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EXTENSION AND SEPARATION OF VECTOR VALUED FUNCTIONS

Zafer Ercan

Abstract

It is proved that: If X is a paracompact Hausdorff space and E is a Fréchet lattice then (X, E) has the separation property. This is employed to extend some varies of functions that are known for spaces of Banach lattice valued functions.

1. Preliminaries:

For a topological space X and a topological Riesz E, we denote by

$C(X, E) = \{f : f$ is a continuous function from X into $E\}$

$C_0(X, E) = \{f : f \in C(K, E)$ and for a given neighborhoad U of E there exists a compact subset K of X such that $f(X \setminus K) \subset U\}$

$C^0(X, E) = \{f : f \in C(K, E)$ and $f(X)$ is order bounded in $E\}$

$C^r(X, E) = \{f : f \in C(X, E)$ and $f(X)$ is relatively compact in $E\}$.

For a set X, $S(X)$ will denote the non-empty subsets of X. Let X and Y be topological spaces. A map $\phi : X \to S(Y)$ is called a lower semicontinuous carrier if $\{x \in X : \phi(x) \cap U \neq \phi\}$ is an open subset of X for each open subset U of Y.

For a proof of the following Lemma we refer the reader to [2] or [3].

Lemma 1.1. Let X be a topological space, E a topological vector space and U be a non-empty open subset of E. Let ϕ, φ be lower semicontinuous carriers from X into $S(E)$ satisfying $\phi(x) \cap (\varphi(x) + U) \neq \phi$ for each $x \in X$. The map $\Phi : X \to S(E)$, defined by, $\Phi(x) = \phi(x) \cap (\varphi(x) + U)$ is also a lower semicontinuous carrier.

We shall need the following theorem which was proved in [3].

Theorem 1.2. *Let X be a paracompact Harsdorff space, E a Fréchet space, and $\phi : X \to 2^E$ with each set $\phi(x)$ being a non-empty closed convex set. Suppose also that for each $\epsilon > 0$ there is a lower semicontinuous carrier $\phi : X \to 2^E$ with each $\phi_\epsilon(x)$ convex, and with $\phi(x) \subseteq \phi_\epsilon(x) \subseteq \phi(x) + B_\epsilon$ for each $x \in X$, where $B_\epsilon = \{e \in E, d(e,0) < \epsilon\}$, ($d$ denoting the metric on E). There is a continuous function $h : X \to E$ with $h(x)\epsilon\phi(x)$ for each $x \in X$.*

For unexplained definitons we refer the reader to [1] and [5].

2. Separation Property

In this section with a suitable modifications of the definitions of upper and lower semicontinuity we will reprove the Hahn-Tong-Kaketov Theorem for Frechlet lattice valued functions defined on a paracompact Hausdorff space.

Definition 2.1 ([3]). *Let X be a topological space and E a locally solid Riesz space. A function $f : X \to E$ is called*

(i) upper semicontinuous if $f^{-1}(U - E_+)$ is open for each open subset U of E

(ii) lower semicontinuous if $f^{-1}(U + E_+)$ is open for each open subset U of E

It is known that a function f from a compact Hausdorff space X into a Banach lattice E is continuous if and only if it is both lower and upper semicontinuous (see [3]). The following is a generalization of that.

Proposition 2.2. *If X is any topological space and E a locally solid Riesz space then a function $f : X \to E$ is continuous if and only if it is both lower and upper semicontinuous.*

Proof. Since $U - E_+$ and $U + E_+$ are open subset of E for each open set $U \subseteq E$ it is clear that a continuous function is also lower and upper semicontinuous. Suppose that $f : X \to E$ is both lower and upper semicontinuous. Let $x_0 \in X, 0 \in V, 0 \in W$ and $W + W \subseteq U$ since

$$x_0 \in f^{-1}(f(x_0) + W - E_+) \cap f^{-1}(f(x_0) + W + E_+)$$

and

$$f(f^{-1}(f(x_0) + W - E_+)) \cap f^{-1}(f(x_0) + W + E_+)) \subseteq f(x_0) + U$$

f is continuous at x_0. Since x_0 is an arbitrary element of X, f is continuous. $\square$

Definition 2.3. *Let X be a topological space, E a locally solid Riesz space. We shall say that the pair (X, E) has the separation property if given any upper semicontinuous function $f : X \to E$ and lower semicontinuous function $g : X \to E$ with $f(x) \leq g(x)$ for all $x \in X$ there exists a continuous function $h : X \to E$ such that $f(x) \leq h(x) \leq g(x)$ for all $x \in X$.*

We now prove our main result, which generalise the Hahn-Tong-Kaketov separation theorem for Banach lattice valued functions on compact Hausdorff space [3]. See also [8] for real valued case.

Proposition 2.4: *If X is a paracompact Hausdorff space and E is a Frechlet lattice. Then (X, E) has the separation property.*

Proof. Let $f : X \to E$ be an upper semicontinuous function and $g : X \to E$ a lower semicontinuous function with $f(x) \leq g(x)$ for all $x \in X$. Let $G : X \to 2^E$ be defined by $G(x) = (f(x) + E_+) \cap (g(x) - E_+)$. $G(x)$ is a non-empty convex set for each $x \in X$. For $\epsilon > 0$ choose an open convex solid set V_ϵ with $0 \epsilon V_\epsilon \subseteq V_\epsilon + V_\epsilon \subseteq B_\epsilon$ where $B_\epsilon = \{e \in E : d(e, 0) < \epsilon\}$. Since for each $\epsilon > 0$ the maps $f + E_+, g - E_+ + V_\epsilon : X \to 2^E$ with $(f + E_+)(x) = f(x) + E_+, (g - E_+, V_\epsilon)(x) = g(x) - E_+ + V_\epsilon$ are lower semicontinuous carriers, the map $G_\epsilon : X \to 2^E$ defined by $G_\epsilon(x) = (f(x) + E_+ + V_\epsilon) \cap (g(x) - E_+ + V_\epsilon)$ is, by Lemma 1.1, also a lower semicontinuous carrier and its values are convex sets. If $y \epsilon G_\epsilon(x)$ then there are $s_1, s_2 \epsilon V_\epsilon$ such that

$$f(x) + s_1 \leq y \leq g(x) + s_2$$

If we set $s = (-s_1) \vee s_2 \vee 0$ then $|s| \leq |s_1| + |s_2| \epsilon V_\epsilon + V_\epsilon$ so $s \epsilon V_\epsilon + V_\epsilon$ and that

$$f(x), y - s \leq g(x), y + s.$$

If we set $u = f(x) \vee (y-s)$, it follows from $f(x), y-s \leq u \leq g(x), y+s$ that $u \in G(\epsilon)$. Since $|u-y| \leq |s| \leq |s_1| + |s_2| \epsilon V_\epsilon + V_\epsilon$, $u - y \epsilon V_\epsilon + V_\epsilon$ $y = u - (u-y) \epsilon G(x) + V_\epsilon + V_\epsilon \subseteq G(x) + B_\epsilon$. We have now established that $G(x) \subseteq G_\epsilon(x) \subseteq G(x) + B_\epsilon$, so we may apply Theorem 1.2 to find a continuous selection $h : X \to E$ to G which will satisfy our requirements, i.e., $f(x) \leq h(x) \leq g(x)$ for all $x \in X$. Hence (X, E) has the separation property. □

We do not know whether the paracompactness of X can be replaced with normality of X in the above proposition. It is affimative if $E = \mathbb{R}$.

3. Extention of Functions

The following theorem is a generalization of Tietze's extension Theorem.

Proposition 3.1: *Let A be a closed subset of a topological space X and E a locally solid Riesz space such that (X, E) has the separation property. If $f : X \to E$ is an upper semicontinuous function, $g : X \to E$ is a lower semicontinuous function and $h : A \to E$ a continuous function with $f(x) \leq g(x)$ for all $x \in X$ and $f(y) \leq h(y) \leq g(y)$ for all $y \epsilon A$. Then there is a continuous extension of h to $\overline{h} : X \to E$ such that $f(x) \leq \overline{h}(x) \leq g(x)$ for all $x \in X$.*

Proof. Define $f', g' : X \to E$ by

$$f'(x) = \begin{cases} -f(x) & if \quad x \epsilon X \setminus A \\ -h(x) & if \quad x \epsilon A \end{cases} \quad \text{and} \quad g'(x) = \begin{cases} g(x) & if \quad x \epsilon X \setminus A \\ h(x) & if \quad x \epsilon A \end{cases}$$

It is easy to see that $-f', g'$ are lower semicontinuous functions, so $-f'$ is upper semicontinuous and $(-f')(x) \leq g'(x)$ for all $x \in X$. Since (X, E) has the separation property there exists a continuous function $\overline{h} : X \to E$ such that $(-f')(x) \leq \overline{h}(x) \leq g'(x)$ for all $x \in X$. Obviously $\overline{h}$ is an extension of h and $f(x) \leq \overline{h}(x) \leq g(x)$ for all $x \in X$. □

Another class of extension results are of importance in the study of $C(K)$ spaces from an order theoretic viewpoint. These are those which involve the continuous extension of functions defined on open subsets of spaces with some strong disconnectedness property.

For real case the proof of the following theorem can be found in [9].

Theorem 3.2. *Let X be a nondiscrete locally compact extremally disconnected Hausdorff space, E a Fréchet lattice. The following are equivalent:*

(i) E *has compact order intervals.*

(ii) $C_0(X,E)$ *is Dedekind complete.*

(iii) If U *is an open subset of* X, $f \epsilon C(U,E)$ *and* $g, h \epsilon C_0(X,E)$ *with* $h(x) \leq f(x) \leq g(x)$ *for all* $x \in U$ *then* f *has a continuous extension* $\overline{f} \epsilon C_0(X,E)$.

Proof. (i) $\Rightarrow$ (ii) $C_0(X)$ is an ideal in $C(\beta X)$ where βX is the stone-Cech compactification of X. Since $C(\beta X)$ is Dedekind complete $C_0(X)$ is also Dedekind complete. Hence Theorem 3.2.6. of [2] implies that $C_0(X,E)$ is Dedekind complete.

(ii) $\Rightarrow$ (iii) We may suppose that $0 \leq f$ in $C(U,E)$ and $0 \leq g$ in $C_0(X,E)$. The set $\mathcal{A} = \{\mathcal{B} \leq S(U) : \mathcal{B}$ is a family of disjoint open-closed subsets of $U\}$ is partially ordered under the ordering $\mathcal{B}_1 \leq \mathcal{B}_2 \Leftrightarrow \mathcal{B}_1 \subseteq \mathcal{B}_2$. Clearly $\{\phi\} \in \mathcal{A}$ and every chain in $\mathcal{A}$ has an upper bound, so by Zorn's lemma $\mathcal{A}$ has a maximal element, say $(W_\gamma)_{\gamma \in I}$ and for each $\gamma \in I$ define

$$f_\gamma(x) = \begin{cases} f(x) & if \quad x \in W_\gamma \\ 0 & if \quad x \notin W_\gamma \end{cases}$$

then $\{f_\gamma : \gamma \in I\}$ is a subset of $C_0(X,E)$ which is bonuded above by g. By (ii), $\sup\limits_{\gamma \in I} f_\gamma = \overline{f}$ exists in $C_0(X,E)$. Note that $U \subset \overline{\bigcup_{\gamma \in I} W_\gamma}$.

Let $\gamma \in I$ be fixed. If $x_0 \in W_\gamma$ then $f(x_0) = f_\gamma(x_0) \leq \overline{f}(x_0)$. We wish to show $f(x_0) = \overline{f}(x_0)$. Suppose $f(x_0) < \overline{f}(x_0)$. Let $\alpha = \frac{1}{2} d(\overline{f}(x_0), f(x_0))$ and $A = \{x \in X : \alpha < d(f(x), \overline{f}(x))\} \cap W_\gamma$ which is open. Choose an open set V with $x_0 \epsilon V \leq \overline{V} \leq A \subseteq W_\gamma$ and define

$$f'(x) = \begin{cases} f(x), & if \quad x \in \overline{V} \\ \overline{f}(x), & if \quad x \notin \overline{V} \end{cases}$$

f' is an upper bound of the set $\{f_\gamma : \gamma \in I\}$ so we must have $\overline{f} \leq f'$. Since $\overline{f}(x_0) \leq f'(x_0) = f(x_0) < \overline{f}(x_0)$, the assumption implies a contradiction. Hence $f_{|W_\gamma} = \overline{f}_{|W_\gamma}$ for all $\gamma \in I$. If $x \in U$ then there is a net $\{x_\gamma\}$ in $\bigcup_{\gamma \in I} W_\gamma$ such that

$x_\gamma \to x$. Since U is open we may suppose that $x_\gamma \in U$ for all γ, so $\overline{f}(x_\gamma) \to \overline{f}(x)$ and $\overline{f}(x_\gamma) = f(x_\gamma) \to f(x)$ which shows that $\overline{f}$ is a continuous extension of f in $C_0(X, E)$.

(iii) $\Rightarrow$ (i) If (i) does not hold then choose $e \epsilon E_+$ such that the order interval $[0, e]$ is not compact and let $\{e_n : n = 1, 2, \cdots\} \subseteq [0, e]$ not be totally bounded. Since X is nondiscrete there exists a compact subset A of X and sequence $\{U_n\}$ of disjoint non-empty open and closed sets U_n with $U_n \subset A$ for all n. Define f on $U = \cup_n U_n$ by $f(x) = e_n$ if $x \in U_n$ and define $g : X \to E$ by $g(x) = \begin{cases} e, & if \quad x \in \overline{U} \\ 0, & if \quad x \notin \overline{U} \end{cases}$ Then $0 \le g$ and $0 \le f(x) \le g(x)$ for all $x \in U$. By (iii) f has a continuous extension $\overline{f} \in C_0(X, E)$. Since, $\{e_n : n \in \mathbb{N}\} \subseteq f(A) = \overline{f}(A) \subseteq \overline{f}(\overline{A})$ it implies that $\{e_n : n \in \mathbb{N}\}$ is totally bounded. This contradiction shows that (iii) $\Rightarrow$ (i), completing the proof □

Similar argument yield the following theorem. We note that for the real case the proof can be found in [9]. See also [3] for Banach lattice case.

Theorem 3.3. *Let X be a non-discrete, locally compact, quasi-extremally disconnected Hausdorff space, E a Fréchet lattice such that (X, E) has the separation property. The following conditions are equivalent:*

(i) E has compact order intervals.

(ii) $C_0(X, E)$ is Dedekind σ-complete

(iii) if U is an open F_σ subset of X, $f \in C(U, E)$ and $g, h \in C_0(X, E)$ such that $h(x) \le g(x) \le g(x)$ for all $x \in U$ then f has a continuous extension $f^\sim \in C_0(X, E)$.

We do not know whether the condition (X, E) has the separation property is dispensable in the above theorem. The answer is affirmative if X is extremally disconnected by Theorem 3.2.

We shall say that a topological space is non-anticompact if there is no infinity compact subset of X.

Theorem 3.4. *Let X be a non-anticompact completely regular, extremally disconnected, Hausdorff space and E a Fréchet lattice. Then the following conditions are equivalent:*

(i) E *has compact order intervals.*

(ii) $C^0(X,E)$ *is Dedekind complete.*

(iii) $C^r(X,E)$ *is Dedekind complete.*

(iv) If U *is an open subset of* X, $f \in C^r(U,E), g,h \in C^r(X,E)$ *such that* $h(x) \leq f(x) \leq g(x)$ *for all* $x \in U$ *then* f *has an continuous extension* $f^{\sim} \in C^r(X,E)$.

(v) If U *is an open subset of* X, $f \in C^0(U,E)$, $g,h \in C^0(X,E)$ *such that* $h(x) \leq f(x) \leq g(x)$ *for all* $x \in U$ *then* f *has a continuous extension* $f^{\sim}C^0(X,E)$

Proof. We refer [4] for equivalence of (i) to (iii). Rest of the proof is very similar to the proof of Theorem 3.2. □

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Zafer ERCAN
Middle East Technical University,
Department of Mathematics,
06531 Ankara-TURKEY

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