THE DUAL OF THE BOCHNER SPACE $L^p(\mu, E)$ FOR ARBITRARY $\mu$

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Abstract

Let $\mu$ be a finite measure, $E$ a Banach space, and $1 \leq p < \infty, 1 < q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. It is known that $L^p(\mu, E)^* \simeq L^q(\mu, E^*)$ if, and only if, $E^*$ has the Radon-Nikodým property with respect to $\mu$. The aim of this article is to generalize the above result to arbitrary measures.

Let $(\Omega, \mathcal{A}, \mu)$ be a positive* measure space, and $E$ a Banach space. If there is no possibility of ambiguity about the underlying measurable space $(\Omega, \mathcal{A})$, for any $1 \leq p \leq \infty$, $L^p(\mu, E)$ will denote the Bochner space $L^p(\Omega, \mathcal{A}, \mu, E)$. For definitions and properties of these spaces we refer to [4]. For two Banach spaces $E$ and $F$, $E \simeq F$ will mean that they are linearly isometric. $E^*$ will denote the topological dual of $E$.

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, $E$ a Banach space, and let $1 \leq p < \infty, 1 < q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $f \in L^p(\mu, E)$ and $g \in L^p(\mu, E^*)$, the function $\langle f, g \rangle$ defined on $\Omega$ by

$$\langle f, g \rangle(\omega) = \langle f(\omega), g(\omega) \rangle = g(\omega)(f(\omega)), \quad \omega \in \Omega,$$

is integrable, and for any fixed $g \in L^q(\mu, E^*)$ the mapping $\phi_g$ defined on $L^p(\mu, E)$ by

$$\phi_g(f) = \int_{\Omega} \langle f, g \rangle \, d\mu, \quad f \in L^p(\mu, E),$$

* Throughout this article all scalar-valued measures are assumed to be positive.
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is a bounded functional of $L^p(μ, E)$ with norm equals $∥g∥_q$. Thus, the mapping $g → φ_g$
is a linear isometry from $L^p(μ, E^*)$ into $L^p(μ, E)^*$.

It is known that the above mentioned isometry $g → φ_g$ is surjective if, and only
if, $E^*$ has the Radon-Nikodym property with respect to $μ$, that is, each $μ$-continuous,
$E^*$-valued measure of bounded variation on $A$ to $E^*$ can be represented (via integral)
by an $E^*$-valued $μ$-integrable function. (This theorem is due to Bochner and Taylor [1]
for the Lebesgue measure on the interval $[0,1]$. It was generalized to $σ$-finite measures by
Gretsky and Uhl[5]. An excellent proof of it can be found in [4, pp. 98-100].)

In [3], Cengiz proves that the preceding theorem can be generalized to arbitrary
measures, but at a price. It is proved that for an arbitrary measure $μ$, if $E^*$ is separable
(hence has the Radon-Nikodym property with respect to $μ$ [4, p. 79]), then $L^p(μ, E)^* ≅
L^q(μ, E^*)$ still holds for $1 < p < ∞$. However, it may fail for $p = 1$ even in the scalar
case (see [6, p. 349]). Instead, we have $L^1(μ, E)^* ≅ L^∞(ν, E^*)$ for some perfect measure
$ν$ on an extremely disconnected locally compact Hausdorff space.

In this article we shall replace the separability condition on $E^*$ by the Radon-Nikodym
property with respect to $μ$. But first, we give some details about the perfect
measure $ν$ mentioned above.

We recall that a Borel measure $μ$ on an extremely disconnected locally compact
Hausdorff space is perfect if every nonempty open set has positive measure, every
nowhere dense Borel set has measure zero, and every nonempty open set contains another
nonempty open set with finite measure (see [2]).

It is proved in [3] that any arbitrary measure space $(T; ∑, λ)$ can be replaced
by a perfect measure space $(Ω, A, ν)$ in the sense that $L^p(λ, E) ≃ L^p(ν, E)$ for every
$1 ≤ p < ∞$ and every Banach space $E$. But $L^∞(ν, E)$ may be enlarged, that is, $L^∞(λ, E)$
is isometric to a subspace of $L^∞(ν, E)$.

Some other additional nice properties of this new measure space $(Ω, A, ν)$ are as follows:

i) $Ω$ is the topological direct sum of a family $\{Ω_i : i ∈ I\}$ of extremely disconnected
compact Hausdorff spaces $Ω_i$, that is, $Ω = ∑_i ⊕ Ω_i$, the spaces $Ω_i$ are mutually
disjoint and the topology on $Ω$ is the weakest topology containing the topologies of
$Ω_i, i ∈ I$. 

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The algebra $A$ contains the Borel algebra. A set $A$ belongs to $A$ if, and only if, $A \cap \Omega_i$ belongs to $A$ for all $i \in I$.

The restriction of $\nu$ to each $\Omega_i$ is a regular Borel measure on $\Omega_i$.

Each $\sigma$-finite measurable set is contained a.e. in the union of a countable subfamily of $\{\Omega_i : i \in I\}$.

$\nu(A) = \sum_i \nu(A \cap \Omega_i)$ for all $A \in A$. Thus every locally null set is actually null.

In view of the above discussion we may, and will assume that the given measure space $(\Omega, A, \mu)$ is perfect and prove the following theorem.

**Theorem** Let $(\Omega, A, \mu)$ be a perfect measure space and $E$ and Banach space. Then, for any $1 \leq p < \infty$, $1 < q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $L^p(\mu, E)^* \simeq L^q(\mu, E^*)$ if, and only if $E^*$ has the Radon-Nikodym property with respect to $\mu$; the isometry being the mapping $g \to \phi_g$, $g \in L^q(\mu, E^*)$.

**Proof.** Let us assume that $E^*$ has the Radon-Nikodym property with respect to $\mu$, and write $\Omega = \sum_i \oplus \Omega_i$. Then, since the theorem is true for finite measures, for each $i \in I$, $L^p(\Omega_i, E)^* \simeq L^q(\Omega_i, E^*)$. Now let $\psi \in L^p(\mu, E)^*$. Then for each $i \in I$, there is a $g_i \in L^q(\Omega_i, E^*)$ such that

$$\psi_i(f) = \int_{\Omega} \langle f, g_i \rangle d\mu \quad \text{for all} \quad f \in L^p(\Omega_i, E),$$

and $\| \psi_i \| = \| g_i \|_q$, where $\psi_i$ denotes the restriction of $\psi$ to the subspace $L^p(\Omega_i, E)$ of $L^p(\mu, E)$.

For any finite subset $J$ of $I$ let

$$\Omega_J = \bigcup_{j \in J} \Omega_j \quad \text{and} \quad g_J = \sum_{j \in J} g_i.$$ 

Since the functions $g_i$ have disjoint supports, it follows that

$$\psi_J(f) = \int_{\Omega} \langle f, g_j \rangle d\mu \quad \text{for} \quad f \in L^p(\Omega_J, E),$$

where $\psi_J$ denotes the restriction of $\psi$ to $L^p(\Omega_J, E)$.  

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If \( p = 1 \), then \( g = \sum_i g_i \) is locally measurable, (i.e., its restriction to each measurable set of finite measure is measurable), and

\[
\| g \|_{\infty} = \sup_i \| g_i \|_{\infty} \leq \psi
\]

which means that \( g \in L^\infty(\mu, E^*) \).

For \( p > 1 \), we have

\[
\sum_{j \in J} \| g_j \|_p = \| g_J \|_p \leq \| g \|_p \leq \| \psi \|_q,
\]

which shows that all but a countable number of the functions \( g_i \) are zero almost everywhere, and therefore, for the sake of simplicity, we may assume that \( I = \{1, 2, 3, \ldots\} \). Consequently, \( g = \sum_i g_i \) is measurable, and

\[
\| g \|_q = \sum_i \| g_i \|_q \leq \| \psi \|_q,
\]

which proves that \( g \in L^q(\mu, E^*) \).

Next we show that

\[
\psi(f) = \int_\Omega \langle f, g \rangle d\mu \text{ for all } f \in L^p(\mu, E).
\]

Let \( f \in L^p(\mu, E) \) and write \( f = \sum_i f_i \), where, for each \( i \in I \), \( f_i = f \) on \( \Omega_i \) and zero outside \( \Omega_i \). Since the support of an integrable function is \( \sigma \)-finite, and since each of \( \sigma \)-finite measurable set is contained in the union of a countable subfamily of \( \{\Omega_i : i \in I\} \), we may again assume that \( I = \{1, 2, 3, \ldots\} \).

For each \( n = 1, 2, 3, \ldots \), let \( h_n = \sum_{i=1}^n f_i \). Then, by the dominated convergence theorem, the sequence \( \langle h_n \rangle \) converges to \( f \) in \( L^p(\mu, E) \). It is clear that for each \( n \),

\[
\psi(h_n) = \int_\Omega \left( \sum_{i=1}^n \langle f_i, g_i \rangle \right) d\mu,
\]

and, since \( f_i \)'s as well as \( g_i \)'s have disjoint supports, it is also clear that
\[
\left| \sum_{i=1}^{n} (f_i(x), g_i(x)) \right| \leq \| f(x) \| \| g(x) \|
\]
for all \( x \in \Omega \) and \( n = 1, 2, 3, \ldots \). Therefore, by the dominated convergence theorem, and the fact that the \( \psi \) is continuous on \( L^p(\mu, E) \), we have

\[
\int_{\Omega} \langle f, g \rangle d\mu = \lim_{n} \int_{\Omega} \left( \sum_{i=1}^{n} (f_i, g_i) \right) d\mu = \lim_{n} \psi(h_n) = \psi(f)
\]
for all \( f \in L^p(\mu, E) \), proving our claim.

Conversely, we now assume that \( L^p(\mu, E)^* \simeq L^q(\mu, E^*) \) for some \( 1 \leq p < \infty \), and show that \( E^* \) has the Radon-Nikodym property with respect to \( \mu \). To this end we let \( \lambda : A \to E^* \) be a \( \mu \)-continuous vector measure of bounded variation. Since \( L^p(\Omega_i, E)^* \simeq L^q(\Omega_i, E^*) \) forall \( i \in I \), and the theorem holds for finite measures, for each \( i \in I \), there is an integrable function \( g_i : \Omega \to E^* \) that vanishes outside \( \Omega_i \) and such that

\[
\lambda(A_i) = \int_{A_i} g_i d\mu \text{ for all } A_i \in A_i,
\]
where \( A_i \) is the trace of \( A \) on \( \Omega_i \). Let \( g = \sum_i g_i \). Obviously \( g \) is locally measurable, but we want to show that it is indeed measurable (i.e., \( \mu \)-essentially separably valued). Since \( \lambda \) is of bounded variation, \( |\lambda|(\Omega) \) is finite which implies that \( |\lambda|(\Omega) = 0 \) for all but countably many \( i \in I \), where \( |\lambda| \) denotes the total variation of \( \lambda \). Thus, here again we may assume that \( I = \{ 1, 2, 3, \ldots \} \), which implies that \( g \) is measurable, and since \( |\lambda|(\Omega) = \int_{\Omega} \| g(\cdot) \| d\mu < \infty \), we conclude that \( g \) is integrable.

Since \( I \) is countable, now by the dominated convergence theorem, it follows that

\[
\lambda(A) = \int_{A} g d\mu
\]
for all \( A \in A \) completing the proof.

Since reflexive Banach spaces have the Radon-Nikodým property with respect to any finite measure [4, p. 76], and the preceding proof can be used to conclude that this
property with respect to any perfect measure, we have the following corollary.

\[ \text{Corollary. For any measure } \mu \text{ and reflexive Banach space } E, L^p(\mu, E)^* \simeq L^q(\mu, E^*) \]

where \(1 \leq p < \infty, 1 < q \leq \infty\) such that \(\frac{1}{p} + \frac{1}{q} = 1\).

**References**