A CHARACTERIZATION OF $\overline{NC}$ – $p$-GROUPS

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Abstract

In this work, presented is a partial characterization of a perfect locally nilpotent $p$-group in which every proper subgroup is nilpotent-by-Chernikov.

1. Introduction

Let $G$ be a locally finite group. If every proper subgroup of $G$ is nilpotent-by-finite ($NF$-group) while $G$ is not nilpotent-by-finite, then $G$ is called a minimal non nilpotent-by-finite group ($\overline{NF}$-group). If in the above definition "finite" is replaced by "Chernikov", then one obtains $NC$-groups and $\overline{NC}$-groups, respectively. Bruno in [4] showed, among other things, that if $G$ is a non-perfect $\overline{NF}$-group for a prime $p$, then $G/G' \cong C_{p\infty}, G'$ is nilpotent and $G'$ is not properly supplemented in $G$. The first example of such a group was constructed by Heineken and Mohamed in [8]. For this reason groups of this type are called $HM$-groups. Later, Meldrum in [9] and Hartley in [6] gave similar constructions. Bruno and Phillips in [5] constructed an $HM$-group with a metabelian derived subgroup. Recently, Menegazzo in [10] constructed $HM$-groups with derived subgroups either abelian of infinite exponent or solvable of arbitrary derived length. Thus the structure of nonperfect $\overline{NF}$ – $p$-groups is known. However it is not known yet whether or not perfect $\overline{NF}$ – $p$-groups exist.

Otal and Peña in [12] extended some of the results of [4] to $\overline{NC}$-groups. Later $\overline{NC}$ – $p$-groups were considered in [1], [2] and [3]. Theorem A of [2] (see also the end of this section) shows that an $\overline{NC}$ – $p$-group must be perfect. Another consequence of the same theorem is that a perfect $\overline{NF}$ – $p$-group has a proper epimorphic image in which every proper subgroup is nilpotent of finite exponent. Of course, the determination of $\overline{NC}$ – $p$-groups will be useful in the investigation of perfect locally nilpotent $p$-groups, about which not much is known yet. In this work the first question is considered. The main results of this work are stated below.

Theorem. Let $G$ be an $\overline{NC}$ – $p$-group. Then $G$ contains a proper normal subgroup $K$ such that every proper subgroup of $G/K$ is a Chernikov extension of a nilpotent subgroup of finite exponent and one of the following holds:
(i) $G/K$ is an $\overline{N^F}$ - $p$-group.

(ii) $G/K = \bigcup_{i=1}^{\infty} (T_i/K)'$, where for each $i \geq 1$, $T_i/K$ is an $HM^*$-subgroup of $G/K$ such that $(T_i/K)/(T_i/K)' \cong C_p^{\infty}$ and $T_i/K \leq T_{i+1}/K$.

(iii) $G/K$ is generated by its normal $HM^*$-subgroups $T/K$ such that $(T/K)/(T/K)' \cong C_p^{\infty}$.

(The definition of an $HM^*$-group is given below)

**Corollary.** Let $G$ be an $\overline{NC}$ - $p$-group satisfying the normalizer condition. Then $G$ contains a proper normal subgroup $K$ such that every proper subgroup of $G/K$ is a Chernikov extension of a nilpotent subgroup of finite exponent and (i) or (iii) of the theorem holds.

It is not known whether or not (ii) and (iii) are necessary in the above results.

In the study of $NC$-groups certain subgroups, similar to the Heineken-Mohamed group, are unavoidable. To single them out, they were called $HM^*$-groups in [2]. By definition a locally nilpotent $p$-group $X \neq 1$ is called an $HM^*$-group if $X'$ is nilpotent and

$$X/X' \cong C_p^{(n)} = C_p^{\infty} \times \cdots \times C_p^{\infty}$$

is $n$ copies for some $n \geq 1$. If $n = 1$, $X' \neq 1$ and every proper subgroup of $X$ is subnormal in $X$ (in which case $X'$ is not properly supplemented in $X$ by Lemma 2.2 (iii)) then $X$ is called a group of Heineken-Mohamed type or an $HM$-group for brevity.

Elementary properties of $HM^*$-groups were collected in [2] and [3] but they will be restated here (see §2) in order to make this work self-contained.

For any group $X$, $X^{\circ}$ denotes the unique maximal normal radicable abelian subgroup of $X$, whenever it exits. We end this section by stating Theorem A of [2].

**Theorem A.** Let $G$ be locally nilpotent $p$-group which does not have any proper subgroup of finite index. Suppose that every proper subgroup of $G$ is an NC-group. Then the following hold.

(i) If $G = G'$, then $G$ is the union of an ascending chain of normal nilpotent subgroups.

Furthermore, $G$ has a normal nilpotent subgroup $N$ such that in $G/N$ every proper subgroup is a Chernikov extension of a nilpotent subgroup of finite exponent and, for every normal nilpotent subgroup $K$ of $G$, $KN/N$ has finite exponent.

(ii) If $G' \neq G$, then $G$ is an NC-group.

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2. Properties of $HM^*$-Groups

Lemma 2.1. Let $A$ be a nilpotent $p$-group and $B$ be a normal subgroup of $A$ of finite exponent $m$ such that $A/B \cong C_p^{(n)} = C_{p^\infty} \times \cdots \times C_{p^\infty}$, $n$ factors for some $n \geq 1$. Then $A = TB$, where $T = A^\circ$.

Proof. We may use induction on the nilpotency class $c$ of $A$. First suppose that $c = 1$. Then $A$ is abelian. Define $f : A \to A$ by $f(a) = a^m$. Then $f$ is a homomorphism with kernel $K = \{a : a^m = 1\}$. Also, $B \leq K$. Therefore $f(A) \cong A/K \cong C_p^{(n)}$ since $K$ has finite exponent. So if we put $T = f(A)$ then $TB/B \cong C_p^{(n)} \cong A/B$ which implies that $TB = A$.

Next suppose that $c > 1$ and the assertion holds for $c - 1$. Let $R = K_c(A)$, the $c$th term of the lower central series of $A$ and put $A = A/R$. Let $\overline{U} = \overline{A}^\circ$. By induction hypothesis $A = \overline{UB}$. Also, since $\overline{U} \cong C_p^{(n)}$ and $R \leq Z(U)$, it is easy to see that $U$ is abelian and so $U = VR$, where $V = U^\circ$ by the first paragraph. Substituting this in $A = UB$ gives that $A = VB$. Also $V = A^\circ$ since $B$ has finite exponent. This completes the proof of the lemma. 

Lemma 2.2. Let $X$ be an $HM^*$-group for a prime $p$. Then the following hold.

(i) $X' = [X, X']$.

(ii) There does not exist any proper normal subgroup $N$ of $X$ satisfying $X = NX'$. In particular, $X/N$ cannot have finite exponent.

(iii) If $X$ satisfies the normalizer condition then $X'$ is not properly supplemented in $X$.

Proof. (i) Let $\overline{X} = X/\{X, X'\}$. Then $\overline{X} \leq Z(\overline{X})$ and so $\overline{X}$ is nilpotent and $\overline{X}/\overline{X}'$ is radicable abelian which implies that $\overline{X}$ is abelian and, hence, $X' \leq [X, X']$ by Theorem 9.23 of [13].

(ii) Assume that $X = NX'$ for some proper normal subgroup $N$ of $X$. Then

$$X' = [X', NX'] = [X', N]X'' = [X', N] \leq N$$

and so $X = N$ by (i) and by Lemma 2.22 of [13], which is a contradiction. Also, if $X/N$ has finite exponent then, as $X/NX'$ is radicable abelian and has finite exponent, it follows that $X = NX'$ and so $X = N$ by the first part of (ii), which is another contradiction.

(iii) Assume that $X = CX'$ for some $C < X$. Let $D = C \cap X'$ and put $\overline{X} = X/DX''$. Then $\overline{X} = \overline{C}\overline{X}'$ and $\overline{C}$ is radicable abelian and Chernikov.
\[ Y = N_{\mathcal{X}}(C). \] Then \( Y = C(Y \cap X') \) which implies that \( Y \) is nilpotent since \( C \) and \( Y \cap X' \) are normal nilpotent subgroups of \( Y \). In particular, \( \overline{C} \leq Z(\overline{Y}) \) by Lemma 3.13 of [13]. Assume that \( \overline{Y} < \overline{X} \) and put \( \overline{V} = N_{\overline{X}}(\overline{Y}) \). Then \( \overline{Y} < \overline{V} \) by hypothesis. But since \( \overline{V} = C(\overline{V} \cap \overline{X'}) \) it follows as in the first case that \( \overline{C} \leq Z(\overline{V}) \) and so \( \overline{V} \leq \overline{Y} \) which is a contradiction. Consequently, it follows that \( \overline{Y} = \overline{X} \) and so \( \overline{C} \leq Z(\overline{X}) \). But now

\[
\overline{X'} = [\overline{X'}, \overline{X}] = [\overline{X'}, \overline{C \overline{X'}}] = [\overline{X'}, \overline{X}] = \overline{X''},
\]

which is possible only if \( \overline{X'} = 1 \) and so \( X' \leq DX'' \leq D \leq C \) by Lemma 2.22 of [13] since \( X' \) is nilpotent. This a contradiction since \( C < X \).

\[ \square \]

**Lemma 2.3.** Let \( X \) be an NC-p-group and \( T \) be an \( H^M_* \)-subgroup of \( X \). If \( N \) is a normal subgroup of \( X \) such that \( X/N \) is Chernikov then \( T' \leq N \).

**Proof.** Let \( D = T' \cap N \) and put \( T = T/D \). Then \( T' \) is nilpotent and Chernikov so the Corollary to Theorem 3.29 (2) of [13] gives that \( T/C_{\mathcal{T}}(T') \) is finite which implies that \( T = C_{\mathcal{T}}(T') \) and hence \( T' = [T', T] = 1 \) by (ii) and (i) of Lemma 2.2. This means that \( T'' \leq D \leq N \), which was to be shown.

\[ \square \]

**Lemma 2.4.** Let \( X \) be an NC-p-group and \( N \) be a normal nilpotent subgroup of finite exponent of \( X \) such that \( X/N \) is infinite Chernikov. Then \( X \) contains a maximal normal \( H^M_* \)-subgroup \( T \) such that \( X/T \) has finite exponent. Furthermore, \( T \) is unique.

**Proof.** We use induction on the nilpotency class \( c \) of \( N \). If \( c = 0 \) then \( N = 1 \) and \( X \) is Chernikov so, then, we may let \( T = X^p \). Now suppose that \( c \geq 1 \) and the assertion holds for \( c - 1 \). Let \( Z = Z(N) \) and put \( \overline{X} = X/Z \). By induction hypothesis \( \overline{X} \) contains a maximal \( H^M_* \)-subgroup \( \overline{T} \) such that \( \overline{X}/\overline{T} \) and, hence, also \( X/T \) has finite exponent. Also \( T'Z = [T', Z] \) since \( \overline{T'} = [\overline{T'}, \overline{T}] \). Moreover, \( T' \leq N \) by Lemma 2.3.

\[ \square \]

Next put \( \overline{T} = T/[T', T] \). Then \( \overline{T}' \leq Z(\overline{T}) \) and so \( \overline{T} \) is nilpotent. Also, \( \overline{T}' = \overline{Z} \) by the preceding paragraph which implies that \( \overline{T}/\overline{Z} \cong C_p^{(n)} \) for some \( n \geq 1 \). Therefore, \( \overline{T} = \overline{S}Z \) by Lemma 2.1 since \( \overline{Z} \) has finite exponent, where \( \overline{S} \) is the maximal radicable abelian subgroup of \( \overline{T} \). In fact \( \overline{S} \cong C_p^{(n)} \) since \( \overline{Z} \) has finite exponent. Furthermore, as \( \overline{T} \) is nilpotent, \( \overline{S} \leq Z(\overline{T}) \) by Lemma 3.13 of [13], which yields that \( \overline{T} \) is abelian and, hence, \( T' = [T', T] \). Consequently it follows that \( S/T' \cong C_p^{(n)} \).

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We claim that $S$ is a normal $HM^*$-subgroup of $X$ such that $X/S$ has finite exponent. Since $S$ is characteristic in $T$ and $T$ is normal in $X$, it follows that $S$ is normal in $X$. Moreover, $T/S$ has finite exponent as does $X/T$, since $T = S \overline{Z}$ and $Z$ has finite exponent. Therefore, $X/S$ has finite exponent. Now put $\overline{T} = T/S'$. Then $\overline{T} = S\overline{Z}$ and so $\overline{T}$ is nilpotent. Also, by Lemma 2.1 $\overline{S} = \overline{V} \overline{T}'$, where $\overline{V}$ is the maximal radicable abelian subgroup of $\overline{S}$ since $T' \leq N$ by Lemma 2.3 and $N$ has finite exponent. Substituting this above yields that $\overline{T} = \overline{V} \overline{T}'$, which yields as before that $\overline{T} = \overline{V} \overline{Z}$ is abelian and hence $T' = S'$. Thus it follows that $S$ is an $HM^*$-subgroup of $X$.

Now let $U$ be any $HM^*$-subgroup of $X$. Then $U \cap S$ is a normal subgroup of $U$ such that $U/U \cap S$ has finite exponent which yields that $U = U \cap S \leq S$ by Lemma 2.2 (ii). Therefore $S$ is the unique maximal $HM^*$-subgroup of $X$ such that $X/S$ has finite exponent.

**Lemma 2.5.** Let $X$ be an $HM^*$-group for a prime $p$ such that $X'$ has finite exponent. Then $X$ is a product of a finite number of normal $HM^*$-subgroups $T$ such that $T/T' \cong C_{p^\infty}$.

**Proof.** There exists an $n \geq 1$ such that

$$X/X' = Y_1/X' \times \cdots \times Y_n/X'$$

and $Y_i/X' \cong C_{p^\infty}$ for $i = 1, \ldots, n$. By Lemma 2.4 each $Y_i$ contains a unique normal $HM^*$-subgroup $T_i$ such that $Y_i/T_i$ has finite exponent and $T_i/T_i' \cong C_{p^\infty}$, since $X'$ has finite exponent. Evidently, $T_1T_2\cdots T_n$ is a normal subgroup of $X$ such that $X/T_1T_2\cdots T_n$ has finite exponent which implies that

$$X = T_1T_2\cdots T_n$$

by Lemma 2.2(ii), as claimed. \hfill \Box

**Proof of the Theorem**

**Lemma 3.1.** Let $G$ be an $NC$–$p$-group. Then $G$ is countably infinite.

**Proof.** By hypothesis $G$ is infinite and every proper subgroup of it, being an $NC$-group, is solvable. However $G$ is not solvable since it is perfect by Theorem A(ii) (see the end of §1). Therefore, for each $n \geq 1$ we can find a finite subgroup $F_n$ of $G$ of derived length equal to $n$. Let $F = \langle F_n : n \geq 1 \rangle$. Then $F$ is countably infinite but not solvable which implies that $F = G$, since every proper subgroup of $G$ is solvable. \hfill \Box
**Lemma 3.2.** Let $G$ be a locally nilpotent $p$-group such that every proper subgroup of $G$ is an NC-subgroup. If every finite subgroup of $G$ is subnormal in $G$, then $F^G$ is nilpotent of finite exponent for every finite subgroup $F$ of $G$.

**Proof.** Assume that every finite subgroup of $G$ is subnormal in $G$. Let $F$ be a finite subgroup of $G$ and put $H = F^G$. First suppose that $H$ is nilpotent. Since $H/H'$, being abelian, has finite exponent it follows from the Corollary to Theorem 2.26 of [13] that $H$ has finite exponent. Thus to complete the proof we must show that $H$ is nilpotent.

If $G$ is perfect, then by Theorem A(i) of [2] $G$ contains a normal nilpotent subgroup $N$ such that $F \leq N$. Then $H = F^G \leq N$, and so $H$ is nilpotent. So suppose that $G$ is not perfect. Then $G$ is an NC-group by Theorem A(ii), that is, $G$ contains a normal nilpotent subgroup $K$ such that $G/K$ is Chernikov. Since $HK / K = F^G K / K = (FK)^G / K$ is Chernikov, $F$ is finite and subnormal, it is easy to see that $HK / K$ is finite. Thus $HK = LK$ for some finite subgroup $L$ of $H$. Moreover, $LK$ is nilpotent by (1) Lemma of [11] since $L$ is subnormal, which implies that $H$ is nilpotent.

\[\square\]

In a locally nilpotent $p$-group $G$ in which every proper subgroup is an NC-group, let $W(G)$ be the set of all $HM^*$-subgroups $T$ of $G$ such that $T/T' \cong C_{p^{\infty}}$ and let $K$ be the subgroup of $G$ which is generated by all the maximal elements of $W(G)$. Of course, $W(G)$ might be empty or it may not have maximal elements, in which case $K$ is not defined.

**Lemma 3.3.** Let $G$ be a locally nilpotent $p$-group such that every proper subgroup of $G$ is a Chernikov extension of a nilpotent subgroup of finite exponent. Suppose that every finite subgroup of $G$ is subnormal in $G$. Then every maximal element of $W(G)$ is normal in $G$.

**Proof.** Let $T$ be a maximal element of $W(G)$. Without loss of generality we may suppose that $T \neq G$. Then $T'$ has finite exponent by hypothesis and by Lemma 2.3. Let $a \in G$ and put $H = a^G$. By Lemma 3.2 $H$ is nilpotent of finite exponent. Put $L = HT$. Since $HT'$ has finite exponent, it must be nilpotent by hypothesis. Thus by Lemma 2.4. $L$ contains a unique maximal $HM^*$-subgroup $R$. Then $T \leq R$ and $HT = HR$. Hence

\[
R/R \cap HT' \cong RH/HT' = TH/HT' \\
\cong T/T \cap HT' \\
\cong C_{p^{\infty}},
\]

which yields that $R/R' \cong C_{p^{\infty}}$, since $HT'$ has finite exponent. Clearly then $R = T$ by the maximality of $T$ and hence $L$ normalizes $T$. In particular, $a$ normalizes $T$ since $a \in L$. Since $a$ is any element of $G$, it follows that $G$ normalizes $T$.

\[\square\]

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Lemma 3.4. Let \( G \) be an \( \overline{N}C - p \)-group such that every proper subgroup of \( G \) is a Chernikov extension of a nilpotent subgroup of finite exponent. Then one of the following holds.

(i) \( G \) has an epimorphic image which is an \( \overline{N}F \)-group.

(ii) \( G = \bigcup_{i=1}^{\infty} T_i \),

where for each \( i \geq 1 \), \( T_i \in W(G) \) and \( T_i \leq T_{i+1} \).

(iii) \( G = K \).

proof Assume that (i) and (ii) do not hold. Partially, order \( W(G) \) by set inclusion. Let \( \{T_i : i \geq 1\} \) be a chain in \( W(G) \) and put

\[
E = \bigcup_{i=1}^{\infty} T_i,
\]

(It suffices to consider only the chains of the above form since \( G \) is countable by Lemma 3.1.) By assumption \( E \neq G \), so by Lemma 2.4 \( E \) contains a unique maximal \( HM^* \)-subgroup \( Y \). Then \( E = Y \), since \( T_i \leq Y \) for all \( i \geq 1 \). Thus \( E \) is an \( HM^* \)-subgroup of \( G \). Also \( E' \) has finite exponent by hypothesis and by Lemma 2.3 and, for the same reason, \( T'_i \leq E' \) for all \( i \geq 1 \). Hence it follows that

\[
E'T_i = E'T_{i+1}
\]

for all \( i \geq 1 \), since \( T_i \leq T_{i+1} \) and \( T_i / T'_i \cong C_{p^\infty} \). Clearly this yields that \( E = E'T_1 \) and so \( E/E' \cong C_{p^\infty} \), that is \( E \in W(G) \). Thus by Zorn’s Lemma \( W(G) \) contains maximal elements and so \( K \) is defined.

By Theorem A(i) every finite subgroup of \( G \) is subnormal in \( G \). Therefore every maximal element of \( W(G) \) is normal in \( G \) by Lemma 3.3. This means that \( K \) is normal in \( G \).

Suppose that \( K \neq G \). Put \( \overline{G} = G/K \). Since \( \overline{G} \) is not an \( \overline{N}F - p \)-group, it contains a proper subgroup \( \overline{X} \) such that \( \overline{X} \) is not nilpotent. Then \( X \) is also not nilpotent. Also, \( X \) contains a normal nilpotent subgroup \( U \) of finite exponent such that \( X/U \) is Chernikov.

Evidently \( X/U \) has infinite exponent since \( X \) is not nilpotent. Thus by Lemma 2.4 \( X \) contains a normal \( HM^* \)-subgroup \( Y \) such \( X/Y \) has finite exponent. But, since \( Y \leq K \) by Lemma 2.5, it follows that \( \overline{X} \) has finite exponent, which is a contradiction.

Proof of the Theorem. By hypothesis and by Theorem A(i) \( G \) contains a proper normal subgroup \( N \) such that every proper subgroup of \( G/N \) is a Chernikov extension of a nilpotent subgroup of finite exponent. Thus \( G/N \) satisfies one of (i), (ii) or (iii) of

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Lemma 3.4. Assume that (i) and (iii) are not satisfied. Then $G/N$ satisfies (ii). Put $\overline{G} = G/N$. Then

$$\overline{G} = \bigcup_{i=1}^{\infty} \overline{T}_i,$$

where for each $i \geq 1$, $\overline{T}_i \in W(\overline{G})$ and $\overline{T}_i \leq \overline{T}_{i+1}$. Put

$$\overline{H} = \bigcup_{i=1}^{\infty} \overline{T}_i'.$$

It is easy to see that $\overline{H}$ is normal in $\overline{G}$ and $\overline{G}/\overline{H}$ is abelian since each $\overline{T}_i\overline{H}/\overline{H}$ is abelian. This implies that $\overline{G}/\overline{H} = 1$ and hence $\overline{G} = \overline{H}$ since $G$ is perfect. This completes the proof of the theorem.

**Proof of the Corollary.** By the theorem $G$ contains a proper normal subgroup $K$ such that every proper subgroup of $G/K$ is a Chernikov extension of a nilpotent subgroup of finite exponent and one of (i), (ii) or (iii) of the Theorem is satisfied. Assume that (ii) is satisfied. Without loss of generality $K = 1$. Then

$$G = \bigcup_{i=1}^{\infty} T_i,$$

where for each $i \geq 1$, $T_i \in W(G)$ and $T_i \leq T_{i+1}$. Also, $T_i T_i' = T_i$ for all $i \geq 1$, since $T_i/T_i' \cong C_{p^\infty}$. But this implies that $T_i = T_i$ for all $i \geq 1$ and hence $G = T_i$ by Lemma 2.2 (iii) which is a contradiction.

**References**


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