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SOME NEW SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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Abstract

In this paper we introduce and examine some properties of three sequence spaces defined by using a sequence of moduli.

Introduction

Ruckle [6] used the idea of a modulus function f to construct the sequence space

$$L(f) = \{x(x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space and Ruckle proved that the intersection of all such $L(f)$ spaces is Φ the space of finite sequences, thereby answering negatively a question of A. Wilansky; "Is there a smallest FK-space in which the set $\{e_1, e_2, ...\}$ of unit vectors in bounded?"

The space $L(f)$ is closely related to the space l_1 which is an $L(f)$ space with $f(x) = x$ for all real $x \geq 0$. Ruckle proved that, for any modulus f,

$$L(f) \subset l_1 \text{ and } L(f)^{\alpha} = 1_{\infty}$$

where

$$L(f)^{\alpha} = \{y = (y_k) : \sum_{k=1}^{\infty} |y_k x_k| < \infty \text{ fo all } x \in L(f)\}$$

is the α-dual of $L(f)$.

Let $A = (a_{nk})$ be an infinite matrix of nonnegative real numbers and let $p = (p_k)$ be a sequence of real numbers such that $p_k > 0$ for all k and $\sup_k \, p_k = H < \infty$. This

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assumption is made throughout the rest of this paper. We write $A_n(x)\sum_k a_{nk}|x_k|^{p_k}$ if the series converges for each n ad $A_{m,n}(x) = \sum_k a_{mk}|x_{k+n}|^{p_k}$ if the series converges for each m and n. (Here and afterwards summation without limits run from 1 to ∞). Following Ruckle [6] and Maddox [2], we recall that a function $f : [0,\infty) \to [0,\infty)$ such that modulus f is

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0, y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

It follows from (ii) and (iv) that f must be continuous everywhere on $[0,\infty)$.

In the present note we introduce and examine some new sequence spaces by using a sequence of moduli.

Definition 1. *Let F be a sequence of moduli and suppose that $A = (a_{mk})$ be a nonnegative regular matrix. We define*

$$w_0[A,p,F] = \left\{x \in w : \lim_m \sum_k a_{mk}\left[f_k\left(|x_{k+n}|\right)\right]^{p_k} = 0 \text{ uniformly in } n\right\},$$

$$w[A,p,F] = \left\{w \in w : \lim_m \sum_k a_{mk}\left[f_k\left(|x_{k+n} - L|\right)\right]^{p_k} = 0, \text{ for some } L, \text{ uniformly in } n\right\}.$$

$$w_\infty[A,p,F] = \left\{x \in w : \sup_{m,n} \sum_k a_{mk}\left[f_k\left(|x_{k+n}|\right)\right]^{p_k} < \infty\right\},$$

For a sequence of moduli $F = (f_k)$ we give following conditions:

(C_1) $\sup_k f_k(t) < \infty$ for all $t > 0$;

(C_2) $\lim_{t\to 0} f_k(t) = 0$ uniformly in $k \geq 1$.

We remark that in case $f_k = f(k \geq 1)$, where f is a modulus, the conditions (C_1) and (C_2) are automatically fulfilled.

When $f_k = f$ and $p_k = 1$ for all k, we denote these sequence spaces by $w_0[A,f], w[A,f]$ and $w_\infty[A,f]$. If $x \in w[A,f]$; we say that x is strongly almost A-summable to L with respect to the modulus f.

When $A = (a_{mk}) = (C,1)$ Cesaro matrix, $f_k = f$ and $p_k = 1$ for all k, we obtain generalization of the sequence spaces $[F_0(f)], [F(f)]$ and $[F_\infty(f)]$ which were defined by Pehlivan [7]. If $x \in [F(f)]$, we say that x is strongly almost convergent to L with respect to the modulus f.

If $[\hat{c}]$ denotes the set of all strongly almost convergent sequences, Maddox [1]

$$[\hat{c}] = \left\{ x : \lim_m \frac{1}{m} \sum_{k=1}^{m} |x_{k+n} - L| = 0, \text{ uniformly in } n \right\}$$

Note that if $A = (C, 1)$ Cesaro matrix $p_k = 1$ and $f_k(x) = x$ for all k, then $w[A, p, f] = [\hat{c}]$. Also in this case $w_\infty[A, p, f] = l_\infty$.

We now establish a number of useful theorems.

Therem 1. *$w_0[A, p, F], w[A, p, F]$ and $w_\infty[A, p, F]$ are linear spaces over the complex field C.*

Proof. We consider only $w[A, p, F]$. Others can be treated similarly. If $H = \sup_k p_k$ and $K = \max(1, 2^{H-1})$, we have Maddox [3] (p. 346).

$$|a_k + b_k|^{p_k} \leq K \cdot (|a_k|^{p_k} + |b_k|^{p_k}) \tag{1}$$

Suppose that $x \to L_1(w[A, p, F])$ and $y \to L_2(w[A, p, F])$. For $\lambda, \mu \in C$, there exists M_λ and N_μ integers such that $|\lambda| \leq M_\lambda$ and $|\mu| \leq N_\mu$. For (1), we write

$$\begin{aligned} A_{m,n}[F(\lambda x + \mu y - (\lambda L_1 + \mu L_2)e)] \leq\ & K(M_\lambda)^H A_{m,n}(F(x - L_1 e)) \\ +\ & K(N_\mu)^H A_{m,n}(F(y - L_2 e)) \end{aligned} \tag{2}$$

Where $A_{m,n}F((x)) = \sum_k a_{mk} \left[f_k \left(|x_{k+n}| \right) \right]^{p_k}$ and $e = (1, 1, 1, ...)$.

If follows from (2) $\lambda x + \mu y \to \lambda L_1 + \mu L_2 (w[A, p, F])$ and completes the proof. □

Theorem 2. *Let A be a nonnegative matrix and $F = (f_k)$ be sequence of moduli. If (C_1) holds then,*

$$w[A, p, F) \subset w_\infty[A, p, F]$$

Proof. It is a direct consequence of Property (1). □

Theorem 3. *$w_0[A, p, F]$ and $w[A, p, F]$ are complete linear topological spaces paranormed by g defined by*

$$g(x) = \sup_{n,m} \left\{ \sum_k a_{mk} \left[f_k \left(|x_{k+n}| \right) \right]^{p_k} \right\}^{\frac{1}{M}}$$

Where $M = \max(1, H = \sup_k p_k)$.

Proof. From Theorem 2, for each $x \in w[A,p,F], g(x)$ exists. Clearly $g(0) = 0, g(x) = g(-x)$ and by Minkowski's inequality $g(x+y) \leq [g(x)] + g(y)$. We now show that the scalar multiplication is continuous. Whence $\lambda \to 0, x \to 0$ imply $g(\lambda x) \to 0$ and also $x \to 0, \lambda$ fixed imply $g(\lambda x) \to 0$. We now show that $\lambda \to 0, x$ fixed imply $g(\lambda x) \to 0$.

Let $x \in w[A,p,f]$, then as $m \to \infty$, □

$$b_{m,n} = \sum_k a_{mk}\left[f_k\left(|x_{k+n} - L|\right)\right]^{p_k} \to 0 \text{ uniformly in } n.$$

For $|\lambda| < 1$ we have

$$\begin{aligned}\left\{\sum_k a_{mk}[f_k(|\lambda x_{k+n}|)]^{p_k}\right\}^{\frac{1}{M}} &= \left\{\sum_k a_{mk}\left[f_k\left(|\lambda x_{k+n} - \lambda L + \lambda L|\right)\right]^{p_k}\right\}^{\frac{1}{M}} \\ &\leq \left\{\sum_k a_{mk}\left[f_k\left(|(\lambda x_{k+n} - \lambda L)|\right) + f_k\left(|(\lambda L)|\right)\right]^{p_k}\right\}^{\frac{1}{M}}\end{aligned}$$

By Minkowski's inequality

$$\begin{aligned}&\leq \left\{\sum_k a_{mk}\left[f_k\left(|(\lambda x_{k+n} - \lambda L)|\right)\right]^{p_k}\right\}^{\frac{1}{M}} + \left\{\sum_k a_{mk}[f_k(|(\lambda L)|)]^{p_k}\right\}^{\frac{1}{M}} \\ &\leq \left\{\sum_{k>n} a_{mk}\left[f_k\left(|x_{k+n} - L|\right)\right]^{p_k}\right\}^{\frac{1}{M}} + \left\{\sum_{k\leq N} a_{mk}\left[f_k\left(|\lambda x_{k+n} - \lambda L|\right)\right]^{p_k}\right\}^{\frac{1}{M}} \\ &\quad + \left\{\sum_k a_{mk}\left[f_k\left(|(\lambda L)|\right)\right]^{p_k}\right\}^{\frac{1}{M}}\end{aligned}$$

Let $\epsilon > 0$ and choose N such that for each n, m and $k > N$ implies $b_{m,n} < \epsilon/2$. For each N, by continuity of f_k for all k, as $\lambda \to 0$,

$$\left\{\sum_{k\leq N} a_{mk}\left[f_k\left(|\lambda(x_{k+n} - L)|\right)\right]^{p_k}\right\}^{\frac{1}{M}} + \left\{\sum_k a_{mk}\left[f_k\left(|\lambda L|\right)\right]^{p_k}\right\}^{\frac{1}{M}} \to 0$$

Then choose $\delta < 1$ such that $|\lambda| < \delta$ implies

$$\left\{\sum_{k\leq N} a_{mk}\left[f_k\left(|\lambda(x_{k+n} - L)|\right)\right]^{p_k}\right\}^{\frac{1}{M}} + \left\{\sum_k a_{mk}\left[f_k\left(|\lambda L|\right)\right]^{p_k}\right\}^{\frac{1}{M}} < \frac{\epsilon}{2}$$

Hence we have

$$\left\{\sum_k a_{mk}\left[f_k\left(|\lambda x_{k+n}|\right)\right]^{p_k}\right\}^{\frac{1}{M}} < \epsilon$$

and $g(\lambda, x) \to 0 (\lambda \to 0)$. Thus $w[A, p, F]$ is paranormed linear topological space by g.

Now, we show that $w[A, p, F]$ is complete with respect to its paranorm topologies.

Let (x^i) be a Cauchy sequence in $w[A, p, F]$. Then, we write

$$G(x^i - x^j) \to 0 \quad \text{as} \quad i, j \to \infty \tag{3.1}$$

Hence for each fixed n and k, as $i, j \to \infty$, we have

$$\left[f_k\left(\left|(x^i_{k+n} - x^j_{k+n})\right|\right)\right]^{p_k} \to 0$$

By continuity of f_k for all k

$$\lim_{i,j\to\infty}\left[f_k\left(\left|(x^i_{k+n} - x^j_{k+n})\right|\right)\right]^{p_k} = \left[f_k\left(\lim_{i,j\to\infty}\left|(x^i_{k+n} - x^j_{k+n})\right|\right)\right]^{p_k} = 0$$

Since f_k is modulus for all k,

$$\lim_{i,j\to\infty}\left|(x^i_{k+n} - x^j_{k+n})\right| = 0$$

and for each fixed n and k (x^i_{k+n}), be a Cauchy sequence in C. Since C is complete, as $i \to \infty (x^i_{k+n}) \to (x_{k+n})$ say. Now from (3.1), we have for $\epsilon > 0$, there exists a natural number T such that

$$\left\{\sum_k a_{mk}\left[f_k\left(\left|(x^i_{k+n} - x^j_{k+n})\right|\right)\right]^{p_k}\right\}^{\frac{1}{M}} < \epsilon \tag{3.2}$$

for all m, n and $i, j > T$. Since for any fixed natural number N, we have from (3.2)

$$\left\{\sum_{k\le N} a_{mk}\left[f_k\left(\left|(x^i_{k+n} - x^j_{k+n})\right|\right)\right]^{p_k}\right\}^{\frac{1}{M}} < \epsilon \tag{3.3}$$

for all m, n and $i, j < T$. By taking $j \to \infty$ in the above expression we obtain

$$\left\{\sum_{k\le N} a_{mk}\left[f_k\left(\left|(x^i_{k+n} - x_{k+n})\right|\right)\right]^{p_k}\right\}^{\frac{1}{M}} < \epsilon$$

for all m, n and $i > T$. Since N is arbitrary, by taking $N \to \infty$ we obtain

$$\left\{ \sum_k a_{mk} \left[f_k \left(|x^i_{k+n} - x_{k+n}| \right) \right]^{p_k} \right\}^{\frac{1}{M}} < \epsilon$$

for all m, n and $i > T$ that is

$$g(x^i - x) \to 0 \text{ as } i \to \infty \text{ and thus } x^i \to x \text{ as } i \to \infty.$$

Also for each i, there exists L^i with

$$\sum_k a_{mk} \left[f_k \left(|(x^i_{k+n} - L^i)| \right) \right]^{p_k} \to 0 \quad (m \to \infty) \tag{3.4}$$

uniformly in n. From Regularity of A and (3.4), we have $[f_k(|L^i - L^j|)] \to 0$ as $i, j \to \infty$, for all k and (L^i) is a cauchy sequence in C, so (L^i) converges to L, say.

Consequently we get

$$\sum_k a_{mk} \left[f_k \left(|(x_{k+n} - L)| \right) \right]^{p_k} \to 0 \quad (m \to \infty)$$

uniformly in n. So that $x = (x_k) \in w[A, p, F]$ and the space is complete.

Theorem 4. *Suppose that A be a nonnegative regular matrix and $F = (f_k)$ be sequence of moduli, then;*

(i) $l_\infty \subset w_\infty[A, p, F]$
(ii) If $F = (f_k)$ is uniformly bounded on $[0, \infty), w_\infty[A, p, F] = w$.
(iii) If $0 < p_k \leq q_k$ and q_k/p_k is bounded, $w[A, q, F] \subset w[A, p, F]$.

Proof. (i) and (ii) are trivial.

(iii) If we take $w_{k,n} = [f_k(|x_{k+n} - L|)]^{p_k}$ for all k and n, then, using the same technique of Theorem 2 of Nanda [5] it is easy to prove (iii) □

Theorem 5. *Let $F = (f_k)$ is uniformly bounded on $[0, \infty)$ and A be a nonnegative regular matrix. When $x \in w_\infty[A, p, F]$*

$\sum_k a_k x_k$ *is convergent if and only if $(a_k) \in \Phi$.*

Proof. The sufficiency is easy. For necessity that $(a_k) \notin \Phi$. Hence there is a strictly increasing $(k(m))$ of positive integers $k(m)$ such that $|a_{k(m)}| > 0, m = 1, 2, 3, ...$

We define the sequence y by

$$y_k = \begin{cases} \frac{1}{a_{k(m)}} & , \quad k = k(m) \\ 0 & , \quad k \neq k(m) \end{cases}$$

Since $F = (f_k)$ is uniformly bounded on $[0, \infty)$ then,

$$\sum_k a_{mk} \left[f_k \left(|y_{k+n}| \right) \right]^{p_k} < \infty$$

hence $y \in w_\infty[A, p, F]$ but $\sum_k a_k \cdot y_k = \sum_m 1 = \infty$. This is a contradiction to $\sum_k a_k \cdot y_k$ convergent. This completes the proof. □

Corollary *Let $F = (f_k)$ is uniformly bounded $[0, \infty)$. Then,*

$$\left[w_\infty[A, p, F] \right]^\beta = \Phi$$

Where $\left[w_\infty[A, p, F] \right]^\beta = \left\{ y = (y_k) : \sum_k y_k x_k < \infty \text{ or all } x \in w_\infty[A, p, F] \right\}$ *is the β-dual of $w_\infty[A, p, F]$ and Φ denotes the space of all finite sequences.*

Theorem 6. *Let $F = (f_k)$ be a sequence of moduli and A be a nonnegative regular matrix. If (C_1) and (C_2) hold then*

$$w[A, p] \subset w[A, p, F]$$

Where $w[A, p] = \left\{ x \in w : lim_m \sum_k a_{mk} \left(|x_{k+n} - L| \right)^{p_k} = 0, \text{ for some} L, \text{ uniformly in } n \right\}$,

Proof. Using the same technique of Theorem 4 of Maddox [2] it is easy to prove of the Theorem. □

Theorem 7. *Let A be a nonnegative regular matrix and $F = (f_k)$ be sequence of moduli. If*

$$\beta = \lim_t (f_k(t)/t) > 0 \text{ for all } k, \text{ then}$$

$$w[A, p] = w[A, p, F]$$

Proof. In Theorem 6, it was shown that $w[A, p] \subset w[A, p, F]$. We must show that $w[A, p, F] \subset w[A, p]$. For any modulus function, the existence of positive limit given with β was given in Maddox [4]. Now $\beta > 0$ and let $x \in w[A, p, F]$. Since $\beta > 0$, for every $t > 0$ we write $f_k(t) \geq \beta.t$ for all k. From this inequality, it is easy to see that $x \in w[A, p]$. This completes the proof. □

Let $F = (f_k)$ and $G = (g_k)$ be sequences of moduli. The next theorem shows the relation between $w[A, p, F]$ and $w[A, p, G]$ for sequences for moduli F and G.

Theorem 8. *Suppose that $F = (f_k)$ and $G = (g_k)$ be sequences of moduli and $g_k \geq f_k$ for all k, then,*

$$\lim_{x \to \infty} \frac{f_k(x)}{g_k(x)} < \infty \text{ implies } w[A, p, G] \subset w[A, p, F]$$

Proof. It is trivial. □

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MODÜLÜS FONKSİYONLARININ BİR DİZİSİ YARDIMIYLA TANIMLANMIŞ BAZI YENİ DİZİ UZAYLARI

Özet

Bu çalışmada modülüs fonksiyonlarının bir dizisi yardımıyla bazı yeni dizi uzayları tanımlanmış ve bu uzayların sağladıı bazı özellikler verilmiştir.

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