INTEGRAL CLOSURE OF AN IDEAL RELATIVE TO A MODULE AND $\Delta$-CLOSURE

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Abstract

The aim in this paper is to give the relation between the $\Delta$-closure of an ideal $I$ in a commutative Noetherian ring $R$, (see [3]), and the integral closure of the ideal $I$ relative to a Noetherian $R$-module (see (1.1). Definition) and to give the closure cancellation law.

1. Introduction

The important ideas of reduction and integral closure of an ideal in a commutative Noetherian ring $R$ (with identity) were introduced by Northcott and Rees [2]; a brief and direct approach to their theory is given in [4, (1.1)] and it is appropriate for me to begin by briefly summarizing some of the main aspects.

Let $a$ be an ideal of $R$. We say that $a$ is a reduction of the ideal $b$ of $R$ if $a \subseteq b$ and there exists $s \in N$ such that $ab^s = b^{s+1}$ (We use $N$ to denote the set of positive integers.). An element $x$ of $R$ is said to be integrally dependent on $a$ if there exists $n \in N$ and elements $c_1, ..., c_n \in R$ with $c_i = a^i$ for $i = 1, ..., n$ such that

$$x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n = 0.$$ 

In fact, this is the case if and only if $a$ is a reduction of $a + Rx$; moreover,

$$\bar{a} = \{y \in R : y \text{ is integrally dependent on } a\}$$

is an ideal of $R$, called the integral closure of $a$, and is the largest ideal of $R$ which has $a$ as a reduction in the sense that $a$ is a reduction of $\bar{a}$ and any ideal of $R$ which has $a$ as a reduction must be contained in $\bar{a}$.

In [6], Sharp, Tiraş and Yassi introduced concepts of reduction and integral closure of an ideal $I$ of a commutative ring $R$ (with identity) relative to a Noetherian $R$-module

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$M$, and they showed that these concepts have properties which reflect those of the classical concepts outlined in the last paragraph. Again, it is appropriate for me to provide a brief review.

**Definition 1.1.** We say that $I$ is a reduction of the ideal $J$ of $R$ relative to $M$ if $I \subseteq J$ and there exists $s \in \mathbb{N}$ such that $I \cdot J^s \cdot M = J^{s+1}M$. An element $x$ of $R$ is said to be integrally dependent on $I$ relative to $M$ if there exists $n \in \mathbb{N}$ such that

$$x^n \cdot M \subseteq \left( \sum_{i=1}^{n} x^{n-i}I^i \right) \cdot M.$$

In fact, this is the case if and only if $I$ is a reduction of $I + Rx$ relative to $M$ [6, (1.5) (iv)]; moreover, $I^- = \{ y \in R : y \text{ is integrally dependent on } I \text{ relative to } M \}$ is an ideal of $R$, called the integral closure of $I$ relative to $M$, and is the largest ideal of $R$ which has $I$ as a reduction relative to $M$. In this paper, I shall indicate the dependence of $I^-$ on the Noetherian $R$-module $M$ by means of the extended notation $I^{(M)}$.

The current paper is concerned with the integral closure of an ideal $I$ of a commutative Noetherian ring $R$ relative to $M$ and the $\Delta$-closure of the ideal $I$. Specifically, for a multiplicatively closed set $\Delta$ of non-zero ideals of a commutative Noetherian ring $R$, $I$ define the $\Delta$-closure $I_\Delta$ of an ideal $I$ of $R$ and prove that, if $\Delta$ is the multiplicatively closed set defined in theorem (2.4) below, then show $I_\Delta = I^{-(M)}$ and also the closure cancellation law:

If $(JK)^{-(M)} = (JK)^{-(M)}$ and $K \in \Delta$ then $I^{-(M)} = J^{-(M)}$

2. The Closure-Cancellation Law

Throughout $R$ will be a Noetherian ring and $M$ will be an non-zero finitely generated $R$-module. I begin with a definition which will be very useful for my aims.

**Definition 2.1.** Let $I$ be an ideal in $R$ and $\Delta$ a multiplicatively closed set of non-zero ideals of $R$. The ascending chain condition guarantees that the set $\{(IKM : KM) : K \in \Delta \}$ has maximal elements, and since for $K$ and $J$ in $\Delta$ $(IJKM : KJM)$ contains both $(IJM : JM)$ and $(IKM : KM)$, we see that the set under consideration in fact contains a unique maximal element. Let $I_\Delta, \Delta$-closure of $I$, denote that unique maximal element.

The following theorem gives some useful properties of the $\Delta$-closure of any ideal of $R$.

**Theorem 2.2.** Let $I$ and $J$ be ideals of $R$. Then

a) $I \subseteq I_\Delta$

b) If $I \subseteq J$ then $I_\Delta \subseteq J_\Delta$
c) $\Delta \subseteq (I)_{\Delta}$

**Proof.** (a) and (b) are very clear so $I$ omit their proof. For (c), let $x \cdot y \in \Delta$ with $x \in \Delta$ and $y \in \Delta$. Then there exist ideals $K_1$ and $K_2$ in $\Delta$ such that $x \in IK_1 : K_1 M$ and $y \in JK_2 : K_2 M$. Therefore $xyK_1 K_2 M \subseteq IJK_1 K_2 M$, so $xy \in (IK_1 K_2 : K_1 K_2 M) \subseteq (IJ)_{\Delta}$, so it follows that (c) holds.

Next I give the first result, which I promised in the introductory section, in two steps.

**Theorem 2.3.** Let $\Delta$ be a multiplicatively closed set of ideals of $R$ such that each ideal in $\Delta$ contains an element of $R$ which is a non-zerodivisor on $M$. Let $I_{\Delta}$ be as in (2.1). Then $I_{\Delta} \subseteq I^{-(M)}$.

**Proof.** Let $I_{\Delta} = (IKM : KM)$ for a suitable $K \in \Delta$ and let $x \in I_{\Delta}$. Suppose that $KM$ is generated by $a_1, \ldots, a_n$. Then for $x \in I_{\Delta}$ and $1 \leq i \leq n$, we have

$$x \cdot a_i = \sum_{j=i}^{n} b_{ij} a_j \text{ with } b_{ij} \in I.$$ 

Now by [5, (13.15)] and since $K \in \Delta$, a standard determinant argument shows that

$$x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n \in (O : R M),$$

where $c_i \in I^i$. This means $\bar{x}$ is integrally dependent on $\bar{I}$ where "\-" refers to the natural ring homomorphism $R \to R/\cdot O : M$. Thus $\bar{x} \in (\bar{I})^\cdot$, the classical integral closure of $\bar{I} = \frac{I + O : R M}{O : R M}$ in $\bar{R}$. Now the result follows from [6, (1.6)].

**Theorem 2.4.** Let $\Delta = \{J : J$ is an ideal of $R$ which contains a non-zerodivisor on $M\}$. Assume that $I \in \Delta$. Let $I_{\Delta}$ be as in (2.3).

Then

$$I_{\Delta} = I^{-(M)}.$$ 

**Proof.** Let $x \in I^{-(M)}$. Then by [6, (1.5) (iv)], $I$ is a reduction of $I + Rx$ relative to $M$. Then there exists $n \in N$ such that $I(I + Rx)^n = (I + Rx)^{n+1} M$.

Suppose $I_{\Delta} = (IKM : KM)$ for a suitable $K \in \Delta$. Then

$$x \cdot (I + Rx)^n \cdot M \subseteq I \cdot (I + Rx)^n \cdot M$$

Since $(I + Rx)^n \in \Delta$ and by the maximality of $I_{\Delta}$, we get $x \in I_{\Delta}$. Now the result follows from (2.3).
Theorem 2.5. Let $\Delta$ and $I$ be as in (2.4). Then
\[ I_\Delta = I_\Delta K M : K M \text{ for all } K \in \Delta. \]

Proof. By the definition of $I_\Delta$ and (2.4), it is readily seen that $I_\Delta K M : K M \subseteq (I_\Delta)_\Delta = (I^{-(M)})^{-}(M)$. Thus $I_\Delta K M : K M \subseteq I_\Delta$ by [6, (1.5) (ix)]. This completes the proof since the reverse is always true.

The following proposition gives another description of $I_\Delta$ and it will be used in the proof of the closure cancellation law (2.8). \qed

Proposition 2.6. Let $\Delta$ and $I$ be as in (2.4).
Then
\[ I_\Delta = I_\Delta K M : K M = (IK)_\Delta M : K M \text{ for all } K \in \Delta. \]

Proof. $I_\Delta = I_\Delta K M : K M \subseteq (IK)_\Delta M : K M$ by (2.5) and (2.2) (c). Let $x \in (IK)_\Delta M : K M$. Then $x K M \subseteq (IK)_\Delta M$. By the definition $(IK)_\Delta = IK JM : JM$ for a suitable $J \in \Delta$. Thus we get $x \in I_\Delta$. This completes the proof. \qed

Remark 2.7. Let $\Delta$ and $I$ be as in (2.4). Also let “−” refer to the natural ring homomorphism $R \to R/O : R M$.

Let $\Delta' = \{ J = J + O : R M : O : R M : J \in \Delta \}$. Then it is easy to see that $I_\Delta = (I_\Delta)'$.

From (2.6) we can easily get that
\[ (I_\Delta)_{\Delta'} = (I_\Delta)_{\Delta'} K M : (IK)_{\Delta'} M : K M \text{ for all } K \in \Delta'. \]

Now $I$ am in the position to give the main theorem which I promised earlier:

Theorem 2.8. (Closure-cancellation law). Let $\Delta$ and $I$ be as in (2.4). Also let $J \in \Delta$. If $(IK)^{-}(M) = (JK)^{-}(M)$, $K \in \Delta$, then $I^{-}(M) = J^{-}(M)$.

Proof. Let “−” and $\Delta'$ be as in (2.7).
Suppose that $(IK)^{-}(M) = (JK)^{-}(M)$.
Let $x \in I^{-}(M)$. Then by [6, (1.6)], $\bar{x} \in T^{-}(M) = \left( I + O : R M \right)^{-}$, the integral closure of the ideal $\bar{I}$ ind $\bar{R}$. Then, as is mentioned in the introductory section, $\bar{T}$ is a reduction of $(\bar{I} + \bar{R} \bar{x})$. Thus there exists $s \in N$ such that $\bar{T} \cdot (\bar{I} + \bar{R} \bar{x})^s = (\bar{I} + \bar{R} \bar{x})^{s+1}$.
Therefore we get
\[ \vec{x}(\vec{I} + \vec{R} \vec{x})^* \subseteq \vec{I}(\vec{I} + \vec{R} \vec{x})^*. \]

Hence
\[ \vec{x} K(\vec{I} + \vec{R} \vec{x})^* M \subseteq \vec{I} K(\vec{I} + \vec{R} \vec{x})^* M \text{ for all } \vec{K} \in \Delta'. \]

Thus
\[ \vec{x} \in (\vec{I} \vec{K}(\vec{I} + \vec{R} \vec{x})^* M : \vec{K}(\vec{I} + \vec{R} \vec{x})^* M). \]

Since \((IK)^{-(M)} = (JK)^{-(M)}, (IK)_\Delta = (JK)_\Delta\) by (2.4) and (2.7). Then
\[ \vec{x} \in ((\vec{I} \vec{K})_\Delta(\vec{I} + \vec{R} \vec{x})^* M : \vec{K}(\vec{I} + \vec{R} \vec{x})^* M) \text{ by (2.2) (a). Thus} \]
\[ \vec{x} \in ((JK)_\Delta(\vec{I} + \vec{R} \vec{x})^* M : \vec{K}(\vec{I} + \vec{R} \vec{x})^* M). \]

Now by (2.7) we get \(x \in J_\Delta = J^{-(M)}\).
Therefore it follows by symmetry that \(I^{-(M)} = J^{-(M)}\) as desired.

As the stronger converse is true as will be shown in the following theorem. \(\square\)

**Theorem 2.9.** Let \(\Delta, I\) and \(J\) be as in (2.8). Then the following are equivalent:

a) \(ILM = JLM\) for some \(L \in \Delta\)

b) \((IK)^{-(M)} = (JK)^{-(M)}\) for all \(K \in \Delta\)

c) \(I^{-(M)} = J^{-(M)}\)

**Proof.** a) \(\rightarrow\) b) This is easy by (2.2) (b), (2.4) and [6, (1.5) (ix)].

b) \(\rightarrow\) c) This is clear by (2.8).

c) \(\rightarrow\) a) \(I^{-(M)} = I_\Delta = IF_1 M : F_1 M = J_\Delta = J^{-(M)} = JF_2 M : F_2 M\) for suitable \(F_1, F_2 \in \Delta\). Let \(L = F_1 F_2\). Then \(F_1 F_2 \in \Delta\) and \(ILM = (ILM : LM)LM = (JLM : LM)LM = JLM\). This completes the proof. \(\square\)

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References


Bir İdealın Bir Modüle Göre İntegral Kapanışı ve Δ-Kapanışı

Özet

Bu makalede temel amaç Noetherian bir halka üzerindeki bir I idealinin [3]'de tanımlanan Δ-kapanışı ile I idealinin bir Noetherian M modülüne göre (1.1) Tanımlı verilen integral kapanışı arasındaki ilişki ve ayrıca kapanış sadeleştirme kuralını vermektir.

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