\) of \(K\) is \(D\)-prime if \(P \neq K\) and \(A \cdot B \subseteq P\), \(A, B \in Id(K)\), implies that \(A \subseteq P\) or \(B \subseteq P\).

For \(A \in Id(K)\), \(A \neq K\), denote by \(R(A)\) the intersection of all \(D\)-semiprime \(D\)-ideals of \(K\) containing \(A\). Put \(r^*(A) = K\) if there are none.

For \(A \in Id(K)\), denote by \(< A >\) the groupoid generated by \(A\). An element of the groupoid \(< A >\) will be denoted by \(f(A)\).

**Definition** A \(D\)-ideal \(H\) of \(K\) is \(D\)-semiprime if \(H \neq K\) and \(f(A) \subseteq H\), \(A \in Id(K)\), \(f(A) \in < A >\), implies that \(A \subseteq H\).

For \(A \in Id(K)\), \(A \neq K\), denote by \(r^w(A)\) the intersection of all \(D\)-semiprime \(D\)-ideals of \(K\) containing \(A\). Put \(r^w(A) = K\) if there are none. It is clear \(r^*(A) \subseteq r^w(A) \subseteq R(A)\) for all \(A \in Id(K)\).

For \(A \in Id(K)\) the \(D\)-ideal \(R(A)\) will be called \(D\)-radical of \(A\).

**Definition** A \(D\)-ideal \(M\) of \(K\) is \(D\)-maximal if \(M \neq K\) and \(M \subseteq B \subseteq K\), \(B \in Id(K)\), implies that \(M = B\) or \(B = K\).

**Proposition 1.2** Let \(K\) be a \(D\)-ring satisfying the ascending chain condition for \(D\)-ideal. Then every \(D\)-ideal of \(K\) is contained in some \(D\)-maximal \(D\)-ideal. In particular, there is a \(D\)-maximal \(D\)-ideal of \(K\).

A proof is standard.

**Proposition 1.3.** Let \(K\) be a \(D\)-ring such that \(K^2 = K\). Then any \(D\)-maximal \(D\)-ideal \(K\) is \(D\)-prime.

**Proof.** Let \(M\) be \(D\)-maximal \(D\)-ideal of \(K\). Suppore that \(A \cdot B \subseteq M\), \(A, B \in Id(K)\). If \(A \nsubseteq M\), then \(A \vee M = K\). Therefore
\[ K \cdot K = (A \cup M) \cdot (B \cup M) = A \cdot B \cup A \cdot M \cup M \cdot B \cup M \cdot M \subseteq M \subseteq K. \]

We obtain \( M = K \). This is a contradiction. \( \square \)

**Remark** The condition of Proposition 1.3 fulfills for \( \mathcal{D} \)-rings with 1.

**Proposition 1.4** Let \( K \) be a \( \mathcal{D} \)-ring satisfying the ascending chain condition for \( \mathcal{D} \)-ideals. Then the following conditions are equivalent:

1. \( K^2 = K \);
2. Every \( \mathcal{D} \)-maximal \( \mathcal{D} \)-ideal of \( K \) is \( \mathcal{D} \)-prime.

**Proof.** (1) \( \Rightarrow \) (2) follows from Proposition 1.3.

(1) \( \Rightarrow \) (2): Assume that \( K^2 \neq K \). By Proposition 1.2 there exists a \( \mathcal{D} \)-maximal \( M \) of \( K \) such that \( K^2 \subseteq M \). It is contradiction since \( M \) is \( \mathcal{D} \)-prime. \( \square \)

For an element \( a \in K \) denote by \( [a] \) the smallest \( \mathcal{D} \)-ideal containing \( a \).

Every sequence \( \{x_0, x_1, \ldots, x_n, \ldots\} \), where \( x_0, x_{n+1} \in [x_n]^2 \), will be called a \( \mathcal{D} \)-sequence of the element \( a \).

**Definition** An element \( a \in K \) is strongly \( \mathcal{D} \)-nilpotent if every its \( \mathcal{D} \)-sequence is ultimately zero.

**Remark** This definition is a generalization of differential rings of the similar definition in ([5], p.55; [1], p.574).

Denote by \( n(0) \) the set of all strongly \( \mathcal{D} \)-nilpotent elements of \( K \), where \( 0 \) is zero ideal of \( K \).

**Theorem 1.5** For any \( \mathcal{D} \)-ring \( K \) the equalities \( n(0) = r^*(0) = r^w(0) = R(0) \) hold.

**Proof.** First we prove that \( n(0) \subseteq r^*(0) \). If there are no \( \mathcal{D}_s \)-semiprime \( \mathcal{D} \)-ideals then \( r^*(0) = K \). Hence \( n(0) \subseteq r^*(0) \). Assume that there is a \( \mathcal{D}_s \)-semiprime \( \mathcal{D} \)-ideal. Let
$a \in n(0)$ and $S$ be a $D_s$-semiprime $D$-ideal. Prove that $a \in S$. Assume that $a \not\in S$. Then $[x_0] \not\subseteq S$, where $x_0 = a$. There exists $x_1 \in [x_0]^2$ such that $x_1 \not\in S$. Continuing in this manner we obtain a $D$-sequence $\{x_0, x_1, \ldots, x_n, \ldots\}$ of the element $a$ such that $x_n \not\in S$ for all $n$. But it is a contradiction since every $D$-sequence of the element $a$ is ultimately zero.

Thus $a \in S$ and $a \in r^*(0)$ since $S$ is any $D_s$-semiprime $D$-ideal. Hence $n(0) \subseteq r^*(0) \subseteq r^w(0) \subseteq R(0)$.

Prove that $R(0) \subseteq n(0)$. If $n(0) = K$ then $n(0) = r^*(0) = r^w(0) = R(0) = K$.

Let $n(0) \not= K$. Let $b \in K$ such that $b \not\in n(0)$. Then there exists a $D$-sequence $X = \{x_0, x_1, \ldots, x_n, \ldots\}$ of the element $b$ such that $X \cap 0 = \emptyset$, where 0 is the zero ideal of $K$.

Denote by $\sum$ the set of $D$-ideals $M$ of $K$ such that $X \cap M = \emptyset$. Then $\sum$ is not empty since $0 \in \sum$. We can apply Zorn’s lemma to the set $\sum$; so there exists a maximal element $P$ of $\sum$. Show that $P$ is $D$-prime.

First, $P$ is proper since $b \in P$. Let $B, C \in Id(K)$, $B \not\subseteq P$, $C \not\subseteq P$. Then $P \cap B \not= P$ and $P \cap C \not= P$. By the maximality of $P$ in $\sum$ we have $P \cap B \not\subseteq \sum$ and $P \cap C \not\subseteq \sum$. Hence there exist $x_m \in X$, $x_q \in X$ such that $x_m \in P \cap B$, $x_q \in P \cap C$. Then

$$[x_m] \subseteq P \cap M, \quad [x_q] \subseteq P \cap C.$$  

Hence

$$x_{m+1} \in [x_m]^2 \subseteq P \cap B, \quad x_{q+1} \in [x_q]^2 \subseteq P \cap C.$$ 

Continuing in this manner we find that

$$x_{m+t} \in P \cap B, \quad x_{q+t} \in P \cap C$$

for all $t$. Put $n = \max(m, q)$. Then

$$x_n \in P \cap B, \quad x_n \in P \cap C.$$  

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Hence

\[ x_{n+1} \in [x_n]^2 \subseteq (P \lor B)(P \lor C) \subseteq P \lor B \cdot C, \]

by Proposition 1.1. But \( x_{n+1} \notin P \). Hence \( B \cdot C \notin P \). Therefore \( P \) is \( D \)-prime. Thus there exists a \( D \)-prime \( D \)-ideal \( P \) such that \( b \notin P \). Then \( n(0) = r^s(0) = r^w(0) = R(0) \).

The \( D \)-ideal \( R(0) \) will be called differential prime radical of \( K \) and will be denoted by \( DPr\text{-}rad(K) \).

**Definition** \( D \)-ideal \( H \) of \( K \) is \( D \)-radical if \( H = R(H) \).

For \( A \in \text{Id}(K) \) denote by \( n(A) \) the set of all elements \( x \in K \) that every \( D \)-sequence of \( x \) meets \( A \).

**Corollary 1** For any \( A \in \text{Id}(K) \) the following equalities hold:

\[ n(A) = r^s(A) = r^w(A) = R(A). \]

In particular, every \( D \)-semiprime \( D \)-ideal is \( D \)-radical.

**Proof.** Applying theorem 1.5 to the quotient \( D \)-ring \( K/A \), we obtain \( n(A) = r^s(A) = r^w(A) = R(A) = DPr\text{-}rad(K/A) \).

**Corollary 2.** For a \( D \)-ring \( K \) the following conditions are equivalent:

1. Every \( D \)-ideal of \( K \) is \( D \)-radical;
2. \( A \cdot B = A \cap B \) for all \( A, B \in \text{Id}(K) \);
3. \( [a]^2 = [a] \) for all \( a \in K \).

**Proof.** We use the following lemma:
Lemma Let $K$ be a $D$-ring. Then

$$R(A \cdot B) = R(A \cap B) = R(A) \cap R(B)$$

for any $A, B \in \text{Id}(K)$. The proof of this lemma follows from proposition 1.6 in [8],

(1) $\Rightarrow$ (2) If every $D$-ideal of $K$ is $D$-radical then using the lemma we obtain

$$A \cdot B = R(A \cdot B) = R(A) \cap R(B) = A \cap B.$$  

(2) $\Rightarrow$ (3) Let $A \cdot B = A \cap B$ for any $A, B \in \text{Id}(K)$. Then $A^2 = A$ for any $A \in \text{Id}(K)$.

(3) $\Rightarrow$ (1) : Prove that every $D$-ideal of $K$ is $D_s$-semiprime. Let $A$ be a $D$-ideal of $K$. Then $A = \bigvee_{a \in A}[a]$. Using proposition 1.1 we have

$$A^2 = (\bigvee_{a \in A}[a])^2 = (\bigvee_{a \in A}[a]^2) \vee (\bigvee_{a,b \in A}[a] \cdot [b]) = (\bigvee_{a \in A}[a]) \vee (\bigvee_{a,b \in A}[a] \cdot [b]) = \bigvee_{a \in A}[a] = A,$$

since $[a] \cdot [b] \subseteq [a] \cap [b]$ for any $a, b \in A$. Thus $A^2 = A$ for any $A \in \text{Id}(K)$. Let $B^2 \subseteq A$, $B \in \text{Id}(K)$. Then $B = B^2 \subseteq A$. Therefore every $D$-ideal $A$ is $D_s$-semiprime. By Corollary 1 $A$ is $D$-radical.

Remark This corollary is a generalization of the similar theorem in ([7], ch.4, §5).

Let $A \in \text{Id}(K)$. Put $A^{(0)} = A$, $A^{(n+1)} = (A^{(n)})^2$.

Corollary 3 For a $D$-ring $K$ the following conditions are equivalent:

1. $\mathcal{D} {\text{Pr-rad}}(K) = 0$
2. If $A^{(n)} = 0$, $A \in \text{Id}(K)$, for some $n$ then $A = 0$;
3. If $A^2 = 0$, $A \in \text{Id}(K)$, then $A = 0$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious.

Prove that (3) $\Rightarrow$ (1). Condition (3) implies that $r^n(0) = 0$. By theorem 1.5 we see $\text{Pr-rad}(K) = r^n(0) = 0$. □
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**Definition** A $\mathcal{D}$-ideal $A$ of $K$ is $s$-nilpotent if $A^{(n)} = 0$ for some $n$.

**Proposition 1.6** Let $K$ be a $\mathcal{D}$-ring and $A, B \in \text{Id}(K)$, $A \subseteq B$. If $A$ is $s$-nilpotent and $B/A$ is $s$-nilpotent in $K/A$ then $B$ is $s$-nilpotent in $K$.

**Proof.** Since $B/A$ is $s$-nilpotent the $B^{(n)} \subseteq A$ for some $n$. Then $B^{(n+m)} = (B^{(n)})^{(m)} = 0$ since $A$ is $s$-nilpotent. $\Box$

**Theorem 1.7** Let $K$ be a $\mathcal{D}$-ring satisfying the ascending chain condition for $\mathcal{D}$-ideals. The $\mathcal{DPr}.rad(K)$ is $s$-nilpotent.

**Proof.** Denote by $\sum$ the set of $s$-nilpotent $\mathcal{D}$-ideals of $K$. $\sum$ is not empty since $0 \in \sum$. There exists a maximal element $P$ in $\sum$. By proposition 1.6 the $\mathcal{D}$-ring $K/P$ have the following property: if $(A/P)^2 = 0$, $A \in \text{Id}(K)$, $P \subseteq A$, then $A/P = 0$. By corollary 3 of theorem 1.5 we have $\mathcal{DPr}.rad(K/P) = 0$. Thus means that $\mathcal{DPr}.rad(K) \subseteq P$. But $P \subseteq \mathcal{DPr}.rad(K)$ since $P$ is $s$-nilpotent. Therefore $\mathcal{DPr}.rad(K) = P$. $\Box$

**Corollary** Let $K$ be a $\mathcal{D}$-ring satisfying the ascending chain condition for $\mathcal{D}$-ideals. Then the followings are equivalent:

1. $K^{(n)} = 0$ for some $n$.
2. $K$ has not a $\mathcal{D}$-prime $\mathcal{D}$-ideal;
3. $K$ has not a $\mathcal{D}_s$-semiprime $\mathcal{D}$-ideal.

Denote by $\text{Id}_r(K)$ the set all $\mathcal{D}$-radical $\mathcal{D}$-ideals of $K$. $\text{Id}_r(K)$ is a complete lattice with respect to the inclusion relation. Denote by $\lor$ and $\land$ the lattice operations in $\text{Id}_r(K)$.

**Theorem 1.8** Let $K$ be a $\mathcal{D}$-ring. Then the lattice $\text{Id}_r(K)$ satisfies the infinite $\land$-distributive condition:

\[ A \land (\lor_{\tau \in T} B_{\tau}) = \lor_{\tau \in T} (A \land B_{\tau}) \]

for any $A, B \in \text{Id}_r(K)$. In particular, $\text{Id}_r(K)$ is distributive.

A proof follows from Theorem 1.3 in [8].
**Theorem 1.9** Let $K$ be a $\mathcal{D}$-ring satisfying the ascending chain condition for $\mathcal{D}$-ideals. Then any $\mathcal{D}$-radical $\mathcal{D}$-ideal $A$ is an intersection of finite $\mathcal{D}$-prime $\mathcal{D}$-ideals and a such representation of $A$ is unique.

**Proof.** First prove the following.

**Lemma** $A \in \text{Id}_r(K)$ is a $\mathcal{D}$-prime $\mathcal{D}$-ideal iff $A$ is an $\wedge$-indecomposable element of the lattice $\text{Id}_r(K)$.

**Proof.** Let $A$ be a $\mathcal{D}$-prime $\mathcal{D}$-ideal of $K$ and $A = A_1 \land A_2$, $A_1, A_2 \in \text{Id}_r(K)$. Then

$$A_1 \land A_2 \subseteq A_1 \cap A_2 \subseteq R(A_1 \land A_2) = A_1 \land A_2 = A.$$ 

Hence $A_1 \subseteq A$ or $A_2 \subseteq A$ since $A$ is $\mathcal{D}$-prime. Then $A = A_1$ or $A = A_2$.

Let $A$ be an $\wedge$-indecomposable element of the lattice $\text{Id}_r(K)$ and $B \cdot C \subseteq A$, $B, C \in \text{Id}_r(K)$. Then $R(B \cdot C) \subseteq A$. By lemma 1.6 in [8] we have $R(B) \land R(C) = R(B \cdot C) \subseteq A$. We obtain

$$A = A \lor (R(B) \land R(C)) = (A \lor R(B)) \land (A \lor R(C))$$

since $\text{Id}_r(K)$ is distributive. Hence $A = A \lor R(B)$ or $A = A \land R(C)$ since $A$ is an $\wedge$-indecomposable. This means that $B \subseteq R(B) \subseteq A$ or $C \subseteq R(C) \subseteq A$. Thus $A$ is $\mathcal{D}$-prime. The lemma is proved.

By the lemma and the corollary in ([4], p.183), we obtain the every $\mathcal{D}$-radical $\mathcal{D}$-ideal of $K$ is an intersection of finite $\mathcal{D}$-prime $\mathcal{D}$-ideals of $K$ and such a representation is unique. □

**Remark** This theorem is a generalization of the similar statement from the theory of associative rings.

Let $A \in \text{Id}(K)$. Put $N_0(A) = A$. Denote by $N_1(A)$ the supremum of all $B \in \text{Id}(K)$ such that $B^{(n)} \subseteq A$ for some $n$ ($n$ depends from $B$). Put $N_\alpha(A) = N_1(N_\beta(A))$ for $\alpha = \beta + 1$ and $N_\alpha(A) = \bigvee_{\beta < \alpha} N_\beta(A)$ for $\alpha$ a limit ordinal.

Put $L(K, \mathcal{D}) = N_\alpha(0)$ for any ordinal $\alpha$ of cardinality $\geq |K|$. 

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Theorem 1.10 \( L(K, \mathcal{D}) = \mathcal{D} \text{Pr.rad}(K) \) for \( \mathcal{D} \)-ring \( K \).

Proof. By theorem 1.5 it is enough to prove that \( L(K, \mathcal{D}) = r^+(0) \).

It is clear that \( N_1(0) \subseteq r^+(0) \). By transfinite induction we obtain \( N_\alpha(0) \subseteq r^+(0) \) for any ordinal \( \alpha \). Hence \( L(K, \mathcal{D}) \subseteq r^+(0) \).

Prove that \( L(K, \mathcal{D}) \) is \( \mathcal{D}_s \)-semiprime. Assume that \( B^2 \subseteq L(K, \mathcal{D}) \). Then there exists an ordinal \( \alpha \) such that \( B^2 \subseteq N_\alpha(0) \). Hence \( B \subseteq N_{\alpha+1}(0) \) by definition of \( N_{\alpha+1}(0) \). Therefore \( B \subseteq L(K, \mathcal{D}) \) and \( L(K, \mathcal{D}) \) is \( \mathcal{D}_s \)-semiprime.

By the definition of \( r^+(0) \) we have \( L(K, \mathcal{D}) = r^+(0) \).

2. \( \mathcal{D} \)-algebras over the field of rational numbers

Let \( \mathbb{Q} \) be the field of rational numbers. Put \( \mathcal{D} \text{Pr.rad}(K) = \text{Pr.rad}(K) \) and \( L(K, \mathcal{D}) = L(K) \) if \( \mathcal{D} = \emptyset \).

\( \mathcal{D} \)-sequence of \( a \in K \) will be called \( n \)-sequence of \( a \) if \( \mathcal{D} = \emptyset \).

For \( a \in K \) we denote its \( n \)-sequence \( \{x_0(a) = a, x_1, \ldots, x_m, \ldots\} \) in the form:

\[ \{x_0(a) = a, x_1(a), \ldots, x_m(a), \ldots\}. \]

If \( R \) is associative then every element \( x_m(a) \in [x_{m+1}(a)]^2 \subseteq [a]^{2m} \) is a finite sum:

\[ x_m(a) = \sum f_1 f_2 \cdots f_s, \]

where \( s = 2^m \) and every \( f_i \) has the form \( f_i = r_1 a r_2 i \). Thus \( x_m(a) \) is a homogeneous polynomial of \( a \) a degree \( 2^m \) with coefficients from \( R \).

If \( R \) is nonassociative then every element \( x_m(a) \) is a homogeneous nonassociative polynomial of degree \( 2^m \) with coefficients from \( R \).

Theorem 2.1 If \( R \) is \( \mathbb{Q} \)-algebra with 1 and \( \delta \) is a differentiation of \( R \), then

\[ \delta(\text{Pr.rad}(R)) \subseteq \text{Pr.rad}(R). \]

Proof. Let \( a \in \text{Pr.rad}(R) \). We consider any \( n \)-sequence of the element \( \delta a \):

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\{x_0(\delta a) = \delta a, x_1(\delta a), \ldots, x_m(\delta a), \ldots\}.

We have

\[\delta^{2m}(x_m(a)) \in q \cdot x_m(\delta a) + [a],\]

where \(q \neq 0\) is an integer. Then \(x_m(\delta a) \in \frac{1}{q}\delta^{2m}(x_m(a)) + [a]\). Since \(a \in \text{Pr.rad}(R)\) then there exist \(m_0\) such that \(x_m(a) = 0\) for all \(m \geq m_0\). Then the following sequence

\[\{y_0(a) = a, y_1(a), \ldots, y_k(a), \ldots\},\]

where \(y_k(a) = x_{m_0+k}(\delta a)\), is an \(n\)-sequence of the element \(a\). Therefore there exists \(k_0\) such that \(y_k(a) = 0\) for all \(k \geq k_0\). This means that any \(n\)-sequence of the form

\[\{x_0(\delta a) = \delta a, x_1(\delta a), \ldots, x_m(\delta a), \ldots\}\]

of element \(\delta a\) is ultimately zero. Thus \(\delta a \in \text{Pr.rad}(R)\).

\[\square\]

**Remark** This theorem is the Ritt’s theorem if \(R\) is a commutative associative \(\mathbb{Q}\)-algebra ([6], p.12). If \(R\) is an associative \(\mathbb{Q}\)-algebra then there is only the formulation of this theorem and outline of its proof in ([10], p.207).

But this instruction is not correct (the member \(n\) in proposition 2.6.28 in ([10], p.207) depends for \(r_{1i}, r_{2i}\)). By this instruction the theorem may be proved for only Noetherian rings.

Further we investigate connections between radical \(\mathcal{D}\)-ideals and \(\mathcal{D}\)-radical \(\mathcal{D}\)-ideals in \(\mathcal{D}\)-rings.

If \(\mathcal{D} = 0\) then in §1 we obtain results for usual rings.

**Corollary** Let \(K\) be a differential \(\mathbb{Q}\)-algebra with 1. Then radical of any \(\mathcal{D}\)-ideal of \(K\) is a \(\mathcal{D}\)-ideal of \(K\).

**Proof.** Let \(H\) be a \(\mathcal{D}\)-ideal of \(K\). Then by Theorem 2.1 \(\text{Pr.rad}(K/H)\) is a \(\mathcal{D}\)-ideal. This means that \(\text{rad}(H)\) is a \(\mathcal{D}\)-ideal of \(K\). \[\square\]
Let $K$ be a ring. For $a \in K$ denote by $(a)$ the intersection of all ideals of $K$ containing $a$.

For ideals $A, B$ of $K$ denote by $A \ast B$ the intersection of all ideals of $K$ containing the set $\{x \in K : x = a \cdot b, \ a \in A, \ b \in B\}$.

**Proposition 2.2** Let $K$ be a ring and $B$ be an ideal of $K$. Then $B$ is prime iff for every $t_1, t_2 \in K \setminus B$ such that $(t_1) \ast (t_2)$.

A proof is obvious.

**Theorem 2.3** Let $H$ be a radical $\mathcal{D}$-ideal of a $\mathcal{D}$-ring $K$. Then $H$ is an intersection of $\mathcal{D}$-prime $\mathcal{D}$-ideals.

**Proof.** Let $x \notin H$. Then there exists a prime ideal $B$ of $K$ such that $H \subseteq B$ and $x \notin B$.

Denote by $\sum$ the set of $\mathcal{D}$-ideals $A$ of $K$ such that $H \subseteq A$ and $A \cap (K \setminus B) = \emptyset$. $\sum \neq \emptyset$ since $H \in \sum$. By Zorn’s lemma there exists a maximal element $P$ in $\sum$. Prove that $P$ is a $\mathcal{D}$-prime $\mathcal{D}$-ideal. $P$ is proper since $x \notin P$.

Let $A_1, A_2 \in \text{Id}(K)$ and $A_1 \not\subseteq P$, $A_2 \not\subseteq P$. Then $P \vee A_1 \neq P$, $P \vee A_2 \neq P$. There exist $t_1 \in K \setminus B$, $t_2 \in K \setminus B$ such that $t_1 \in P \vee A_1$, $t_2 \in P \vee A_2$. Then

$$(t_1) \subseteq P \vee A_1, \quad (t_2) \subseteq P \vee A_2.$$  

By proposition 1.1 we have

$$(t_1) \ast (t_2) \subseteq (P \vee A_1) \cdot (P \vee A_2) \subseteq P \vee A_1 \cdot A_2.$$  

By proposition 2.2 for $t_1, t_2 \in K \setminus B$ there exists $t \in K \setminus B$ such that $t \in (t_1) \ast (t_2)$. Then $t \in P \vee A_1 \cdot A_2$. This means that $A_1 \cdot A_2 \not\subseteq P$. Therefore $P$ is a $\mathcal{D}$-prime $\mathcal{D}$-ideal and $H \subseteq P$. Thus for any $x \notin H$ there exists a $\mathcal{D}$-prime $\mathcal{D}$-ideal $P$ such that $H \subseteq P$ and $x \notin P$. This means that $H$ is an intersection of all $\mathcal{D}$-prime $\mathcal{D}$-ideals containing $H$. \qed
Theorem 2.4 Let $K$ be a differential $Q$-algebra with 1. Then $Pr\text{-rad}(K)$ is an intersection of all $D$-prime $D$-ideals of $K$ containing $Pr\text{-rad}(K)$ and $DPr\text{-rad}(K) \subseteq Pr\text{-rad}(K)$.

Proof. By theorem 2.1 $Pr\text{-rad}(K)$ is a $D$-ideal of $K$. By theorem 2.3 $Pr\text{-rad}(K)$ is an intersection of all $D$-prime $D$-ideals of $K$ containing $Pr\text{-rad}(K)$. Hence $DPr\text{-rad}(K) \subseteq Pr\text{-rad}(K)$. \(\square\)

Theorem 2.5 Let $K$ be a differential $Q$-algebra with 1. Assume that $K$ satisfies the ascending chain condition for ideals. Then $DPr\text{-rad}(K) = Pr\text{-rad}(K)$.

Proof. In this case $Pr\text{-rad}(K)$ is $s$-nilpotent by theorem 1.7 $D = \emptyset$. Therefore $Pr\text{-rad}(K))^{(n)} = 0$ for some $n$. Since $Pr\text{-rad}(K)$ is a $D$-ideal $[Pr\text{-rad}(K)]^{(r)} = (Pr\text{-rad}(K))^{(n)} = 0$. Then by theorem 1.10 $Pr\text{-rad}(K) \subseteq DPr\text{-rad}(K)$. Thus $DPr\text{-rad}(K) = Pr\text{-rad}(K)$. \(\square\)

Corollary Let $K$ be a differential $Q$-algebra with 1. Assume that $K$ satisfies the ascending chain condition for ideals. Then every $D$-radical $D$-ideal of $K$ is radical.

Proof. The statement follows from theorem 2.5. \(\square\)

Remark Theorems 2.3-2.5 are known for commutative differential rings [1].

References


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