A NOTE ON FINITE HYPERBOLIC PLANES
OBTAINED FROM PROJEKTIVE PLANES

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Abstract

Let $\Pi$ be a finite projective plane of order $n$ and $\mathcal{M}$ be a set, $|\mathcal{M}| = m$, of any lines of $\Pi$ which contains three non-concurrent lines. Consider the hyperbolic plane $\Pi_m$ obtained from $\Pi$ by removing all lines (including all points on them) of $\mathcal{M}$. In this paper, we obtain larger values than the known maximum value of $m$ and determine the line classes of some hyperbolic planes of type $\Pi_m$. Furthermore we give an answer to a question in Bumcrot [1] about hyperbolic planes containing two-point lines.

1. Preliminary Definitions and Propositions

An incidence structure is an ordered triple of sets $(\mathcal{P}, \mathcal{L}, I)$, where $\mathcal{P} \cap \mathcal{L} = \emptyset, I \subset \mathcal{P} \times \mathcal{L}$. For $P$ in $\mathcal{P}$ and $l$ in $\mathcal{L}$, $P I l$ is read “$P$ is on $l$”. We also write this $l I P$.

Definition 1.1. A linear space is an incidence structure $(\mathcal{P}, \mathcal{L}, I)$ satisfying:

L1. Each two distinct points are on exactly one line.

L2. Each line is on at least two points.

Definition 1.2. A hyperbolic plane is a linear space $(\mathcal{P}, \mathcal{L}, I)$ satisfying:

H1. Through each point $P$, not on a line $l$, there pass at least two lines not meeting $l$.

H2. There exist at least four points, no three of which are collinear.

H3. If a subset $S$ of $\mathcal{P}$ contains three non-collinear points and contains all points on the lines through pairs of distinct points of $S$, then $S$ contains all points of $\mathcal{P}$.

If $S = (\mathcal{P}, \mathcal{L}, I)$ is a linear space, we define as usual $\nu = |\mathcal{P}|, b = |\mathcal{L}|$, where $| |$ denotes cardinality. For each point $P$ and line $l$ of $S$, let

$$r(P) = |\{l \in \mathcal{L} : P I l\}|$$

$$k(l) = |\{P \in \mathcal{P} : P I l\}|$$
If $\nu$ is finite, then all of these numbers as well as $|I|$ are finite, in this case we say $S$ is finite. For finite $S$ we further define

\[
\begin{align*}
k_m &= \min\{k(l) : l \in \mathcal{L}\} \\
k_M &= \max\{k(l) : l \in \mathcal{L}\} \\
r_m &= \min\{r(P) : P \in \mathcal{P}\} \\
r_M &= \max\{r(P) : P \in \mathcal{P}\}
\end{align*}
\]

The following proposition is easy consequence of axiom H1.

**Proposition 1.1 (Bumcrot [1]).** If a two-dimensional linear space $S$ contains three distinct points $P_1, P_2, P_3$ such that $k(P_iP_j) = 2$ for $1 \leq i < j \leq 3$, then $S$ is not a hyperbolic plane.

**Proposition 1.2 (Bumcrot [1]).** Any finite linear space satisfying:

1. $r_m \geq k_M + 2$
2. $k_m(k_{m-1}) \geq r_M$

is a hyperbolic plane.

2. Some Hyperbolic Planes Obtained From Projective Planes by Removing Some Lines.

   Let $\Pi = (\mathcal{P}, \mathcal{L}, I)$ be a finite projective plane of order $n$, and $\mathcal{M}$ be a set of lines in $\Pi$ satisfying:

   C. Every line of $\Pi$ meets lines of $\mathcal{M}$ in at least two distinct points.

   Let $Q$ be the set of all points of $\mathcal{P}$ that are on at least one line of $\mathcal{M}$. Then the substructure

   \[
   \Pi_m = (\mathcal{P} - Q, \mathcal{L} - \mathcal{M}, I \cap (\mathcal{P} - Q) \times (\mathcal{L} - \mathcal{M}))
   \]

   is a hyperbolic plane, if $3 \leq m \leq n + \frac{1}{3}(1 - \sqrt{4n + 5})$ and $n \geq 5$, where $m = |\mathcal{M}|$.

   Here, the inequality $m \leq n + \frac{1}{3}(1 - \sqrt{4n + 5})$ is a sufficient but not necessary condition. In fact, when $\mathcal{M}$ consists of lines such that no three of them are concurrent, define a corner point as an intersection point of any two lines in $\mathcal{M}$. If the minimum number of corner points on any line of $\Pi_m$ is $r$ and

   \[
   3 \leq m \leq n + r + \frac{1}{2}(1 - \sqrt{4n + 5}), \quad n \geq 5,
   \]

   then $\Pi_m$ is a hyperbolic plane (see Kaya-Özcan [2]).

   Recall that in this case, if $n$ is odd, $m \leq n + 1$ and if $n$ is even, $m \leq n + 2$.

Therefore there are two cases:
Case I: Let $n$ be odd. If $m = n + 1$, that is, the elements of $\mathcal{M}$ are tangent lines of an oval $\mathcal{O}$, then $\Pi_m = \Pi_{n+1}^{\mathcal{O}}$ is the hyperbolic plane model of Ostrom [4].

If $n$ elements of $\mathcal{M}$ are removed from $\Pi$, then $\Pi_n^{\mathcal{O}}$ is not a hyperbolic plane, since each tangent line in $\Pi_n^{\mathcal{O}}$ contains only one point in $\Pi_n^{\mathcal{O}}$. On the other hand, if $m$ tangent lines of $\mathcal{O}$ are removed from $\Pi$, where $3 \leq m \leq n - 1$, then $\Pi_m^{\mathcal{O}}$ is a hyperbolic plane which can be easily shown.

Case II. Let $n$ be even. If $m = n + 2$, then it is known that $\Pi_{n+2}$ is hyperbolic plane (Olgun [3]). However, $\Pi_{n+1}, \Pi_n$ and $\Pi_{n-1}$ are not hyperbolic planes since $\Pi_{n+1}$ contains a line which has no point in $\Pi_{n+1}, \Pi_n$ contains two lines, each of which has only one point in $\Pi_n$ and $\Pi_{n-1}$ contains three lines which form a triangle and every one of which has only two points in $\Pi_{n-1}$. (See Prop.1.1).

On the other hand, one can easily show that $\Pi_m$ is a hyperbolic plane for $3 \leq m \leq n - 2$. Now we give a proposition which will show that the proposition is still true for larger values of $m$.

**Proposition 2.1.** Let $\Pi$ be a projective plane of order $n$ and $\mathcal{M}$ be a set of $m$ lines that contains $n + 1$ or $n + 2$ lines which no three concurrent, according as $n$ is odd or even, respectively. Denote by $\Pi_m^{\mathcal{O}}$ the structure obtained from $\Pi$ by removing all lines (including all points on them) of $\mathcal{M}$.

If $n + 1 \leq m \leq \frac{3n - \sqrt{4n + 5}}{2}$ for $n \geq 7$, when $n$ is odd

or

$n + 2 \leq m \leq \frac{3n + 3 - \sqrt{4n + 5}}{2}$ for $n \geq 8$, when $n$ is even,

then $\Pi_m^{\mathcal{O}}$ is a hyperbolic plane.

**Proof.** We give the proof for $n$ odd.

L1. Clearly any two distinct points in $\Pi_m$, as points of $\Pi$, are on exactly one line in $\Pi_m$.

L2. Any line of $\Pi_m$ is on at least

$$n + 1 - \left[ \frac{n - 1}{2} + \left( m - 2 \left( \frac{n - 1}{2} \right) \right) \right] = \frac{3n + 1}{2} - m$$

points in $\Pi_m$ since it has at most $\frac{n - 1}{2}$ corner points.

$$\frac{3n + 1}{2} - m = \frac{3n + 1}{2} - \frac{3n - \sqrt{4n + 5}}{2} = \frac{1 + \sqrt{4n + 5}}{2} > 2, \ (n \geq 7)$$

even if

$$m = \frac{3n - \sqrt{4n + 5}}{2}.$$ 

So, any line of $\Pi_m$ has more than two points. Thus, L1 and L2 imply that $\Pi_m$ is a linear space.
In $\Pi_m$, since $r_m = r_M = n + 1$ and $k_M \leq \frac{n + 3}{2}$,  

$$r_m > k_M + 2$$  \hspace{1cm} (1)

Since $k_m = \frac{3n + 1}{2} - m$,  

$$k_m(k_m - 1) = \frac{3n + 1 - 2m}{2} \left( \frac{3n + 1 - 2m}{2} - 1 \right) = \frac{3n + 1 - 2m}{2} \cdot \frac{3n - 1 - 2m}{2}$$  

Thus $k_m(k_m - 1) \geq n + 1$, since $m \leq \frac{3n - \sqrt{4n + 5}}{2}$. Therefore we have  

$$k_m(k_m - 1) \geq r_M$$  \hspace{1cm} (2)

It follows from Proposition 1.2 the inequalities (1)-(2) imply that the linear space $\Pi_m$ is a hyperbolic plane.  

If $n$ is even the proof is similar. \hfill \Box

**Open question:** Let $\Pi_m$ be the hyperbolic plane obtained in Proposition 2.1. Is there any value of $m$ which is larger than obtained in Proposition 2.1?

The classification of sets of lines with respect to the number of points on each line of a hyperbolic plane obtained from $\Pi$ by removing some lines, in particular, no three of them are concurrent is an important subject. In the mean time, the lines of some hyperbolic planes of this type have been classified by some authors. For example, line classes of the hyperbolic planes $\Pi_3, \Pi_4, \Pi_5, \Pi_6$ and $\Pi_7$, have been determined. Also the line classes of the hyperbolic planes $\Pi_{n-1}^0$ and $\Pi_{n-2}^0$ are examined in Özcan-Olgun-Kaya [4].

**The Classification of Lines.**

Let $\Pi_m$ be a hyperbolic plane obtained from $\Pi$ by removing $m$ lines such that no three of them are concurrent. Then the lines of $\Pi_m$ are classified as follows.

The set of lines of $\Pi_m$, every one of which contains exactly $s$ corner points, is called a class and denoted by $C_s$.

Unless otherwise stated, $\Pi_m$ will be understood as defined above.

Now, we recall, without proof, two results on line classes of the hyperbolic planes $\Pi_3, \Pi_4, \Pi_5, \Pi_6$ and $\Pi_7$ given in Özcan-Olgun-Kaya [5].

**Corollary 2.1.** For any hyperbolic plane $\Pi_m$ with $m \in \{3, 4, 5\}$,  

(i) $q_2 = \frac{1}{2} \binom{m}{2} \binom{m - 2}{2}$

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(ii) \( q_1 = \binom{m}{2} (n-1) - \binom{m-2}{2} \)

(iii) \( q_0 = n^2 + n \left( 1 - \frac{(m-2)}{2} \right) + \frac{(m-1)}{2} \left( \frac{m}{2} \right) \left( \frac{m-2}{2} \right) \)

where \( q_s = |C_s|, s = 0,1,2 \).

**Corollary 2.2.** The number of lines in \( C_0, C_1, C_2 \) of any hyperbolic plane of type \( \Pi_6 \) or \( \Pi_7 \) can be determined in terms of the number of lines in \( C_3 \) as follows:

\[
\begin{align*}
q_2 & = 45 - 3q_3 & q_2 & = 105 - 3q_3 \\
q_1 & = 15(n - 7) + 3q_3 & \text{or} & q_1 & = 21(n - 11) + 3q_3 \\
q_0 & = n^2 - 14n + 55 - q_3 & q_0 & = n^2 - 20n + 120 - q_3
\end{align*}
\]

respectively.

**Proposition 2.2.** Let \( \Pi \) be a projective plane of even order \( n \). Let \( H \) be a dual hyperoval in \( \Pi \) and \( \Pi_m \) be a hyperbolic plane obtained from \( \Pi \) by removing \( m \) lines, \( m \in \{n-2,n-3\} \), of \( H \). Then there exist three line classes in \( \Pi_{n-2} \) and \( \Pi_{n-3} \), namely, \( C_{\frac{n-2}{2}}, C_{\frac{n-4}{2}} \) and \( C_{\frac{n-2}{2}} \) which are in \( \Pi_{n-2} \) and \( C_{\frac{n-8}{2}}, C_{\frac{n-10}{2}} \) and \( C_{\frac{n-8}{2}} \) which are in \( \Pi_{n-3} \). Furthermore,

\[
q_{\frac{n-6}{2}} = q_0, q_{\frac{n-4}{2}} = q_1, q_{\frac{n-2}{2}} = q_2 \quad \text{and} \quad q_{\frac{n-8}{2}} = q_0', q_{\frac{n-10}{2}} = q_1', q_{\frac{n-8}{2}} = q_2'.
\]

where \( q_s \) is related to \( \Pi_4 \) and \( q_s' \) is related to \( \Pi_5 \) as in Corollary 2.1, \( s = 0,1,2 \).

**Sketch of proof.** The proof of the first part is straightforward. For the second part, consider the lines which complete \( m \) lines \( m \in \{n-2,n-3\} \) to \( n+2 \) lines of \( H \). Then, the classes \( C_{\frac{n-2}{2}}, C_{\frac{n-4}{2}}, C_{\frac{n-2}{2}} \) of \( \Pi_{n-2} \) and the classes \( C_0, C_1, C_2 \) of \( \Pi_4 \) are the same, that is, \( C_{\frac{n-2}{2}} = C_0, C_{\frac{n-4}{2}} = C_1 \) and \( C_{\frac{n-2}{2}} = C_2 \).

Likewise, the classes \( C_{\frac{n-8}{2}}, C_{\frac{n-10}{2}}, C_{\frac{n-8}{2}} \) of \( \Pi_{n-3} \) and the classes \( C_0', C_1', C_2' \) of \( \Pi_5 \) are the same, that is, \( C_{\frac{n-8}{2}} = C_0', C_{\frac{n-10}{2}} = C_1' \) and \( C_{\frac{n-8}{2}} = C_2' \).

(Recall that \( q_s \) and \( q_s' \), \( s = 0,1,2 \) is known for \( m = 4 \) and \( m = 5 \), from Corollary 2.1, respectively). Thus the proof is finished.

□

**Proposition 2.3.** Let \( \Pi \) be a projective plane of order \( n \), with \( n \) even, \( H \) be a dual hyperoval in \( \Pi \) and \( \Pi_m \) be a hyperbolic plane obtained from \( \Pi \) by removing \( m \) lines \( m \in \{n-4, n-5\} \) of \( H \), then there exist for line classes in \( \Pi_{n-4} \) and \( \Pi_{n-5} \). Then there exist for line classes in \( \Pi_{n-4} \) and \( \Pi_{n-5} \), namely, \( C_{\frac{n-2}{2}}, i \in \{4,6,8,10\} \) which are in \( \Pi_{n-4} \) and \( C_{\frac{n-8}{2}}, i \in \{6,8,10,12\} \) which are in \( \Pi_{n-5} \).

Furthermore
\[ q_{\frac{n}{2}} = 45 - 3q_{\frac{n-4}{2}}, q_{\frac{n}{2}} = 15(n - 7) + 3q_{\frac{n-4}{2}}, q_{\frac{n}{2}-10} = n^2 - 14n + 55 - q_{\frac{n}{2}} \quad \text{in} \quad \Pi_{n-4}, \quad \text{and} \quad q'_{\frac{n}{2}} = 105 - 3q'_{\frac{n}{2}-6}, q'_{\frac{n}{2}-10} = 21(n-11) + 3q'_{\frac{n}{2}-6}, q'_{\frac{n}{2}-12} = n^2 - 20n + 120 - q'_{\frac{n}{2}} \quad \text{in} \quad \Pi_{n-5}. \]

**Sketch of proof.** The proof of the first part is straightforward. For the second part, consider again the lines which complete \( m \) lines, \( m \in \{n - 4, n - 5\} \), to \( n + 2 \) lines belong to \( \mathbf{H} \). Then, the classes \( C_{\frac{n}{2} - (a-k)} = C_{\frac{n}{2} + k} \) of \( \Pi_{n-4} \) and the classes \( C_k \) of \( \Pi_6, k \in \{0, 1, 2, 3\} \) are completely the same. Therefore, one can easily write

\[ q_{\frac{n}{2}} = 45 - 3q_{\frac{n-4}{2}}, q_{\frac{n}{2}} = 15(n - 7) + 3q_{\frac{n-4}{2}}, q_{\frac{n}{2}-10} = n^2 - 14n + 55 - q_{\frac{n}{2}} \]

from Corollary 2.2 The proof for \( \Pi_{n-5} \) is similar to above.

3. Some Models of the Hyperbolik Planes Containing two-Point Lines

The following proposition constitutes an answer for question in Bumcroft [1]: “How many two-point lines can be on a given point in a hyperbolic plane?”

**Proposition 3.1.** Let \( \Pi \) be a projective plane of order \( n \). Remove \( n - 1 \) lines (including all points on them) from \( \Pi \) such that \( n - 2 \) of them are concurrent. Denote this substructure as \( \Pi_{n-1} \). If \( \Pi \) is not a Fano plane, that is, the diagonal points of any complete quadrangle in \( \Pi \) are not collinear and \( n \geq 5 \), then \( \Pi_{n-1} \) is a hyperbolic plane.

**Proof.** Let \( l_i, l, (i = 1, 2, 3, \ldots, n-2) \), be removed lines from \( \Pi \) to obtain the hyperbolic plane \( \Pi_{n-1} \) and the lines \( l_i, (i = 1, 2, \ldots, n-2) \) be concurrent at the point \( Q_0 \). Denote, \( l_i \cap l = Q_i, (i = 1, 2, \ldots, n-2) \), and the other points of \( l \) as \( Q_{n-1}, Q_n, Q_{n+1} \).

It is clear that any line of \( \Pi_{n-1} \) on a point \( Q_j, (j = n - 1, n, n + 1) \), but not on \( Q_0 \) is a line including only two points of \( \Pi_{n-1} \) (two-point line). So, there exist exactly two two-point lines on any point of \( \Pi_{n-1} \).

We now start with the satisfaction of the hyperbolic plane conditions.

L1. Clearly, any two distinct points in \( \Pi_{n-1} \) as points of \( \Pi \) are on exactly one line in \( \Pi_{n-1} \).

L2. Let \( \Pi_{n-1} = (P', L', I') \) and \( d \in L' \).

If \( d \) does not contain \( l_i \cap l \) for \( 1 \leq i \leq n - 2 \), then \( k(d) = 2 \) or \( k(d) = n - 1 \). Hence \( k_n = 2 \). Therefore, each line of \( \Pi_{n-1} \) contains at least two points.

H1. \( \Pi_{n-1} \) has \( b = n^2 + 2 \) lines, \( 3n - 3 \) of them have degree \( 2 \), \( n - 3n + 2 \) of them have degree of \( 3 \), and the remaining three of them have degree \( n - 1 \). So through each point which is not on a line \( d \) there exists at least two lines that do not meet \( d \).

H2. All lines, which are of degree \( n - 1 \) do not meet in \( \Pi_{n-1} \). So there exist at least four points with no three collinear.

H3. Let \( S \) be a subset of \( P' \).
Suppose that $S$ contains three non-collinear points and all point on the lines through pairs of any distinct points of $S$.

If $S$ contains three distinct non-collinear points $A, B, C$ such that $k(AB) = k(AC) = 2$ in $\Pi_{n-1}$ and $AQ_{n+1}Q_0$ then $\{A, Q_0, Q_{n-1}, Q_{n+1}\}$ is a complete quadrangle in $\Pi$ and the diagonal points, $Q_n, B, C$ of this quadrangle in $\Pi$ are not collinear, since $\Pi$ is not a Fano plane. So the line $BC$ is not on $Q_n$. Therefore $k(BC) = 3$ in $\Pi_{n-1}$ and $|S| \geq 4$.

Suppose first that $|S| = 4$ and $S$ consists of all points on the triangle $\{A, B, C\}$. $k(BC) = 3$. Therefore $S$ contains all points of the line $AD$.

If $E'F'AD$ then $S$ contains all points of the line $BE$ and the line $CE$. So $k(BE) = k(CE) = 3$.

If $G'I'BE$ and $F'I'CE$, then $G'I'_{n-1}$ and $F'I'_{n+1}$. Therefore $S$ contains all points of $P'$. The proof for $|S| > 4$ is entirely similar to the proof for $|S| = 4$.

Remark: Consider the hyperbolic plane $\Pi_{n-1}^O$ in the case I of section 2. $\Pi_{n-1}^O$ has exactly two two-point lines meeting at a point of $\Pi_{n-1}^O$ (two tangent lines, not removed, of $O$).

References


PROJEKTİF DÜZLEMLERDEN ELDE EDİLEN SONLU HİPERBOLİK DÜZLEMLER ÜZERİNE BİR NOT

Özet

İI mertebesi n olan sonlu bir projektif düzlem, \( M \) de İI nin noktası olmayan üç doğru kapsayan herhangi doğrular cümlesi, \( |M| = m \), olsun. İI den \( M \) nin tüm doğrularının üzerindeki tüm noktalaryla birlikte atulmasıyla elde edilen \( \Pi_m \) hiperbolik düzlemi için \( m \) nin bilinen değerlerinden daha büyük olan bazı değerler elde edildi. \( \Pi_m \) tipi bazı hiperbolik düzlemlerin doğru sınıfları belirlendi. Bundan başka Buncrot[1] de iki noktası doğrular kapsayan hiperbolik düzlemlere dair bir soruya cevap verildi.

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