TIMELIKE Ruled Surfaces in the Minkowski 3-Space—II

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Abstract
This paper is devoted to a study of timelike ruled surfaces in three dimensional Minkowski space obtained by a spacelike straight line moving along a timelike curve. The central point, the curve of striction and the distribution parameter of a timelike ruled surface in Minkowski 3-space are considered, and some theorems relating to their structure are obtained. In addition, some results about developable timelike ruled surfaces are also given.

Introduction
A surface in the 3-dimensional Minkowski space \( \mathbb{R}^3_1 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2) \) is called a timelike surface if the induced metric on the surface is a Lorentz metric [1]. If the tangent vector at every point of a given curve in \( \mathbb{R}^3_1 \) is a spacelike vector (timelike vector), then the given curve is called a spacelike curve (timelike curve) [2].

A ruled surface is a surface swept out by a straight line \( \ell \) moving along a curve \( \alpha \). The various positions of the generating line \( \ell \) are called the rulings of the surface. Such a surface, thus, has a parametrization in ruled form
\[
\varphi(t, v) = \alpha(t) + vZ(t),
\]
where we call \( \alpha \) the base curve and \( Z \) the director vector of \( \ell \). If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable surface. All other ruled surfaces are called skew surfaces. If there exists a common perpendicular to two preceding rulings of a skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the curve of striction. If there is a curve which meets perpendicularly each one of the rulings, then this curve is called an orthogonal trajectory of the ruled surface. In \( \mathbb{R}^3_1 \), we define the exterior product of vectors by \( W \wedge V = -(i_W i_V dx \wedge dy \wedge dz)^\# \), where \( i_W \) denotes the interior product with respect to \( W \) and \( \# \) stands for the operation of raising indices by the metric \( dx^2 + dy^2 - dz^2 \). Here we choose the sign \( \ll - \gg \) so that \( \partial_x \wedge \partial_y = \partial_z \) holds.

The notation and fundamental concepts used in this study are the same as in [3].
1. Timelike Ruled Surfaces

Let

\[ \alpha : I \to \mathbb{R}^3_1 \]
\[ t \to \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \]

where \( \{0\} \subset I \), be a differentiable timelike curve in Minkowski 3-space parameterized by arc-length. The tangent vector field of \( \alpha \) will be denoted by \( T \).

A spacelike straight line,

\[ \ell : \mathbb{R} \to \mathbb{R}^3_1 \]
\[ v \to \ell(v) = (\alpha_1(t) + va_1(t), \alpha_2(t) + va_2(t), \alpha_3(t) + va_3(t)) \]

where the scalars \( a_i(t) \in \mathbb{R} \) for all \( 1 \leq i \leq 3 \), are the components of the director vector at the point \( \alpha(t) \), can be chosen so that the director vector of \( \ell \) and the tangent vector of \( \alpha \) are linearly independent at every point of the curve \( \alpha \).

As \( \ell \) moves along \( \alpha \) it generates a ruled surface given by the parametrization \((I \times \mathbb{R}, \varphi)\), where

\[ \varphi : I \times \mathbb{R} \to \mathbb{R}^3_1 \]
\[ (t, v) \to \varphi(t, v) = (\alpha_1(t) + va_1(t), \alpha_2(t) + va_2(t), \alpha_3(t) + va_3(t)) \]

which can be obtained in the Minkowski 3-space. This ruled surface will be denoted by \( M \). An orthonormal base \( \{T, X\} \) of \( \chi(M) \), the space of tangent vector fields of \( M \), can be obtained; thus, \( N = T \wedge X \) where \( N \) is the unit normal vector field of \( M \). Hence, \( \{X, N, T\} \) is an orthonormal frame field along \( \alpha \) in \( \mathbb{R}^3_1 \). Let \( D \) be the Levi-Civita connection on \( \mathbb{R}^3_1 \). The variation formulae of this system along \( \alpha \) in \( \mathbb{R}^3_1 \) are

\[ D_TX = cN + aT \]
\[ D_TN = -cX + bT \]
\[ D_TT = aX + bN, \]

where \( a = -\langle T, DTX \rangle = -T[\langle T, X \rangle] + \langle DTX, T \rangle = \langle DTX, X \rangle \), etc.

\[ B = \begin{bmatrix} 0 & c & a \\ -c & 0 & b \\ a & b & 0 \end{bmatrix} \]

is a skew-adjoint matrix, since \( B^T = -\epsilon B \epsilon \), where

\[ \epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \]
In view of the parametrization \( \varphi(t, v) = \alpha(t) + vX(t) \) we have

\[
E = \left( \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial t} \right) = -(1 + av)^2 + c^2v^2, \quad F = \left( \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial v} \right) = 0, \quad G = \left( \frac{\partial \varphi}{\partial v}, \frac{\partial \varphi}{\partial v} \right) = 1.
\]

The induced metric on the ruled surface is a Lorentz metric in the case where \( E \) is negative.

\[
\min \left\{ -\frac{1}{a-c}, -\frac{1}{a+c} \right\} \quad \text{and} \quad \max \left\{ -\frac{1}{a-c}, -\frac{1}{a+c} \right\}
\]

are roots of \( E \), where \( c^2 - a^2 = \langle D_T X, D_T X \rangle \).

Note that:

1) If \( D_T X \) is a timelike vector field, then

\[-\infty < v < \min \left\{ -\frac{1}{a-c}, -\frac{1}{a+c} \right\} \quad \text{or} \quad \max \left\{ -\frac{1}{a-c}, -\frac{1}{a+c} \right\} < v < \infty.\]

2) If \( D_T X \) is a spacelike vector field, then

\[\min \left\{ -\frac{1}{a-c}, -\frac{1}{a+c} \right\} < v < \max \left\{ -\frac{1}{a-c}, -\frac{1}{a+c} \right\}.\]

3) Let \( D_T X \) be the null vector field on \( \mathbb{R}^4_1 \).

If \( a > 0 \), then \( v < -\frac{1}{am} \), and if \( a < 0 \), then \( v > -\frac{1}{am} \).

Therefore, in all three cases above, the domain of the parameter \( v \) is not the whole of \( \mathbb{R} \) but is one of the above intervals. Let us denote the domain of \( v \) by \( J \). If we fix the parameter \( v \) in \( J \), then the curve

\[
\varphi_v : I \times \{v\} \rightarrow M \quad (t, v) \rightarrow \varphi_v(t, v) = \alpha(t) + vX(t)
\]

can be obtained on \( M \). The tangent vector field of this curve is

\[A = (1 + av)T + cvN.\]
2. Developable Timelike Ruled Surfaces

Let $M$ be a timelike ruled surface. Along any ruling of $M$, if all of the tangent planes of $M$ are the same (coincide) then we call $M$ as a developable surface.

**Theorem 1.** Let $M$ be a timelike ruled surface. The tangent planes along any ruling of $M$ coincide if and only if $c = 0$.

**Proof.** Trivial. \(\square\)

Now, we will a criterion for timelike ruled surfaces to be developable in $\mathbb{R}^{3}$.

**Corollary 1.** The timelike ruled surface $M$ is developable if and only if $c = 0$.

**Lemma 1.** $c = -\det(T, X, DTX)$ for the timelike ruled surface $M$.

3. Position Vector of a Central Point

If the distance between the central point and the base curve of a skew timelike ruled surface is $\pi$, then the position vector $\mathbf{\pi}(t)$ can be expressed in the form

$$\mathbf{\pi}(t, \pi) = \mathbf{\alpha}(t) + \pi \mathbf{X}(t),$$

where $\mathbf{\alpha}(t)$ is the position vector of the base curve and $\mathbf{X}(t)$ is the director vector belonging to the ruling. The parameter $\pi$ can be expressed in terms of the position vector of the base curve and the directed vector of the ruling. Take three neighbouring rulings of a timelike ruled surface such that the first and second are $\mathbf{X}(t)$ and $\mathbf{X}(t) + d\mathbf{X}(t)$ respectively. Let $P, P'$ and $Q, Q'$ be the feet on the rulings of the common perpendicular to two neighbouring rulings. The common perpendicular to $\mathbf{X}(t)$ and $\mathbf{X}(t) + d\mathbf{X}(t)$ is $\mathbf{X}(t) \wedge d\mathbf{X}(t)$.

![Figure 1](image-url)
The vector $\vec{PQ}$ coincides with the vector $\vec{PP'}$ in the limiting position, and $\vec{PQ}$ will be the tangent vector to the curve of striction. Thus, we have $\langle D_T X, \vec{PQ} \rangle = 0$. Therefore, we get
\[
\mathbf{u} = -\frac{\langle T, D_T X \rangle}{\langle D_T X, D_T X \rangle} = \frac{a}{c^2 - a^2}.
\]

Hence the curve of striction is given by
\[
\mathbf{r}(t) = \mathbf{r}(t) - \frac{\langle T, D_T X \rangle}{\langle D_T X, D_T X \rangle} X(t) \quad (1)
\]
where $\langle D_T X, D_T X \rangle \neq 0$. $\mathbf{u} = \frac{a}{c^2 - a^2}$ is constant since $\langle \frac{dT}{dt}, X \rangle = 0$.

**Theorem 2.** The curve of striction $\mathbf{r}$ does not depend on the choice of the base curve $\alpha$ for the skew timelike surface.

**Proof.** Let $\beta$ be another base curve of the skew timelike surface; that is, let, for all $(t, v)$,
\[
\varphi(t, v) = \alpha(t) + v X(t) = \beta(t) + s X(t)
\]
for some function $s = s(v)$. Then from (1) we obtain
\[
\mathbf{r}(t) - \beta(t) = \alpha(t) - \beta(t) - \frac{\langle T - \frac{ds}{dt} X, D_T X \rangle}{\langle D_T X, D_T X \rangle} X(t) = 0
\]
since $\langle X, D_T X \rangle = 0$. This proves our claim. \hfill \square

**Theorem 3.** Let $M$ be a skew timelike surface. The point $\varphi(t, v_0)$ on the ruling through the point $\alpha(t)$ is the central point if and only if $D_T X$ is a normal vector of the tangent plane at $\varphi(t, v_0)$.

**Proof.** Let $D_T X$ be a normal of the tangent plane at $\varphi(t, v_0)$ on the ruling through $\alpha(t)$. Thus $\langle D_T X, A \rangle = 0$. Hence, we get $v_0 = \frac{a}{c^2 - a^2}$. Therefore, $\varphi(t, v_0)$ is the central point of $M$.

Conversely, let $\varphi(t, v_0)$ be the central point on the ruling through $\alpha(t)$. Then, we have $\langle D_T X, A \rangle = -a + (-a^2 + c^2)v = 0$.

On the other hand, $\langle D_T X, X \rangle = 0$. Therefore, $D_T X$ is a normal vector of the tangent plane at $\varphi(t, v_0)$.

$D_T X$ is a spacelike vector at the central point since $D_T X$ is a normal vector of the tangent plane at the central point. Thus, $\langle D_T X, D_T X \rangle = -a^2 + c^2 > 0$.
Theorem 4. The curve of striction of a skew timelike surface

\[ \alpha(t) = \alpha(t) + \frac{a}{c^2 - a^2} X(t) \]

is a timelike curve.

Proof. Straightforward calculation. \qed

Theorem 5. Assume that \( M \) is a timelike ruled surface in \( \mathbb{R}^3_1 \). There exists a unique orthogonal trajectory of \( M \) through each point of \( M \).

Proof. Let \( \varphi : I \times J \to \mathbb{R}^3_1 \)

\[ (t, v) \to \varphi(t, v) = \alpha(t) + vZ(t), \]

be a parametrization of \( M \). An orthogonal trajectory of \( M \) is given by

\[ \beta : \quad \tilde{I} \to M \]

\[ s \to \beta(s) = \alpha(s) + f(s)Z(s), \]

where \( \langle Z(s), Z(s) \rangle = 1 \). We may assume that \( \tilde{I} \subset I \). Since

\[ \langle \frac{d\beta(s)}{ds}, Z(s) \rangle = 0, \]

we obtain

\[ f(s) = -\int \langle \frac{d\alpha(s)}{ds}, Z(s) \rangle ds + h, \]

where \( h \) is a real constant. Hence \( h = f(s_0) - F(s_0) \), where

\[ F(s) = -\int \langle \frac{d\alpha(s)}{ds}, Z(s) \rangle ds. \]

Therefore the orthogonal trajectory of \( M \) through the point \( P_0 \) is unique. Thus, we have \( \tilde{I} = I \) since the orthogonal trajectory of \( M \) meets each one of the rulings of \( M \). \qed

Theorem 6. Suppose that \( M \) is a skew timelike surface. The longest distance between two rulings is the distance measured only on the curve of striction which is one of the orthogonal trajectories.
Proof. Fixing two rulings, say for \( t_1 < t_2 \), we compute the length \( j(v) \) of an orthogonal trajectory between these two rulings by

\[
j(v) = \int_{t_1}^{t_2} \sqrt{|[A,A]|} \, dt = \int_{t_1}^{t_2} [(a^2 - c^2)v^2 + 2av + 1]^{1/2} \, dt.
\]

To find the value of \( s \) which maximizes \( j(v) \), we use \( \frac{\partial j(v)}{\partial v} = 0 \) which gives \( v = \frac{a}{c-a} \).

This completes the proof.

4. The Distribution Parameter of a Timelike Ruled Surface

Let the curve of striction be the base curve of a timelike ruled surface. Then \( \overrightarrow{a} = 0 \); that is, \( \overrightarrow{a} = 0 \). Thus, we have \( a = 0 \). Hence, \( DTX \) and \( N \) are linearly dependent; that is, \( \lambda DTX = N \) where \( DTX = aT + cN \) and \( N = T \wedge X = \overrightarrow{aT} \wedge X \). Thus, we obtain

\[
\lambda = \frac{\langle T \wedge X, DTX \rangle}{\langle DTX, DTX \rangle} = \frac{\det(T, X, DTX)}{\langle DTX, DTX \rangle}.
\]

(2)

\( \lambda \) is called the distribution parameter of the timelike ruled surface, and is denoted by \( \lambda \) or \( P_X \). Note that \( \langle DTX, DTX \rangle \neq 0 \) since \( DTX \) is a timelike vector field.

Theorem 7. A timelike ruled surface is a developable surface if and only if the distribution parameter is zero.

Proof. Straightforward.

Theorem 8. Let \( M \) be a timelike ruled surface in \( R^3_1 \). Each one of the rulings of \( M \) is an asymptotic line and a geodesic in \( M \).

Proof. Each one of the rulings is geodesic in \( R^3_1 \) since each one of the rulings is a straight line in \( R^3_1 \). Thus, we have \( DX = 0 \). The Gaussian curvature is

\[ DX = \nabla_X + \langle S(X), X \rangle N \]

where \( \nabla \) is the Levi-Civita connection on \( M \), and \( S \) is the shape operator of \( M \) derived from \( N \). Furthermore, \( \langle DX, X \rangle \in \chi(M) \) and \( \langle (S(X), X)N \rangle \in \chi^+(M) \). Since \( M \) is a timelike surface; that is, \( M \) has a nondegenerate metric, and we can write

\[ \chi(R^3_1) = \chi(M) \oplus \chi^+(M) \quad \text{and} \quad \chi(M) \cap \chi^+(M) = \{0\}. \]

Then, we obtain

\[ DX = 0 \quad \text{and} \quad \langle S(X), X \rangle = 0. \]

This completes the proof of the theorem.
Theorem 9. Let $M$ be a timelike ruled surface in $\mathbb{R}^3_1$. Then the Gaussian curvature function $K(p)$ satisfies

$$K(p) \geq 0,$$

at each point $p \in M$.

Proof. Let $X$ be the spacelike vector field of the rulings through the point $p \in M$. An orthogonal base $\{X, Y\}$ of $\chi(M)$ can be obtained in which $Y$ is a timelike vector field. The matrix corresponding to the shape operator of $M$ derived from $N$ is

$$S = \begin{bmatrix}
\langle S(X), X \rangle & -\langle S(X), Y \rangle \\
\langle S(Y), X \rangle & -\langle S(Y), Y \rangle
\end{bmatrix}$$

Hence, the Gaussian curvature

$$K = \det S = (\langle S(X), Y \rangle)^2$$

can be obtained from Theorem 8 since $S$ is self-adjoint. Thus, $K(p) \geq 0$ for each point $p \in M$. 

Lemma 2. Assume that $M$ is a timelike ruled surface. Let the unit tangent vector field of the base curve, the unit tangent vector field (director vector) of the rulings and the unit normal vector field of $M$ be $T, X, N$, respectively. Then,

$$T \wedge X = N,$$

$$T \wedge N = -X,$$

$$X \wedge N = -T.$$ 

Proof. Straightforward calculation.

Theorem 10. Let $M$ be a skew timelike surface. The Gaussian curvature function has its minimum value at the central point on each one the rulings.

Proof. $\{A_0, X\}$ is an orthonormal base of $\chi(M)$, where

$$A_0 = \frac{A}{\|A\|} = \frac{(1 + av)T + cvN}{(a^2 - c^2)v^2 + 2av + 1}^{1/2}.$$ 

Denote the normal vector of $M$ at $\varphi(t, v)$ by $\tilde{N} = N_{\varphi(t, v)}$. Thus,

$$\tilde{N} = A_0 \wedge X = \frac{1}{\|A\|} \{cvT + (1 + av)N\}.$$
from Lemma 2, and $\langle \tilde{N}, \tilde{N} \rangle = 1$. Therefore, the Gaussian curvature is

$$K = (\langle S(A_0), X \rangle)^2.$$ 

On the other hand,

$$S(A_0) = -D_{A_0} \tilde{N} = -\frac{1}{\|A\|} \left\{ \left( \frac{1}{\|A\|} \right)^* (cv) + \frac{\dot{w}}{\|A\|} + \frac{b(1 + av)}{\|A\|} \right \} T$$

$$- \frac{c}{\|A\|} X + \left( \frac{1}{\|A\|} \right)^* (1 + av) + \frac{\dot{w}}{\|A\|} + \frac{bcv}{\|A\|} N \right \},$$

where $(\cdot)$ denotes the derivative with respect to the parameter. Thus, we obtain

$$K(t, v) = \frac{c^2}{(c^2 - a^2)v^2 - 2av - 1}.$$  

(3)

Hence, we have

$$\frac{\partial K(t, v)}{\partial v} = -\frac{4c^2[(c^2 - a^2)v - a]}{[\langle A, A \rangle]^3}.$$ 

Thus, $v = \frac{a}{c^2 - a^2}$ gives us the minimum of $K(t, v)$ since

$$\left. \frac{\partial^2 K(t, v)}{\partial^2 v} \right|_{v=\frac{a}{c^2 - a^2}} > 0.$$ 

The Gaussian curvature has its minimum value at the central point on each of the rulings since the central point corresponds to the value $v = \frac{a}{c^2 - a^2}$.

\[ \square \]

**Theorem 11.** Let $M$ be a timelike ruled surface. Then $M$ is developable if and only if the Gaussian curvature function of $M$ is zero.

**Proof.** This follows easily from (3) and Corollary 1. \[ \square \]

**Theorem 12.** The distribution parameter of a timelike ruled surface depends only on the rulings.

**Proof.** We obtain $K_{\min} = \frac{(c^2 - a^2)^2}{c^2 - a^2}$ if we write $v = \frac{a}{c^2 - a^2}$ in (3). Thus, we get

$$K_{\min} = c^2 = \left( \frac{1}{P_X} \right)^2.$$
from (2), since $a = 0$ at the central point. Therefore, we have

$$P_X = \frac{1}{\sqrt{K_{\text{min}}}}.$$ 

The value of $K_{\text{min}}$ is unique along a ruling. Therefore, the value of the distribution parameter is unique along a ruling; that is, the distribution parameter depends only on the rulings.

An important theorem concerning the central point of any skew surface in 3-dimensional Euclidean space was given by Chasles in 1839. Next, we will give a corresponding theorem for any skew timelike surface in $\mathbb{R}^3$.

**Theorem 13.** Let $M$ be a skew timelike surface, and let $\theta$ be the angle between the normal vector at a point of a ruling and the normal vector at the central point of this ruling, then $\tan \theta$ is proportional to the distance between these two points, and the coefficient of proportionality is the inverse of the distribution parameter.

**Proof.** If $v = 0$, this gives the central point on a particular ruling; that is, if we take our orthogonal curve $\alpha$ through this central point, then $D_T X$ is the normal vector at $v = 0$, whence $a = 0$. Thus, the distribution parameter is $P_X = \frac{1}{c}$, and the normal $N_v$ along the ruling is given by

$$N(v) = \frac{N + cvT}{\sqrt{1 - c^2 v^2}}.$$ 

On the other hand, $N$ and $N_v$ are unit spacelike vectors. Therefore, we obtain

$$\langle N, N_v \rangle = \frac{1}{\sqrt{1 - c^2 v^2}}.$$ 

Thus, we get

$$\cos \theta = \frac{1}{\sqrt{1 - \left(\frac{v}{P_X}\right)^2}}.$$ 

Hence, we have $\tan \theta = \frac{v}{P_X}$. 

**Corollary 2.** The tangent plane turns evenly through $180^\circ$ along a ruling for $-\frac{1}{c} < v < \frac{1}{c}$ in a skew timelike surface.

**Proof.** Let $\ell_p$ be a ruling through the central point $p$, and let $N_p$ and $N_q$ be the normal vectors at $p$ and $q$, respectively. If the angle between $N_p$ and $N_q$ is $\theta$ and the distance
between $p$ and $q$ is $v$, then \( \tan \theta = \frac{v}{r} \) from Theorem 13. Since $D_T X$ is a spacelike vector at the central point, we get

\[
\min \left\{ -\frac{1}{c}, \frac{1}{c} \right\} < v < \max \left\{ -\frac{1}{c}, \frac{1}{c} \right\}.
\]

If $v = 0$, then the distance between $p$ and $q$ is zero. Hence, $p = q$. Thus, we get $\theta = 0$. If $0 < v < \max \left\{ -\frac{1}{c}, \frac{1}{c} \right\}$ then we get $0 < \theta \leq \frac{\pi}{2}$. If $\min \left\{ -\frac{1}{c}, \frac{1}{c} \right\} < v < 0$ then we have $-\frac{\pi}{2} \leq \theta < 0$. 

\[\Box\]

**Example 1.** (The helicoid of the 2nd kind). This is a timelike ruled surface parametrized by,

\[
\varphi(t, v) = \left( -\left( \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}} + v \right) \right) ch \sqrt{\kappa^2 - \tau^2} t, \quad \frac{\tau t}{\sqrt{\kappa^2 - \tau^2}}, \quad -\left( \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}} + v \right) sh \sqrt{\kappa^2 - \tau^2} t\),
\]

[4], where $\kappa$ and $\tau$ are the curvature and the torsion of $\alpha$ respectively, and $|\kappa| > |\tau|$. The base curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$, where $I$ is an open interval, such that

\[
\alpha(t) = \left( -\left( \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}} \right) ch \sqrt{\kappa^2 - \tau^2} t, \quad \frac{\tau t}{\sqrt{\kappa^2 - \tau^2}}, \quad -\left( \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}} \right) sh \sqrt{\kappa^2 - \tau^2} t \right) \quad \forall t \in I
\]

is a timelike curve since $(\frac{da}{dt}, \frac{da}{dt}) = -1$, and each one of its rulings is a spacelike line. Now,

\[
v < \min \left\{ -\frac{1}{\kappa - \tau}, -\frac{1}{\kappa + \tau} \right\} \quad \text{or} \quad v > \max \left\{ -\frac{1}{\kappa - \tau}, -\frac{1}{\kappa + \tau} \right\}
\]

since $D_T X$ is a timelike vector field. Furthermore, $\det(T, X, D_T X) = -\tau$. The helicoid of the 2nd kind is developable if and only if $\tau = 0$.

**Example 2.** (The helicoid of the 1st kind). This is a timelike ruled surface parametrized by,

\[
\varphi(t, v) = \left( \left( \frac{\kappa}{\tau^2 - \kappa^2} - v \right) \cos \sqrt{\tau^2 - \kappa^2} t, \left( \frac{\kappa}{\tau^2 - \kappa^2} - v \right) \sin \sqrt{\tau^2 - \kappa^2} t, \frac{\tau t}{\sqrt{\tau^2 - \kappa^2}} \right),
\]

[4], where $|\tau| > |\kappa|$. The base curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$, where $I$ is an open interval, such that

\[
\alpha(t) = \left( \left( \frac{\kappa}{\tau^2 - \kappa^2} \right) \cos \sqrt{\tau^2 - \kappa^2} t, \left( \frac{\kappa}{\tau^2 - \kappa^2} \right) \sin \sqrt{\tau^2 - \kappa^2} t, \frac{\tau t}{\sqrt{\tau^2 - \kappa^2}} \right) \quad \forall t \in I
\]

is a timelike curve, and each one of its rulings is a spacelike line. Now,

\[
\min \left\{ -\frac{1}{\kappa - \tau}, -\frac{1}{\kappa + \tau} \right\} < v < \max \left\{ -\frac{1}{\kappa - \tau}, -\frac{1}{\kappa + \tau} \right\}
\]

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since $D_TX$ is a spacelike vector field. The curve of striction is given by

$$\overline{\alpha}(t) = \alpha(t) + \frac{\kappa}{\tau^2 - \kappa^2} X(t),$$

and $\overline{\alpha}(t)$ is a timelike curve. Furthermore, $\det(T, X, D_TX) = \tau$. The helicoid of the 1st kind is developable if and only if $\tau = 0$. Thus, the distribution parameter of the helicoid of the 1st kind is $P_X = -\frac{1}{\tau}$. 

Figure 2. The helicoid of the 2nd kind

Figure 3. The helicoid of the 1st kind

Example 3. (The conjugate surface of Enneper of the 2nd kind). This is a timelike ruled surface parametrized by,

$$\varphi(t, v) = \left(\frac{\kappa t^2}{2} + v, \frac{-\kappa t^3}{6} - \tau tv, \frac{-\kappa^2 t^3}{6} + t + \kappa vt\right),$$

[4], where $|\kappa| = |\tau| \neq 0$. The base curve $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}^2$, where $I$ is an open interval, such that

$$\alpha(t) = \left(\frac{\kappa t^2}{2}, \frac{-\kappa t^3}{6}, \frac{-\kappa^2 t^3}{6} + t\right) \quad \forall \ t \in I$$

is a timelike curve, and each one of its rulings is a spacelike line. Now,

$$v > -\frac{1}{2\kappa} \quad \text{if} \quad \kappa > 0$$

$$v < -\frac{1}{2\kappa} \quad \text{if} \quad \kappa < 0$$
since $D_TX$ is the null vector field. Furthermore, $\det(T, X, D_TX) = -\tau$. The conjugate surface of Enneper of the 2nd kind is developable if and only if $\tau = 0$.

**Figure 4.** The conjugate surface of Enneper of the 2nd kind

**Example 4.** This is timelike ruled surface parametrized by,

$$\varphi(t, v) = \alpha(t) + vX(t) = (0, 0, t) + v(t, 0, 0),$$

[4]. The base curve is a timelike curve, and each one of its rulings is a spacelike line. This ruled surface is developable.
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