ON THE CURVES OF CONSTANT BREADTH IN $E^4$ SPACE

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Abstract

In this paper, the concepts concerning the space curves of constant breadth were extended to $E^4$-space. The integral of third curvature of the curve was obtained as $\int_0^{2\pi} \sigma ds = 2k\pi (k \in \mathbb{Z})$. In addition, the relation $\int_0^{2\pi} g(\kappa, \tau, \sigma)ds = 0$ was obtained between the curvatures of curves of constant breadth in $E^4$.

Key words and phrases: Curvature, Constant Breadth, Integral Characterization of Curve, Spherical Curves.

1. Introduction

Curves of constant breadth were introduced by L. Euler [4]. F. Reuleaux gave a method obtaining some curves of constant breadth and has found use in the kinematics of machinery [11]. In mathematics, many geometers have obtained the geometric properties of plane curves of constant breadth [3], [8], [10], [14].

Furthermore, W. Blascke defined the curve of constant breadth on the sphere [1] and M. Fujivara had obtained a problem to determine whether there exist "space curve of constant breadth" or not, an he defined "breadth" for space curves and obtained these curves on a surface of constant breadth [5]. Ö. Köse presented some concepts for space curves of constant breadth [9]. M. Sezer investigated differential equations characterizing space curves of constant breadth and gave a criterion for these curves [12]. But, the work of these papers are in $E^2$ or $E^3$. A. R. Forsyt had given the theory of curves in $E^4$[6].

In this paper, this kind of curves were extented to the $E^4$-space and some characterizations were obtained.

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2. The Curves of Constant Breadth

Let \( \vec{X} = \vec{X}(s) \) be a simple closed curve in \( E^4 \)-space. These curves will be denoted by \( (C) \). The normal plane at every point \( P \) on the curve meets the curve at a single point \( Q \) other then \( P \). We call the point \( Q \) the opposite point of \( P \). We consider a curve in the class \( \Gamma \) as in [5] having parallel tangents \( \vec{T} \) and \( \vec{T}^* \) in opposite directions at the opposite points \( X \) and \( X^* \) of the curve. A simple closed curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented by the equation

\[
\vec{X}^*(s) = \vec{X}(s) + m_1(s)\vec{T} + m_2(s)\vec{N} + m_3(s)\vec{B} + m_4(s)\vec{E} \quad (0 \leq s \leq L),
\]

where \( \vec{X} \) and \( \vec{X}^* \) are opposite points and \( \vec{T}, \vec{N}, \vec{B}, \vec{E} \) denote the Frenet - Serret frame in \( E^4 \)-space [7]. We have from equation (1)

\[
\frac{dX^*}{ds} = \frac{dX^*}{ds} = T^* \frac{ds^*}{ds} = (1 + \frac{dm_1}{ds} - m_2\kappa)\vec{T}
\]

\[
+ (m_1\kappa + \frac{dm_2}{ds} - m_3\tau)\vec{N} + (\frac{dm_3}{ds} + m_2\tau - m_4\sigma)\vec{B} + (\frac{dm_4}{ds} + m_3\sigma)\vec{E},
\]

where \( \kappa, \tau \) and \( \sigma \) are that first, the second and the third curvatures of the curve, respectively [7]. Since \( T^* = -T \), we obtain

\[
\begin{align*}
1 + \frac{dm_1}{ds} - m_2\kappa &= -\frac{ds^*}{ds} \\
\frac{m_1\kappa}{ds} + \frac{dm_2}{ds} - m_3\tau &= 0 \\
\frac{dm_3}{ds} + m_2\tau - m_4\sigma &= 0 \\
\frac{dm_4}{ds} + m_3\sigma &= 0.
\end{align*}
\]

If we call \( \phi \) as the angle between the tangent of the curve \( (C) \) at point \( X(s) \) with a given fixed direction and consider \( \frac{d\phi}{ds} = \kappa \), we can rewrite equations (3) as

\[
\begin{align*}
\frac{dm_1}{d\phi} &= m_2 - f(\phi) \\
\frac{dm_2}{d\phi} &= -m_1 + \rho \tau m_3 \\
\frac{dm_3}{d\phi} &= -\rho \tau m_3 + \rho \tau m_4 \\
\frac{dm_4}{d\phi} &= -\rho \sigma m_3.
\end{align*}
\]

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where \( f(\phi) = \rho + \rho^* \cdot \rho = \frac{k}{\kappa} \) and \( \rho^* = \frac{1}{\rho^*} \) denote the radii of curvatures at \( X \) and \( X^* \), respectively. If \( m_2, m_3, m_4 \) and their derivatives are eliminated in equations (4), we obtain the following equation with respect to \( m_1 \):

\[
\frac{d}{d\phi} \left\{ \frac{d}{d\phi} \left[ \frac{1}{\rho \tau} \left( \frac{d^2 m_1}{d\phi^2} + m_1 \right) \frac{\tau d m_1}{\sigma d\phi} \right] + \frac{\sigma}{\tau} \left( \frac{d^2 m_1}{d\phi^2} + m_1 \right) \right\} + \frac{d}{d\phi} \left[ \frac{1}{\rho \sigma \phi} \left( \frac{1}{d\phi} \frac{df}{d\phi} \right) + \frac{\tau f}{\sigma} \right] + \frac{\sigma}{\tau} \frac{df}{d\phi} = 0.
\]

This equation is a characterization for \( X^* \). If the distance between the opposite points of \( (C) \) and \( (C^*) \) is constant, then

\[
||X^* - X||^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 = k^2, \quad k \in \mathbb{R}.
\]

Hence, we write

\[
m_1 \frac{d m_1}{d\phi} + m_2 \frac{d m_2}{d\phi} + m_3 \frac{d m_3}{d\phi} + m_4 \frac{d m_4}{d\phi} = 0.
\]

By considering system (4),

\[
m_1 \left( \frac{d m_1}{d\phi} - m_2 \right) = 0.
\]

Thus, we write \( m_1 = 0 \) or \( \frac{d m_1}{d\phi} = m_2 \). If \( \frac{d m_1}{d\phi} = m_2 \) then \( f(\phi) = 0 \). In this case, \( (C^*) \) is translated by the constant vector

\[
i = m_1 \vec{T} + m_2 \vec{N} + m_3 \vec{B} + m_4 \vec{E}
\]

of \( (C) \). We write from system (4)

\[
f(\phi) = 0, \quad m_3 = \frac{m_1}{\rho r}, \quad \frac{d m_3}{d\phi} = \rho \sigma m_4, \quad \frac{d m_4}{d\phi} = -\rho \sigma m_3
\]

by letting \( m_1 = \text{constant} = c \) and \( m_2 = 0 \). The change of variable \( t(\phi) = \int_0^\phi \rho \sigma d\phi \)
gives

\[
\frac{d^2 m_3}{d t^2} + m_3 = 0
\]

General solution for \( m_3(t) \) is

\[
m_3 = A \cos \int_0^\phi \rho \sigma dt + B \sin \int_0^\phi \rho \sigma dt
\]

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If we consider $m_3$, then

$$m_4 = -A \sin \int_0^\phi \rho \sigma dt + B \cos \int_0^\phi \rho \sigma dt,$$

where $A$ and $B$ are arbitrary constants. Therefore, the general solution set of the system (9) is

$$\{ m_1 = c, m_2 = 0, m_3 = A \cos \int_0^\phi \rho \sigma dt + B \sin \int_0^\phi \rho \sigma dt, $$

$$m_4 = -A \sin \int_0^\phi \rho \sigma dt + B \cos \int_0^\phi \rho \sigma dt \}. \quad (10)$$

Thus, equation (1) becomes

$$\ddot{X}^* = \dddot{X} + c \ddot{\tilde{\tau}} + (A \cos \int_0^\phi \rho \sigma dt + B \sin \int_0^\phi \rho \sigma dt) \ddot{\tilde{\eta}}$$

$$+ (-A \sin \int_0^\phi \rho \sigma dt + B \cos \int_0^\phi \rho \sigma dt) \ddot{\tilde{\xi}}. \quad (11)$$

The distance between the opposite point of these curves is $\sqrt{c^2 + A^2 + B^2}$. Since $m_3 = \frac{m_4}{\rho \tau} = \frac{c}{\rho \tau} = c \frac{\xi}{x}$, we obtain

$$\frac{\kappa}{\tau} = A_1 \cos \int_0^\phi \rho \sigma dt + B_1 \sin \int_0^\phi \rho \sigma dt, \quad (12)$$

where $A_1 = A/C$ and $B_1 = B/C$.

In the case $m_1 = 0$, we write from equations (4)

$$m_1 = f(\phi), \quad f(\phi) \neq 0$$

$$\frac{dm_2}{d\phi} = \rho \tau m_3$$

$$\frac{dm_3}{d\phi} = -\rho \tau m_2 + \rho \sigma m_4$$

$$\frac{dm_4}{d\phi} = -\rho \tau m_3. \quad (13)$$

Let us consider system (13) with $\lambda = \rho \tau$, $\mu = \rho \sigma$ and $u = \int_0^\phi \mu(t)dt$. Hence, we obtain the differential equation
\[
\frac{d^2 m_3}{du^2} + m_3 = -\frac{d}{du} \left( \frac{\lambda}{\mu} m_2 \right). \tag{14}
\]

General solution of (14) is

\[
m_3 = F_1(\phi) - \int_0^\phi \cos[u(\phi) - u(t)]\rho(t)\tau(t)f(t)dt, \tag{15}
\]

where

\[
F_1(\phi) = A_2 \cos \int_0^\phi \rho \sigma dt + B_2 \sin \int_0^\phi \rho \sigma dt \tag{16}
\]

[2]. From equations (13) and (15), we write

\[
m_4 = F_2(\phi) + \int_0^\phi \sin[u(t) - u(\phi)]\rho(t)\tau(t)f(t)dt, \tag{17}
\]

where

\[
F_2(\phi) = -A_2 \sin \int_0^\phi \rho \sigma dt + B_2 \cos \int_0^\phi \rho \sigma dt.
\]

Therefore, the general solution set of system (9) is

\[
\{m_1 = 0, m_2 = f(\phi), m_3 = F_1(\phi) - \int_0^\phi \cos[u(\phi) - u(t)]\rho(t)\tau(t)f(t)dt, \quad m_4 = F_2(\phi) + \int_0^\phi \sin[u(t) - u(\phi)]\rho(t)\tau(t)f(t)dt \}. \tag{18}
\]

Consequently, we obtain by using (18) in (1) the curve \((C^*)\) as

\[
\tilde{X}^* = \tilde{X} + \left\{ \int_0^\phi \rho \tau[F_1(\phi) - \int_0^\phi \cos[u(\phi) - u(t)]\rho(t)\tau(t)f(t)dt]d\phi \right\} \tilde{N}
+ \left\{ F_1(\phi) - \int_0^\phi \cos[u(\phi) - u(t)]\rho(t)\tau(t)f(t)dt \right\} \tilde{B}
+ \left\{ F_2(\phi) + \int_0^\phi \sin[u(t) - u(\phi)]\rho(t)\tau(t)f(t)dt \right\} \tilde{E}. \tag{19}
\]

If we write \(m_1 = 0\) in (5),

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\[
\frac{d}{d\phi} \left[ \frac{1}{\rho \phi} \frac{d}{d\phi} \left( \frac{1}{\rho \phi} \frac{df}{d\phi} \right) + \frac{\tau f}{\sigma} \right] + \frac{\sigma df}{\tau d\phi} = 0. \tag{20}
\]

By means of transformation of the independent variable \( \phi \) of the form \( w = \int_0^\phi \rho r d\phi \), we rewrite (20) \((\tau \neq 0, \rho \neq 0)\)

\[
\frac{d}{dw} \left\{ \frac{\tau}{\sigma} \left[ \frac{d^2 f}{dw^2} + f \right] \right\} + \frac{\sigma df}{\tau dw} = 0. \tag{21}
\]

The function \( f \) of (19) satisfies (21). Since the curves of constant breadth is simple closed we have the fact that \( X^*(0) = X^*(2\pi), 0 \leq \phi \leq 2\pi \). Therefore, we give the following results.

**Corollary 1.** Let \( (C^*) \) be a curve in \( E^4 \)-space, such that \( \kappa > 0, \tau \) and \( \sigma \) are continuous periodic functions. If \( (C^*) \) is a curve of constant breadth, then

\[
\int_0^{2\pi} \sigma ds = 2k\pi, \quad k \in \mathbb{Z} \tag{22}
\]

**Corollary 2.** For the curves of constant breadth in \( E^4 \)-space

\[
\int_0^{2\pi} \cos[u(2\pi) - u(t)]\rho(t)\tau(t)f(t)dt = 0, \\
\int_0^{2\pi} \sin[u(t) - u(2\pi)]\rho(t)\tau(t)f(t)dt = 0.
\]

Corollary shows that, the curvatures of the curves of constant breadth in \( E^4 \)-space satisfy

\[
\int_0^{2\pi} g(\kappa, \tau, \sigma)ds = 0. \tag{23}
\]

With respect to this result, if \( \int_0^{2\pi} \cos[u(2\pi) - u(t)]\rho(t)\tau(t)f(t)dt = 0 \) or \( \int_0^{2\pi} g(\kappa, \tau, \sigma)ds = 0 \) is used in (19), we can write

\[
f^2 + \{ A_2 \cos \int_0^\phi \rho \sigma dt + B_2 \sin \int_0^\phi \rho \sigma dt \}^2 + \{ -A_2 \sin \int_0^\phi \rho \sigma dt + B_2 \cos \int_0^\phi \rho \sigma dt \}^2 = C^2.
\]

A curve satisfying this condition lies on an \( E^4 \) sphere of radius \( C/13 \). Thus \( \int_0^{2\pi} g(\kappa, \tau, \sigma)ds = 0 \) characterizes that the curve of constant breadth lies on \( E^4 \) sphere.
Corollary 3. If \( \frac{r}{\sigma} \) is a constant in equation (21), we write

\[
a \frac{d}{dw} \left[ \frac{d^2 f}{dw^2} + f \right] + \frac{1}{a} \frac{df}{dw} = 0
\]
or

\[
\frac{d^3 f}{dw^3} + K^2 \frac{df}{dw} = 0, \quad K^2 = \frac{1 + \alpha^2}{a^2}, \quad K \neq \mp 1.
\]

(24)

From the equation (24), we obtain

\[
f = A_3 \sin \int_0^\phi K r \rho d\phi + B_3 \cos \int_0^\phi K r \rho d\phi + D,
\]

(25)

where \( A_3, B_3 \) and \( D \) are constants [14]. In this case, the curvature of the \((C^*)\) is

\[
\rho^* = -\rho + A_3 \sin \int_0^\phi K r \rho d\phi + B_3 \cos \int_0^\phi K r \rho d\phi + D.
\]

We find from (25) for the curves of constant breadth that

\[
\int_0^{2\pi} \tau ds = \frac{2\pi}{K}, \quad k \in Z.
\]

The final equality shows that total torsion of \((C^*)\) is a constant. From corollary 3, if \( \frac{r}{\sigma} = a \) (constant) then we write

\[
\int_0^{2\pi} \tau ds = a \int_0^{2\pi} \sigma ds = 2k\pi, \quad k \in Z.
\]

That is, corollary 1 generalize this, on the other hand this result shows that the integral of third curvature of such a curve depends only on an integer \( k \).

References


$E^4$ Uzayında Sabit Genişlikli Eğriler Üzerine

Özet

Bu çalışmada, sabit genişlikli eğriler $E^4$ uzayına genişletilerek sabit genişlikli bir eğrinin toplam üçüncü eğriliği $\int_{0}^{2\pi} \sigma ds = 2k\pi$ olarak elde edildi. Ayrıca sabit genişlikli eğrilerin eğriliklerinin $\int_{0}^{2\pi} g(\kappa, \tau, \sigma)ds = 0$ eşitliğini sağladıkları gösterildi.

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