SOME COMMUTATIVITY PROPERTIES FOR RINGS WITH UNITY

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Abstract

In this paper, we prove the commutativity of a ring $R$ with unity satisfying one of the following ring properties:

(P1) For each $x, y$ in $R$, \(1 - h(yx^r))\{x, yx^r - f(yx^r)\}(1 - g(yx^r)) = 0\) for some \(f(X) \in X^2\mathbb{Z}[X]\) and \(g(X), h(X) \in X\mathbb{Z}[X]\).

(P2) Given $x, y$ in $R$, \(1 - h(yx^r))\{x, yx^r - f(x^r)y\}(1 - g(yx^r)) = 0\) and \(1 - \tilde{h}(xy^r))\{y, y^x - \tilde{f}(xy^r)\}(1 - \tilde{g}(xy^r)) = 0\) for some \(f(X), \tilde{f}(X) \in X^2\mathbb{Z}[X]\) and \(g(X), \tilde{g}(X), h(X), \tilde{h}(X) \in X\mathbb{Z}[X]\).

(P3) For each $x, y \in R$, \(|x, yx^r - x^4f(y)x^t| = 0\) for some \(f(X) \in X^2\mathbb{Z}[X]\).

Introduction

Throughout this paper $R$ will represent a ring with unity 1, $N(R)$ the set of nilpotent elements in $R$, $N'(R)$ the subset of $N(R)$ consisting of all elements $a \in R$ with $a^2 = 0$ and $U(R)$ the group of units in $R$. For $x, y \in R$, the commutator $xy - yx$ will be denoted by $[x, y]$. Let $\mathbb{Z}$ be the ring of integers and let $r$, $s$ and $t$ be non-negative integers.

In this paper we consider the following properties:

(P1) For each $x, y \in R$, \(1 - h(yx^r))\{x, yx^r - f(yx^r)\}(1 - g(yx^r)) = 0\) for some \(f(X) \in X^2\mathbb{Z}[X]\) and \(g(X), h(X) \in X\mathbb{Z}[X]\).

(P2) Given $x, y$ in $R$, \(1 - h(yx^r))\{x, yx^r - f(x^r)y\}(1 - g(yx^r)) = 0\) and \(1 - \tilde{h}(xy^r))\{y, y^x - \tilde{f}(xy^r)\}(1 - \tilde{g}(xy^r)) = 0\) for some \(f(X), \tilde{f}(X) \in X^2\mathbb{Z}[X]\) and \(g(X), \tilde{g}(X), h(X), \tilde{h}(X) \in X\mathbb{Z}[X]\).

(P3) For each $x, y \in R$, \(|x, yx^r - x^4f(y)x^t| = 0\) for some \(f(X) \in X^2\mathbb{Z}[X]\).
Main Results

The main results of this paper are stated as follows:

**Theorem 1.** Let $R$ be a ring with unity $1$. If $R$ satisfies $(P_1)$, then $R$ is commutative.

**Theorem 2.** Let $R$ be a ring with unity $1$. If $R$ satisfies $(P_2)$, then $R$ is commutative.

**Theorem 3.** Let $R$ be a ring with unity $1$. If $R$ satisfies $(P_3)$, then $R$ is commutative.

As is easily seen from the proof of [5, Korollar (1)], if $R$ is a non-commutative ring, then there exists a factor subring of $R$ which is of type (a), (b), (c), (d) or (e):

(a) \( \begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix} \), $p$ a prime.

(b) \( M_{\sigma}(F) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \mid \alpha, \beta \in F \right\} \), where $F$ is a finite field with a non-trivial automorphism $\sigma$.

(c) A non-commutative division ring.

(d) A domain $S = (1) + T, T$ is a simple radical subring of $S$.

(e) $S = (1) + T, T$ is a non-commutative subring of $S$ such that $T[T, T] = [T, T]T = 0$.

The following result plays an essential role in our subsequent study:

**Meta Theorem.** Let $P$ be a ring property which is inherited by factor subrings. If no rings of type (a), (b), (c), (d), or (e) satisfy $P$, then every ring with $1$ and satisfying $P$ is commutative.

**Proof of Theorem 1.** In view of Meta Theorem, it suffices to show that $R$ cannot be of type (a), (b), (c), (d) or (e). For each $f(X) \in X^2Z[X]$ and $g(X), h(X) \in XZ[X]$, we set

\[ x = e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad y = e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

in the hypothesis to get

\[ \{1 - h(e_{21}e_{11}^r)\}[e_{11}, e_{21}e_{11}^r - f(e_{21}e_{11}^r)]\{1 - g(e_{21}e_{11}^r)\} = e_{21} \neq 0. \]

This is a contradiction.

Suppose that $R = M_{\sigma}(F)$, and put

\[ x = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}, \quad (\sigma(\alpha) \neq \alpha) \quad \text{and} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = e_{21}. \]

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Then for each \( f(X) \in X^2\mathbb{Z}[X] \) and \( g(X), h(X) \in \mathbb{Z}[X] \) we have
\[
\{1 - h(yz^r]\}[[x, yz^r - f(yz^r)]\{1 - g(yz^r)] = \epsilon_{21}(\alpha^r (\alpha - \sigma(\alpha)) \neq 0,
\]
which is a contradiction.

Suppose that \( R \) is a division ring. For any \( x \neq 0 \) and \( y \) in \( R \), there exist \( f(X) \in X^2\mathbb{Z}[X] \) and \( g(X), h(X) \in \mathbb{Z}[X] \) such that
\[
\{1 - h(yz^r]\}[[x, yz^r - f(yz^r)]\{1 - g(yz^r)] = 0.
\]

Thus
\[
\{1 - h(y)\}[[x, y - f(y)]\{1 - g(y)] = 0.
\]

Then either \( y, y - f(y) = 0 \) or \( y - yg(y) = 0 \) or \( y - yh(y) = 0 \). Therefore \( R \) is commutative by [3, Theorem 3].

Let \( u, v \in T \). Then \( u = 1 + v \in U(R) \), and there exist \( f(X) \in X^2\mathbb{Z}[X] \) and \( g(X), h(X) \in \mathbb{Z}[X] \) such that
\[
0 = \{1 - h(wv^ru^r]\}[[u, wv^ru^r - f(wv^ru^r)]\{1 - g(wv^ru^r)]\}
\]

Thus
\[
0 = \{1 - h(w]\}[[u, w - f(w)]\{1 - g(w)]
\]

Then, either \( [u, w - f(w)] = 0 \), \( w - wzg(w) = 0 \) or \( w - wgh(w) = 0 \). Hence, \( T \) is commutative by [3, Theorem 3]. But this is a contradiction.

Finally, suppose that \( R \) is of type (e). Let \( v, v \in T \). Then \( u = 1 + v \in U(R) \) and there exist \( f(X) \in X^2\mathbb{Z}[X] \) and \( g(X), h(X) \in \mathbb{Z}[X] \). In the hypothesis, replace \( x \) by \( 1 + v \) and \( y \) by \( w \), to get
\[
0 = \{1 - h(w(1 + v)^r]\}[[1 + v), w(1 + v)^r - f(w(1 + v)^r)]\{1 - g(w(1 + v))]\}
\]

Thus
\[
0 = \{1 - h(w(1 + v)^r]\}[[1 + v)\{1 + v)^r][v, w]\{1 - g(w(1 + v))]\}
\]

This implies that \( [v, w] = 0 \). Therefore \( T \) is commutative. This is a contradiction.

**Proof of Theorem 2.** Let \( x, y \in R \) and let \( f(X) \in X^2\mathbb{Z}[X] \) and \( g(X), h(X) \in \mathbb{Z}[X] \) such that
\[
\{1 - h(yx^r]\}[[x, yx^r - f(yx^r)]\{1 - g(yx^r)] = 0.
\]

Let \( z \in R \) such that \( f(x^r) = x^rz \) and \( f(yx^r) = xz^r \). Now we choose \( f(X) \in X^2\mathbb{Z}[X] \) and \( \tilde{g}(X) f(X) \in \mathbb{Z}[X] \) such that
\[
0 = \{1 - \tilde{h}(f(yx^r]\}[[x, f(x^r) y - \tilde{f}(f(yx^r))]\{1 - \tilde{g}(f(yx^r))]\}
\]

and
\[
0 = \{1 - \tilde{h}(xz^r]\}[[x, x^r z - \tilde{f}(xz^r)]\{1 - \tilde{g}(xz^r)]\}
\]

Now, combining (1) and (2) gives
\[
0 = \{1 - \tilde{h}(f(yx^r]\}[[1 - h(yx^r)]\{1 - h(yx^r)]\{1 - g(yx^r)]\{1 - g(yx^r)]\}
\]

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This implies that
\[ 0 = \{ 1 - h(yx^r) \} [x, yx^r - f(yx^r)] \{ 1 - g(yx^r) \}. \]

By using the same line of the proof of Theorem 1, we prove the commutativity of \( R \).

In preparation for proving Theorem 1, we first establish the following Lemmas:

**Lemma 1.** If \( R \) satisfies \((P_3)\) and \( x \) is in \( U(R) \), then for each \( y \in R \), there exists \( h(X) \in X^2Z[X] \) such that \( [x, y - h(y)] = 0 \).

**Proof.** Let \( x \in U(R) \) and \( y \in R \). Then we choose \( f(X) \in X^2Z[X] \) such that
\[ [x^{-1}, yx^{-r} - x^{-s} f(y)x^{-t}] = 0. \]
Hence \( [x, yx^{-r} - x^{-s} f(y)x^{-t}] = 0 \) and \( [x, y] x^{-r} = x^{-s}[x, f(y)] x^{-t} \). Thus
\[ x^s[x, y] x^t = [x, f(y)] x^r. \] (3)

Now let \( g(X) \in X^2Z[X] \) such that \( [x, f(y) x^{-r} - x^{-s} f(g(y)) x^{-t}] = 0. \) Since \( g(f(X)) \in X^2Z[X] \) and
\[ [x, f(y)] x^{-r} = x^s[x, h(y)] x^t. \] (4)

Combining equation (3) and (4) gives \( x^s[x, y] x^t = x^s[x, h(y)] x^t \) and so \( [x, y] = [x, h(y)] \). Therefore, \( [x, y - h(y)] = 0 \). \( \square \)

**Lemma 2.** If \( R \) satisfies \((P_3)\) and \( R \) is a division ring, then \( R \) is commutative.

**Proof.** Let \( R \) be a division ring. By Lemma 1, for each \( x, y \in R \), there exists \( f(X) \in X^2Z[X] \) such that \( [x, y - f(y)] = 0. \) Thus \( R \) is commutative by [3, Theorem 3]. \( \square \)

**Lemma 3.** If \( R \) satisfies \((P_3)\) and \( R = (1) + T \), \( T \) is a radical subring of \( R \), then \( R \) is commutative.

**Proof.** Let \( v, w \in T. \) Then \( 1 - v \) is a unit and by Lemma 1, there exists \( f(X) \in X^2Z[X] \) such that
\[ [v, w - f(w)] = [1 - v, w - f(w)] = [v, w - f(w)] = [1 - v, w - f(w)] = 0. \]

Thus \( R \) is commutative by [3, Theorem 3]. \( \square \)

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Proof of Theorem 3. In view of Lemma 2 and Lemma 3, no rings of type (c) or (d) satisfy \( (P_3) \). In \( R = M_2(GF(p)) \), where \( GF(p) \) is the Galois field over a prime \( p \), we see that \( [e_{11}, e_{21}e_{11} - e_{11}f(e_{21})e_{11}] = e_{21} \neq 0 \), for every \( f(X) \in X^2 \mathbb{Z}[X] \). Thus we have a contradiction. Hence, no rings of type (a) satisfy \( (P_3) \).

Next consider the ring \( M_2(F) \). Let
\[
x = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}, \quad (\sigma(\alpha)) \neq \alpha \quad \text{and} \quad y = e_{21}.
\]
Then
\[
[x, yx^r - x^sf(y)x^t] = [x, y]^t
= (\sigma(\alpha) - \alpha)y\alpha^r
= (\alpha - \sigma(\alpha))\alpha^r y \neq 0
\]
for every \( f(X) \in X^2 \mathbb{Z}[X] \).

Finally, suppose that \( R \) is of type (e). For each \( v, w \in T \), there exists \( f(X) \in X^2 \mathbb{Z}[X] \) such that
\[
[v, w] = [v, w](v + 1)^r - (v + 1)^s[v, f(w)](v + 1)^t = 0.
\]
This is a contradiction.

We have thus seen that no rings (a), (b), (c), (d) or (e) satisfy \( (P_3) \). Hence \( R \) is commutative by Meta Theorem.

References


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