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TIMELIKE RULED SURFACES IN THE MINKOWSKI 3-SPACE-II

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Abstract

This paper is devoted to a study of timelike ruled surfaces in three dimensional Minkowski space obtained by a spacelike straight line moving along a timelike curve. The central point, the curve of striction and the distribution parameter of a timelike ruled surface in Minkowski 3-space are considered, and some theorems relating to their structure are obtained. In addition, some results about developable timelike ruled surfaces are also given.

Introduction

A surface in the 3-dimensional Minkowski space $\mathbb{R}^3_1 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$ is called a timelike surface if the induced metric on the surface is a Lorentz metric [1]. If the tangent vector at every point of a given curve in $\mathbb{R}^3_1$ is a spacelike vector (timelike vector), then the given curve is called a spacelike curve (timelike curve) [2].

A ruled surface is a surface swept out by a straight line ℓ moving along a curve α. The various positions of the generating line ℓ are called the rulings of the surface. Such a surface, thus, has a parametrization in ruled form

$$\varphi(t, v) = \alpha(t) + vZ(t),$$

where we call α the base curve and Z the director vector of ℓ. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable surface. All other ruled surfaces are called skew surfaces. If there exists a common perpendicular to two preceding rulings of a skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the curve of striction. If there is a curve which meets perpendicularly each one of the rulings, then this curve is called an orthogonal trajectory of the ruled surface. In $\mathbb{R}^3_1$, we define the exterior product of vectors by $W \wedge V = -(i_V i_W dx \wedge dy \wedge dz)^{\#}$, where i_W denotes the interior product with respect to W and $\#$ stands for the operation of raisign indices by the metric $dx^2 + dy^2 - dz^2$. Here we choose the sign $\ll - \gg$ so that $\partial_x \wedge \partial_y = \partial_z$ holds.

The notation and fundamental concepts used in this study are the same as in [3].

1. Timelike Ruled Surfaces

Let

$$\begin{aligned} \alpha : \quad & I \to \mathbb{R}^3_1 \\ & t \to \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \end{aligned}$$

where $\{0\} \subset I$, be a differentiable timelike curve in Minkowski 3-space parameterized by arc-length. The tangent vector field of α will be denoted by T.

A spacelike straight line,

$$\begin{aligned} \ell : \quad & \mathbb{R} \to \mathbb{R}^3_1 \\ & v \to \ell(v) = (\alpha_1(t) + va_1(t), \alpha_2(t) + va_2(t), \alpha_3(t) + va_3(t)), \end{aligned}$$

where the scalars $a_i(t) \in \mathbb{R}$ for all $1 \leq i \leq 3$, are the components of the director vector at the point $\alpha(t)$, can be chosen so that the director vector of ℓ and the tangent vector of α are linearly independent at every point of the curve α.

As ℓ moves along α it generates a ruled surface given by the parametrization $(I \times \mathbb{R}, \varphi)$, where

$$\begin{aligned} \varphi : \quad & I \times \mathbb{R} \to \mathbb{R}^3_1 \\ & (t, v) \to \varphi(t, v) = (\alpha_1(t) + va_1(t), \alpha_2(t) + va_2(t), \alpha_3(t) + va_3(t)), \end{aligned}$$

which can be obtained in the Minkowski 3-space. This ruled surface will be denoted by M. An orthonormal base $\{T, X\}$ of $\chi(M)$, the space of tangent vector fields of M, can be obtained; thus, $N = T \wedge X$ where N is the unit normal vector field of M. Hence, $\{X, N, T\}$ is an orthonormal frame field along α in $\mathbb{R}^3_1$. Let D be the Levi-Civita connection on $\mathbb{R}^3_1$. The variation formulae of this system along α in $\mathbb{R}^3_1$ are

$$\begin{aligned} D_T X &= cN + aT \\ D_T N &= -cX + bT \\ D_T T &= aX + bN, \end{aligned}$$

where $a = -\langle T, D_T X\rangle = -T[\langle T, X\rangle] + \langle D_T T, X\rangle = \langle D_T T, X\rangle$, etc.

$$B = \begin{bmatrix} 0 & c & a \\ -c & 0 & b \\ a & b & 0 \end{bmatrix}$$

is a skew-adjoint matrix, since $B^T = -\epsilon B \epsilon$, where

$$\epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In view of the parametrization $\varphi(t,v)=\alpha(t)+vX(t)$ we have

$$E=\langle\frac{\partial\varphi}{\partial t},\frac{\partial\varphi}{\partial t}\rangle=-(1+av)^2+c^2v^2,\quad F=\langle\frac{\partial\varphi}{\partial t},\frac{\partial\varphi}{\partial v}\rangle=0,\quad G=\langle\frac{\partial\varphi}{\partial v},\frac{\partial\varphi}{\partial v}\rangle=1.$$

The induced metric on the ruled surface is a Lorentz metric in the case where E is negative.

$$\min\left\{-\frac{1}{a-c},-\frac{1}{a+c}\right\}\quad\text{and}\quad\max\left\{-\frac{1}{a-c},-\frac{1}{a+c}\right\}$$

are roots of E, where $c^2-a^2=\langle D_TX,D_TX\rangle$.
Note that:

1) If D_TX is a timelike vector field, then

$$-\infty<v<\min\left\{-\frac{1}{a-c},-\frac{1}{a+c}\right\}\quad\text{or}\quad\max\left\{-\frac{1}{a-c},-\frac{1}{a+c}\right\}<v<\infty.$$

2) If D_TX is a spacelike vector field, then

$$\min\left\{-\frac{1}{a-c},-\frac{1}{a+c}\right\}<v<\max\left\{-\frac{1}{a-c},-\frac{1}{a+c}\right\}.$$

3) Let D_TX be the null vector field on $\mathbb{R}^3_1$.
If $a>0$, then $v<-\frac{1}{2a}$, and if $a<0$, then $v>-\frac{1}{2a}$.

Therefore, in all three cases above, the domain of the parameter v is not the whole of $\mathbb{R}$ but is one of the above intervals. Let us denote the domain of v by J. If we fix the parameter v in J, then the curve

$$\begin{aligned}\varphi_v:\quad & I\times\{v\}\to M\\ & (t,v)\to\varphi_v(t,v)=\alpha(t)+vX(t)\end{aligned}$$

can be obtained on M. The tangent vector field of this curve is

$$A=(1+av)T+cvN.$$

2. Developable Timelike Ruled Surfaces

Let M be a timelike ruled surface. Along any ruling of M, if all of the tangent planes of M are the same (coincide) then we call M as a developable surface.

Theorem 1. *Let M be a timelike ruled surface. The tangent planes along any ruling of M coincide if and only if $c = 0$.*

Proof. Trivial. □

Now, we will a criterion for timelike ruled surfaces to be developable in $\mathbb{R}^3_1$.

Corollary 1. *The timelike ruled surface M is developable if and only if $c = 0$.*

Lemma 1. $c = -\det(T, X, D_T X)$ for the timelike ruled surface M.

3. Position Vector of a Central Point

If the distance between the central point and the base curve of a skew timelike ruled surface is $\overline{u}$, then the position vector $\overline{\alpha}(t)$ can be expressed in the form

$$\overline{\alpha}(t, \overline{u}) = \alpha(t) + \overline{u}X(t),$$

where $\alpha(t)$ is the position vector of the base curve and $X(t)$ is the director vector belonging to the ruling. The parameter $\overline{u}$ can be expressed in terms of the position vector of the base curve and the directed vector of the ruling. Take three neighbouring rulings of a timelike ruled surface such that the first and second are $X(t)$ and $X(t)+dX(t)$ respectively. Let P, P' and Q, Q' be the feet on the rulings of the common perpendicular to two neighbouring rulings. The common perpendicular to $X(t)$ and $X(t) + dX(t)$ is $X(t) \wedge dX(t)$.

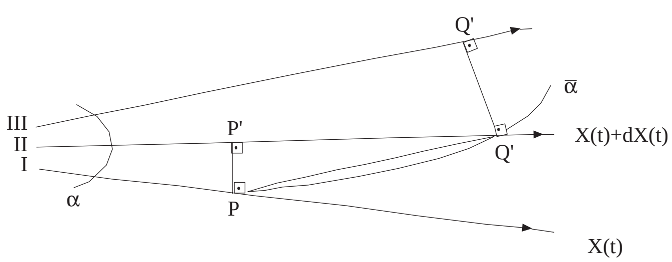

Figure 1.

The vector $\vec{PQ}$ coincides with the vector $\vec{PP'}$ in the limiting position, and $\vec{PQ}$ will be the tangent vector to the curve of striction. Thus, we have $\langle D_T X, \vec{PQ}\rangle = 0$. Therefore, we get

$$\overline{u} = -\frac{\langle T, D_T X\rangle}{\langle D_T X, D_T X\rangle} = \frac{a}{c^2 - a^2}.$$

Hence the curve of striction is given by

$$\overline{\alpha}(t) = \alpha(t) - \frac{\langle T, D_T X\rangle}{\langle D_T X, D_T X\rangle} X(t) \tag{1}$$

where $\langle D_T X, D_T X\rangle \neq 0$. $\overline{u} = \frac{a}{c^2-a^2}$ is constant since $\langle \frac{d\overline{\alpha}}{dt}, X\rangle = 0$.

Theorem 2. *The curve of striction $\overline{\alpha}$ does not depend on the choice of the base curve α for the skew timelike surface.*

Proof. Let β be another base curve of the skew timelike surface; that is, let, for all (t, v),

$$\varphi(t, v) = \alpha(t) + vX(t) = \beta(t) + sX(t)$$

for some function $s = s(v)$. Then from (1) we obtain

$$\overline{\alpha}(t) - \overline{\beta}(t) = \alpha(t) - \beta(t) - \frac{\langle T - \frac{d\beta}{dt}, D_T X\rangle}{\langle D_T X, D_T X\rangle} X(t) = 0$$

since $\langle X, D_T X\rangle = 0$. This proves our claim. □

Theorem 3. *Let M be a skew timelike surface. The point $\varphi(t, v_0)$ on the ruling through the point $\alpha(t)$ is the central point if and only if $D_T X$ is a normal vector of the tangent plane at $\varphi(t, v_0)$.*

Proof. Let $D_T X$ be a normal of the tangent plane at $\varphi(t, v_0)$ on the ruling through $\alpha(t)$. Thus $\langle D_T X, A\rangle = 0$. Hence, we get $v_0 = \frac{a}{c^2-a^2}$. Therefore, $\varphi(t, v_0)$ is the central point of M.

Conversely, let $\varphi(t, v_0)$ be the central point on the ruling through $\alpha(t)$. Then, we have $\langle D_T X, A\rangle = -a + (-a^2 + c^2)v = 0$.

On the other hand, $\langle D_T X, X\rangle = 0$. Therefore, $D_T X$ is a normal vector of the tangent plane at $\varphi(t, v_0)$. □

$D_T X$ is a spacelike vector at the central point since $D_T X$ is a normal vector of the tangent plane at the central point. Thus, $\langle D_T X, D_T X\rangle = -a^2 + c^2 > 0$.

Theorem 4. *The curve of striction of a skew timelike surface*

$$\overline{\alpha}(t)=\alpha(t)+\frac{a}{c^2-a^2}X(t)$$

is a timelike curve.

Proof. Straightforward calculation. □

Theorem 5. *Assume that M is a timelike ruled surface in $\mathbb{R}^3_1$. There exists a unique orthogonal trajectory of M through each point of M.*

Proof. Let

$$\begin{aligned}\varphi: \quad & I\times J\to\mathbb{R}^3_1\\ & (t,v)\to\varphi(t,v)=\alpha(t)+vZ(t),\end{aligned}$$

be a parametrization of M. An orthogonal trajectory of M is given by

$$\begin{aligned}\beta: \quad & \tilde{I}\to M\\ & s\to\beta(s)=\alpha(s)+f(s)Z(s),\end{aligned}$$

where $\langle Z(s),Z(s)\rangle=1$. We may assume that $\tilde{I}\subset I$. Since

$$\langle\frac{d\beta(s)}{ds},\quad Z(s)\rangle=0,$$

we obtain

$$f(s)=-\int\langle\frac{d\alpha(s)}{ds},\quad Z(s)\rangle ds+h,$$

where h is a real constant. Hence $h=f(s_0)-F(s_0)$, where

$$F(s)=-\int\langle\frac{d\alpha(s)}{ds},\quad Z(s)\rangle ds.$$

Therefore the orthogonal trajectory of M through the point P_0 is unique. Thus, we have $\tilde{I}=I$ since the orthogonal trajectory of M meets each one of the rulings of M. □

Theorem 6. *Suppose that M is a skew timelike surface. The longest distance between two rulings is the distance measured only on the curve of striction which is one of the orthogonal trajectories.*

Proof. Fixing two rulings, say for $t_1 < t_2$, we compute the length $j(v)$ of an orthogonal trajectory between these two rulings by

$$j(v) = \int_{t_1}^{t_2} \|A\| dt = \int_{t_1}^{t_2} \sqrt{|\langle A, A\rangle|} dt = \int_{t_1}^{t_2} [(a^2 - c^2)v^2 + 2av + 1]^{1/2}\, dt.$$

To find the value of s which maximizes $j(v)$, we use $\frac{\partial j(v)}{\partial v} = 0$ which gives $v = \frac{a}{c^2-a^2}$. This completes the proof. □

4. The Distribution Parameter of a Timelike Ruled Surface

Let the curve of striction be the base curve of a timelike ruled surface. Then $\overline{u} = 0$; that is, $\frac{a}{c^2-a^2} = 0$. Thus, we have $a = 0$. Hence, $D_T X$ and N are linearly dependent; that is, $\lambda D_T X = N$ where $D_T X = aT + cN$ and $N = T \wedge X = \frac{d\overline{\alpha}}{dt} \wedge X$. Thus, we obtain

$$\lambda = \frac{\langle T \wedge X, D_T X\rangle}{\langle D_T X, D_T X\rangle} = -\frac{\det(T, X, D_T X)}{\langle D_T X, D_T X\rangle}. \tag{2}$$

λ is called the distribution parameter of the timelike ruled surface, and is denoted by λ or P_X. Note that $\langle D_T X, D_T X\rangle \neq 0$ since $D_T X$ is a timelike vector field.

Theorem 7. *A timelike ruled surface is a developable surface if and only if the distribution parameter is zero.*

Proof. Straightforward. □

Theorem 8. *Let M be a timelike ruled surface in $\mathbb{R}^3_1$. Each one of the rulings of M is an asymptotic line and a geodesic in M.*

Proof. Each one of the rulings is geodesic in $\mathbb{R}^3_1$ since each one of the rulings is a straight line in $\mathbb{R}^3_1$. Thus, we have $D_X X = 0$. The Gaussian curvature is

$$D_X X = \overline{D}_X X + \langle S(X), X\rangle N$$

where $\overline{D}$ is the Levi-Civita connection on M, and S is the shape operator of M derived from N. Furthermore, $(\overline{D}_X X) \in \chi(M)$ and $(\langle S(X), X\rangle N) \in \chi^{\perp}(M)$ [2]. Since M is a timelike surface; that is, M has a nondegenerate metric, and we can write

$$\chi(\mathbb{R}^3_1) = \chi(M) \oplus \chi^{\perp}(M) \quad \text{and} \quad \chi(M) \cap \chi^{\perp}(M) = \{0\}.$$

Then, we obtain

$$\overline{D}_X X = 0 \quad \text{and} \quad \langle S(X), X\rangle = 0.$$

This completes the proof of the theorem. □

Theorem 9. *Let M be a timelike ruled surface in $I\!R^3_1$. Then the Gaussian curvature function $K(p)$ satisfies*

$$K(p) \geq 0,$$

at each point $p \in M$.

Proof. Let X be the spacelike vector field of the rulings through the point $p \in M$. An orthogonal base $\{X, Y\}$ of $\chi(M)$ can be obtained in which Y is a timelike vector field. The matrix corresponding to the shape operator of M derived from N is

$$\mathcal{S} = \begin{bmatrix} \langle S(X), X\rangle & -\langle S(X), Y\rangle \\ \langle S(Y), X\rangle & -\langle S(Y), Y\rangle \end{bmatrix}$$

Hence, the Gaussian curvature

$$K = \det \mathcal{S} = (\langle S(X), Y\rangle)^2$$

can be obtained from Theorem 8 since S is self-adjoint. Thus, $K(p) \geq 0$ for each point $p \in M$. □

Lemma 2. Assume that M is a timelike ruled surface. Let the unit tangent vector field of the base curve, the unit tangent vector field (director vector) of the rulings and the unit normal vector field of M be T, X, N, respectively. Then,

$$\begin{aligned} T \wedge X &= N, \\ T \wedge N &= -X, \\ X \wedge N &= -T. \end{aligned}$$

Proof. Straightforward calculation. □

Theorem 10. *Let M be a skew timelike surface. The Gaussian curvature function has its minimum value at the central point on each one the rulings.*

Proof. $\{A_0, X\}$ is an orthonormal base of $\chi(M)$, where

$$A_0 = \frac{A}{\|A\|} = \frac{(1+av)T + cvN}{[(a^2-c^2)v^2 + 2av + 1]^{1/2}}.$$

Denote the normal vector of M at $\varphi(t,v)$ by $\tilde{N} = N_{\varphi(t,v)}$. Thus,

$$\tilde{N} = A_0 \wedge X = \frac{1}{\|A\|}\{cvT + (1+av)N\}$$

from Lemma 2, and $\langle \tilde{N}, \tilde{N} \rangle = 1$. Therefore, the Gaussian curvature is

$$K = (\langle S(A_0), X \rangle)^2.$$

On the other hand,

$$\begin{aligned} S(A_0) = -D_{A_0}\tilde{N} &= -\frac{1}{\|A\|}\left\{\left[\left(\frac{1}{\|A\|}\right)^{\bullet}(cv) + \frac{\dot{c}v}{\|A\|} + \frac{b(1+av)}{\|A\|}\right]T \right. \\ & \left. -\frac{c}{\|A\|}X + \left[\left(\frac{1}{\|A\|}\right)^{\bullet}(1+av) + \frac{\dot{a}v}{\|A\|} + \frac{bcv}{\|A\|}\right]N\right\}, \end{aligned}$$

where $(\cdot)$ denotes the derivative with respect to the parameter. Thus, we obtain

$$K(t,v) = \frac{c^2}{[(c^2-a^2)v^2 - 2av - 1]^2}. \tag{3}$$

Hence, we have

$$\frac{\partial K(t,v)}{\partial v} = -\frac{4c^2[(c^2-a^2)v - a]}{[\langle A, A \rangle]^3}.$$

Thus, $v = \frac{a}{c^2-a^2}$ gives us the minimum of $K(t,v)$ since

$$\left.\frac{\partial^2 K(t,v)}{\partial^2 v}\right|_{v=\frac{a}{c^2-a^2}} > 0.$$

The Gaussian curvature has its minimum value at the central point on each of the rulings since the central point corresponds to the value $v = \frac{a}{c^2-a^2}$. □

Theorem 11. *Let M be a timelike ruled surface. Then M is developable if and only if the Gaussian curvature function of M is zero.*

Proof. This follows easily from (3) and Corollary 1. □

Theorem 12. *The distribution parameter of a timelike ruled surface depends only on the rulings.*

Proof. We obtain $K_{\min} = \frac{(c^2-a^2)^2}{c^2}$ if we write $v = \frac{a}{c^2-a^2}$ in (3). Thus, we get

$$K_{\min} = c^2 = \left(\frac{1}{P_X}\right)^2$$

from (2), since $a = 0$ at the central point. Therefore, we have

$$P_X = \frac{1}{\sqrt{K_{\min}}}.$$

□

The value of $K_{\min}$ is unique along a ruling. Therefore, the value of the distribution parameter is unique along a ruling; that is, the distribution parameter depends only on the rulings.

An important theorem concerning the central point of any skew surface in 3-dimensional Euclidean space was given by Chasles in 1839. Next, we will give a corresponding theorem for any skew timelike surface in $\mathbb{R}_1^3$.

Theorem 13. *Let M be a skew timelike surface, and let θ be the angle between the normal vector at a point of a ruling and the normal vector at the central point of this ruling, then $\tan\theta$ is proportional to the distance between these two points, and the coefficient of proportionality is the inverse of the distribution parameter.*

Proof. If $v = 0$, this gives the central point on a particular ruling; that is, if we take our orthogonal curve α through this central point, then $D_T X$ is the normal vector at $v = 0$, whence $a = 0$. Thus, the distribution parameter is $P_X = \frac{1}{c}$, and the normal N_v along the ruling is given by

$$N(v) = \frac{N + cvT}{\sqrt{1 - c^2v^2}}.$$

On the other hand, N and N_V are unit spacelike vectors. Therefore, we obtain

$$\langle N, N_V \rangle = \frac{1}{\sqrt{1 - c^2v^2}}.$$

Thus, we get

$$\cos\theta = \frac{1}{\sqrt{1 - \left(\frac{v}{P_X}\right)^2}}.$$

Hence, we have $\tan\theta = \frac{v}{P_X}$. □

Corollary 2. *The tangent plane turns evenly through 180° along a ruling for $-\frac{1}{c} < v < \frac{1}{c}$ in a skew timelike surface.*

Proof. Let ℓ_p be a ruling through the central point p, and let N_p and N_q be the normal vectors at p and q, respectively. If the angle between N_p and N_q is θ and the distance between p and q is v, then $\tan\theta = \frac{v}{P_X}$ from Theorem 13. Since $D_T X$ is a spacelike vector at the central point, we get

$$\min\left\{-\frac{1}{c}, \frac{1}{c}\right\} < v < \max\left\{-\frac{1}{c}, \frac{1}{c}\right\}.$$

If $v = 0$, then the distance between p and q is zero. Hence, $p = q$. Thus, we get $\theta = 0$. If $0 < v < \max\{-\frac{1}{c}, \frac{1}{c}\}$ then we get $0 < \theta \leq \frac{\pi}{2}$. If $\min\{-\frac{1}{c}, \frac{1}{c}\} < v < 0$ then we have $-\frac{\pi}{2} \leq \theta < 0$. □

Example 1. (The helicoid of the 2nd kind). This is a timelike ruled surface parametrized by,

$$\varphi(t,v) = \left(-\left(\frac{\kappa}{\kappa^2-\tau^2} + v\right) ch\sqrt{\kappa^2-\tau^2}t,\ \frac{\tau t}{\sqrt{\kappa^2-\tau^2}},\ -\left(\frac{\kappa}{\kappa^2-\tau^2} + v\right)\ sh\sqrt{\kappa^2-\tau^2}t\right),$$

[4], where κ and τ are the curvature and the torsion of α respectively, and $|\kappa| > |\tau|$. The base curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3_1$, where I is an open interval, such that

$$\alpha(t) = \left(-\left(\frac{\kappa}{\kappa^2-\tau^2}\right) ch\sqrt{\kappa^2-\tau^2}t,\ \frac{\tau t}{\sqrt{\kappa^2-\tau^2}},\ -\left(\frac{\kappa}{\kappa^2-\tau^2}\right) sh\sqrt{\kappa^2-\tau^2}t\right)\quad \forall\, t \in I$$

is a timelike curve since $\langle \frac{da}{dt}, \frac{da}{dt}\rangle = -1$, and each one of its rulings is a spacelike line. Now,

$$v < \min\left\{-\frac{1}{\kappa-\tau}, -\frac{1}{\kappa+\tau}\right\} \quad \text{or} \quad v > \max\left\{-\frac{1}{\kappa-\tau}, -\frac{1}{\kappa+\tau}\right\}$$

since $D_T X$ is a timelike vector field. Furthermore, $\det(T, X, D_T X) = -\tau$. The helicoid of the 2nd kind is developable if and only if $\tau = 0$.

Example 2. (The helicoid of the 1st kind). This is a timelike ruled surface parametrized by,

$$\varphi(t,v) = \left(\left(\frac{\kappa}{\tau^2-\kappa^2} - v\right)\cos\sqrt{\tau^2-\kappa^2}t,\ \left(\frac{\kappa}{\tau^2-\kappa^2} - v\right)\sin\sqrt{\tau^2-\kappa^2}t,\ \frac{\tau t}{\sqrt{\tau^2-\kappa^2}}\right),$$

[4], where $|\tau| > |\kappa|$. The base curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3_1$, where I is an open interval, such that

$$\alpha(t) = \left(\left(\frac{\kappa}{\tau^2-\kappa^2}\right)\cos\sqrt{\tau^2-\kappa^2}t,\ \left(\frac{\kappa}{\tau^2-\kappa^2}\right)\sin\sqrt{\tau^2-\kappa^2}t,\ \frac{\tau t}{\sqrt{\tau^2-\kappa^2}}\right)\quad \forall t \in I$$

is a timelike curve, and each one of its rulings is a spacelike line. Now,

$$\min\left\{\frac{-1}{\kappa-\tau}, \frac{-1}{\kappa+\tau}\right\} < v < \max\left\{\frac{-1}{\kappa-\tau}, \frac{-1}{\kappa+\tau}\right\}$$

since $D_T X$ is a spacelike vector field. The curve of striction is given by

$$\overline{\alpha}(t) = \alpha(t) + \frac{\kappa}{\tau^2 - \kappa^2} X(t),$$

and $\overline{\alpha}(t)$ is a timelike curve. Furthermore, $\det(T, X, D_T X) = \tau$. The helicoid of the 1st kind is developable if and only if $\tau = 0$. Thus, the distribution parameter of the helicoid of the 1st kind is $P_X = -\frac{\tau}{\tau^2 - \kappa^2}$.

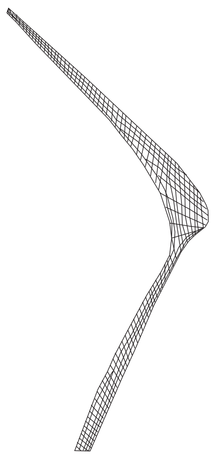

Figure 2. The helicoid of the 2 nd kind

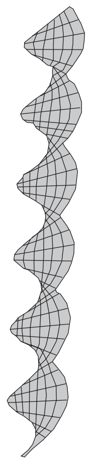

Figure 3. The helicoid of the 1st kind

Example 3. (The conjugate surface of Enneper of the 2nd kind). This is a timelike ruled surface parametrized by,

$$\varphi(t, v) = \left(\frac{\kappa t^2}{2} + v,\ \frac{-\kappa \tau t^3}{6} - \tau t v,\ \frac{\kappa^2 t^3}{6} + t + \kappa v t \right),$$

[4], where $|\kappa| = |\tau| \neq 0$. The base curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3_1$, where I is an open interval, such that

$$\alpha(t) = \left(\frac{\kappa t^2}{2}, \frac{-\kappa \tau t^3}{6}, \frac{\kappa^2 t^3}{6} + t \right) \quad \forall\ t \in I$$

is a timelike curve, and each one of its rulings is a spacelike line. Now,

$$\begin{aligned} v &> -\frac{1}{2\kappa} \quad \text{if} \quad \kappa > 0 \\ v &< -\frac{1}{2\kappa} \quad \text{if} \quad \kappa < 0 \end{aligned}$$

since $D_T X$ is the null vector field. Furthermore, $\det(T, X, D_T X) = -\tau$. The conjugate surface of Enneper of the 2nd kind is developable if and only if $\tau = 0$.

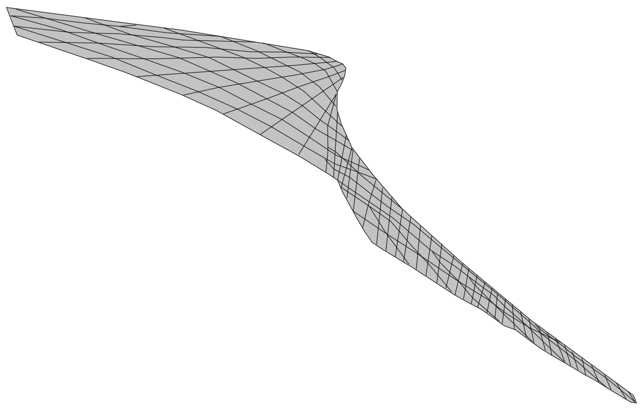

Figure 4. The conjugate surface of Enneper of the 2nd kind

Example 4. This is timelike ruled surface parametrized by,

$$\varphi(t, v) = \alpha(t) + vX(t) = (0, 0, t) + v(t, 0, 0),$$

[4]. The base curve is a timelike curve, and each one of its rulings is a spacelike line. This ruled surface is developable.

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