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CESS-MODULES

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Abstract

In this paper, we investigate generalizations of CS-modules, namely CESS-modules, weak CS-modules and modules satisfying a condition (P). Several results are given to show the relationships between the classes of these modules.

Definitions and Notation

All modules are assumed to be unital right modules over a ring R containing an identity. If we let M be a module then $N \leq M$ will indicate that N is a submodule of M, while $N \leq_e M$ will indicate that N is an essential submodule of M. A complement (closed) submodule N of M, written as $N \leq_c M$, is one which has no proper essential extensions in M. We will write $N \leq_d M$ to indicate that N is a direct summand of M.

Given any $N \leq M$, by Zorn's Lemma there exist submodules L and K such that $N \leq_e L \leq_c M$ and K is maximal with respect to the property $N \cap K = 0$. In this case, L is called a closure of N in M and K is called a complement of N in M. Following [8], we say that M is a UC-module if each of its submodule has a unique closure in M. If every complement of M is a direct summand then M is called a CS-module (or extending module). CS-modules have been studied extensively and generalized in several ways (see [5], [6], [7], [8]). In this note we will be interested in the class of modules given in the following definitions.

(1) The module M is called a CESS-module if every complement in M with essential socle is a direct summand of M.

(2) The module M is called a weak CS-module if every semisimple submodule of M is essential in a direct summand of M.

(3) The module M is said to satisfy condition (P) if for any submodule N of M there exists a direct summand K of M such that $soc(K) \leq N \leq K$.

We will use Z and Q to denote the ring of integers and rationals, respectively.

Weak CS-Modules

For ease of reference, we begin with some known facts.

Lemma 1.1. *Every CS-module is a CESS-module, and every CESS-module is a weak CS-module.*

The following example shows that the converses of the statements in Lemma 1.1. are not true in general.

Example 1.1. Let p be a prime integer. Then the Z-modules $Z/Zp \oplus Z/Zp^3$ is a weak CS-module which is not a CESS-module (see [9]).

Example 1.2. Again let p be prime. Then the Z-module $M = (Z/Zp) \oplus Q$ is a CESS-module which is not a CS-module (see [10], Example 10).

Lemma 1.2. *Any direct summand of a CS-module (CESS-module) is also a CS(CESS)-module.*

Proof. This is clear from [9]. □

P.F. Smith has asked in [9, Question 1.4] whether every direct summand of a weak CS-module M is also weak CS. In Lemma 1.4, we answer this positively under the additional assumption that M is UC. First we record, for later use, a characterization of UC-modules.

Lemma 1.3. *For a module M, the following conditions are equivalent:*

(i) *M is a UC-module.*

(ii) *For any $K \leq_c M$ and $N \leq M$ we have $K \cap N \leq_c N$.*

(iii) *There does not exists an R-module X with a proper essential submodule Y such that the module $(X/Y) \oplus X$ embeds in M.*

For proof, see [8].

Lemma 1.4. *Let M be a UC-module. If M is a weak CS-module then every direct summand of M is also weak CS.*

Proof. Let $K \leq_d M$ and N be a semisimple submodule of K. Since M is weak CS, there exists a direct summand M_1 of M such that $N \leq_e M_1$. Let L denote the closure of N in K, so that $N \leq_e L \leq_c K$. Then (see for example [3], 1.10), we have $L \leq_c M$. Thus $N \leq_e M_1 \leq_c M$ and also $N \leq_e L \leq_c M$ and so, since M is UC, we have $L = M_1$. Hence the closure L of N in K is a direct summand of K showing that K is weak CS, as required. □

Next we look at the direct sum of two weak CS-modules.

Proposition 1.1. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are both weak CS-modules and M_1 is M_2-injective. Then M is a weak CS-module.*

Proof. Let N be a semisimple submodules of M. We prove N is essential in a direct summand of M by considering two cases.

Case 1. $N \cap M_1 = 0$.

In this case, by [4, Lemma 5] there exists a direct summand C of M such that C is isomorphic to $M_2, N \leq C$ and $M = M_1 \oplus C$. Then C is a weak CS-module, and so $N \leq_e K \leq_d C$ for some $K \leq C$ as required.

Case 2. $N \cap M_1 \neq 0$.

Let N' be a submodule of N such that $N = (N \cap M_1) \oplus N'$. Since M_1 is a weak CS-module, $N \cap M_1 \leq_e K_1 \leq_d M_1 = K_1 \oplus K_2$ for submodules K_1 and K_2 of M_1. Since $N' \cap M_1 = 0$, as in case (1) there exists $C_1 \leq_d M$ such that C_1 isomorphic to $M_2, N' \leq C_1,\ M = C_1 \oplus M_1$ and $C_1 = C_2 \oplus C_3$ with $N' \leq_e C_2$ for some submodules C_2, C_3 of C_1. Hence $K_1 \oplus C_2 \leq_d M$. Thus M is a weak CS-module. □

Lemma 1.5. *Let $M = M_1 \oplus M_2$ be a UC-module such that $Soc(M_1) \leq_e M_1$ and $Soc(M_2) = 0$. Then $Hom(K, M_1) = 0$ whenever $K \leq M_j$ with $\{i, j\} = \{1, 2\}$.*

Proof. Let K be a submodule of M_2 and suppose that $f : K \to M_1$ is a nonzero homomorphism. Then, since $Soc(M_1) \leq_e M_1$, $f(K)$ contains a simple submodule U. Set $L = f^{-1}(U) \cap \ker f$. Then L is a maximal submodule of $f^{-1}(U)$.

If L is not essential in $f^{-1}(U)$ then $f^{-1}(U) = L \oplus L_1$ for some simple submodule L_1 of M_2, a contradiction since $Soc(M_2) = 0$.Thus L must be essential in $f^{-1}(U)$. However, since $(f^{-1}(U)/L) \oplus f^{-1}(U)$ can be embedded in $M_1 \oplus M_2 = M$ this gives a contradiction by Lemma 1.3.(iii). Thus $Hom(K, M_1) = 0$.

On the other hand, if $K \leq M_1$ then it follows from the proof of (ii) $\Rightarrow$ (iii) of [2, Lemma 2.3] that $Hom(K, M_2) = 0$. □

Corollary 1.1. *Let $M = M_1 \oplus M_2$ be a UC-module such that $Soc(M_1) \leq_e M_1$ and $Soc(M_2) = 0$. Then M is weak CS-module if and only if M_1 and M_2 are weak CS.*

Proof. The necessity is clear from Lemma 1.4. and the sufficiency follows by Lemma 1.5 and Proposition 1.1. □

Corollary 1.2. *Let $M = M_1 \oplus M_2$ be a UC-module such that $Soc(M_1) \leq_e M_1$ and $Soc(M_2) = 0$. Then M is CS-module if and only if M_1 and M_2 are CS-modules.*

Proof. This is clear from Lemma 1.5. and [3, Theorem 8]. □

Corollary 1.3. *Let M be a UC-module with essential socle. Then the following statements are equivalent.*

(i) *M is weak CS-module.*

(ii) *M is CESS-module.*

(iii) *M is CS-module.*

Proof. This is clear from [2, Lemma 1.4] and Corollary 1.1. □

Lemma 1.6. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a direct sum of finite many uniform submodules M_i of M. Suppose that for any complement K in M there exists an i such that $K \cap M_i \neq 0$. If M is UC-module then M is CS-module.*

Proof. Let K be a complement in M and suppose, without loss of generality, that $K \cap M_1 \neq 0$. By Lemma 1.3 (ii), $K \cap M_1 \leq_c M_1$ and so, since M_1 is uniform $K \cap M_1 = M_1$. Thus $K = M_1 \oplus (K \cap (M_2 \oplus \cdots \oplus M_n))$ and by Lemma 1.3 (ii)

$$K \cap (M_2 \oplus \cdots \oplus M_n) \leq_c M_2 \oplus \cdots \oplus M_n$$

if we set $L = (M_2 \oplus \cdots M_n) \cap K$ then, since $M_2 \oplus \cdots \oplus M_n$ also satisfies our hypotheses, if $L \neq 0$, then we have $L \cap M_i \neq 0$ for some $i \geq 2$ and so $L \cap M_i = M_i \leq_d K$. Then repeating the argument, we get eventually that either $M = K$ or $K \leq_d M$. Hence M is a CS-module. □

Lemma 1.7. *Let M be a module such that $M/Soc(M)$ is simple. Then M is a CESS-module if and only if M is a CS-module.*

Proof. Assume that M is CESS and let K be a complement in M. By hypothesis $Soc(M)$ is maximal submodule of M and so either $K \leq Soc(M)$ or $K + Soc(M) = M$. In the former case, since M is CESS, we have $K \leq_d M$. In the latter case there exists a submodule B of $Soc(M)$ such that $Soc(M) = (K \cap Soc(M)) \oplus B$. Then $M = K + Soc(M) = K \oplus B$. Then M is a CS-module. □

Modules Satisfying Condition (P)

Let M denote the Z-module $(Z/Z2) \oplus Q$. Then M has uniform dimension two and it is well known that M is not a CS-module (see [7]). We now show that M is CESS-module but that it does not satisfy condition (P). Firstly, let K be a complement in M with $Soc(K) \leq_e K$. Since $Soc(M)$ is the simple submodule $Z/Z2$, we must have $Soc(M) = Soc(K)$ and that K is uniform module. It follows that $K \cap Q = 0$ and so $K \leq_d M$. Hence M CESS-module.

To prove that M does not satisfy (P), we assume to the contrary and let $K \leq_c M$. Then there exists a direct summand L of M such that $Soc(L) \leq K \leq L$. If $L = M$

then, as in the proceeding paragraph, K is a direct summand of M. So assume $L \neq M$. Then L has uniform dimension one, and so $K \leq_e L$. Thus $K = L$ and so $K \leq_d M$. It follows that M is CS-module, but this is a contradiction.

We now prove a more general result.

Lemma 2.1. *Let M be a module uniform dimension two such that $Soc(M)$ is a nonzero direct summand of M and M is not a CS-module. Then*

(i) M does not satisfy condition (P) and

(ii) M is CESS-module.

Proof. (i) By hypothesis $M = Soc(M) \oplus T$ for some non zero $T \leq M$. Assume to the contrary that M does not satisfy condition (P). Let K be a complement in M which is not a direct summand of M. Then there exists a submodules L, L_1 of M such that $Soc(L) \leq K \leq L \leq_d M = L \oplus L_1$. We now consider two cases.

Case 1. $L = M$. Here $Soc(M) \leq K \leq M$ and $Soc(M) \neq K$. Hence $K \cap T \neq 0$, and so $Soc(M) \oplus (K \cap T) \leq K$. Since M has dimension two, it follow that $K \leq_e M$. Thus $K = M$ and this a contradiction.

Case 2. $L \neq M$. Here L is uniform and so since $K \leq L$ and $K \leq_c M$, it follows that $K = L$, a summand. This a contradiction shows that M does not satisfy (P).

(ii) Let K be a complement in M with $Soc(K) \leq_e K$. Since $Soc(M) \leq_d M$ we have $Soc(K) \leq_d M$ and so $K = Soc(K) \leq_d M$.

This completes the proof. □

Theorem 2.1. *Let M be UC-module. Then the following conditions are equivalent.*

(i) M satisfy condition (P).

(ii) M is a CESS-module.

(iii) M is a weak CS-module.

Proof. (i) $\Rightarrow$ (ii). Let $K \leq_c M$ with $Soc(K) \leq_e K$. By (i) there exists a direct summand L of M such that $Soc(L) \leq K \leq L$. Then by [1, Proposition 10], $M = M_1 \oplus M_2$ where $Soc(M_1) \leq_e M_1, M_1$ is CS-module and $Soc(M_2) = 0$. Hence $Soc(K) = Soc(L) \leq M_1$. Since M_1 is CS-module, we can find a direct summand U of M such that $Soc(K) \leq_e U$. Then since M is UC, we get that $K = U$ and so $K \leq_d M$. Hence M is CESS.

(ii) $\Rightarrow$ (i). By [9, Corollary 1.6] $M = M_1 \oplus M_2$ where M_1 is a CS, $Soc(M_1) \leq_e M_1$ and $Soc(M_2) = 0$. Let K be a complement in M. We consider two cases.

Case 1. $Soc(K) = 0$. Since $Soc(M) = Soc(M_1) \leq_e M_1$, we have $K \cap M_2 \neq 0$. Moreover, $K \cap M_2 \leq_c M_2$ by Lemma 1.3 (ii). Then there exists $V \leq M_2$ such that

$V \oplus (K \cap M_2) \leq_e M_2$. Then $M_1 \cap (V \oplus K) = 0$ and $(M_1 \oplus V) \cap K = 0$. It follows that $M_1 \cap (K + M_2) = 0$, and so $K \leq M_2$.

Case 2. $Soc(K) \neq 0$. $Soc(K) = (Soc(M_1)) \cap K \leq K \cap M_1$. By Lemma 1.3 (ii) $K \cap M_1 \leq_c M_1$, and since M_1 is CS-module, $K \cap M_1 \leq_d M_1$, say $M_1 = (K \cap M_1) \oplus L$ for some submodule L of M_1. Setting $T = K \cap (L \oplus M_2)$ we have $K = (K \cap M_1) \oplus T$. Then T is a complement in M and $Soc(T) = 0$. As in Case 1 we may prove that T is contained in M_2. Hence $K \leq (K \cap M_1) \oplus M_2 \leq_d M$ with $Soc((K \cap M_1) \oplus M_2) \leq K$. Thus M satisfy (P).

(ii) $\Rightarrow$ (iii). This is clear from Lemma 1.1.

(iii) $\Rightarrow$ (ii). Let K be a complement in M with $Soc(K) \leq_e K$. Then there exist a direct summand L of M such that $Soc(K) \leq_e L$ by (iii). Since M is UC, $K = L$, a direct summand as required. This completes the proofs. □

Corollary 2.1. *Let R be a commutative Noetherian domain. Then R is Dedekind if and only if every UC-module over R satisfies condition (P).*

Proof. If M be a UC-module over a Dedekind domain R then by [2, Theorem 3.4] M is CESS-module and so satisfy (P) by Theorem 2.1. □

Conversely, if every UC-module over R satisfy (P) then, by Theorem 2.1, every UC-module is CESS-module. Hence R is Dedekind by [2, Theorem 3.4.]

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