THE HESSIAN TENSOR ON A HYPERSURFACE IN EUCLIDEAN SPACE AND OTSUKI’S LEMMA

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Abstract

The purpose of this paper is to obtain a condition for a hypersurface in Euclidean space with belongs to Hessian Tensor and is to give an alternative proof of Otsuki’s lemma by applying this condition.

1. Introduction

Let \( M^n \) be n-dimensional manifold and let \( h : M^n \to R \) be a differentiable function. The linear operator \( Hess h : T_pM \to T_pM \) given by

\[
(Hess h)Y = \nabla_Y \text{grad } h, \quad Y \in T_pM,
\]

is called hessian of \( h \) at \( p \in M \), where \( \nabla \) is the Rianmanian connection of \( M \). If \( X, Y \in T_pM \), then the Hessian Tensor of \( h \) is defined by

\[
(HessH)(X, Y) = \langle Hess h, X, Y \rangle.
\]

We observe that a Hessian Tensor has the property

\[
HessH \leq 0
\]

at a point of maximum of \( h \).

2. A Condition of Hessian Tensor on a Manifold

In this section, we shall prove a lemma which give a relation between Hessian Tensor and the second fundamental form of an immersion on a hypersurface in \( R^{n+1} \). Here we shall denote the connection on \( M^n \) by \( \nabla \) and the connection on \( R^{n+1} \) by \( \nabla \).
Lemma 2.1. Let $M^n$ be a n-dimensional hypersurface in $R^{n+1}$. Let $x$ denote the position vector in $R^{n+1}$ and consider the distance function $f(x) = \langle x, x \rangle$ on $M^n$. Then at any $x \in M^n$ and for any unit vector $V \in T_x M$, the Hessian of $f$ at $x$ in the direction $V$ is

$$Hess f (V, V) = 2 < B(V, V), x > + 2$$

where $B$ denotes the second fundamental form of $M^n$ in $R^{n+1}$.

Proof. Let $X \in T_x M$. For the distance function $g : R^{n+1} \to \mathbb{R}$ which is $f = g|_M$, we have

$$< \nabla f, X > (x) = df(x)X = < \nabla g, X > (x). \quad (1)$$

Since, at $x$,

$$\nabla g = (\nabla g)^T + (\nabla g)^\perp,$$

by (1) we get

$$(\nabla g)^T = \nabla f$$

where $(\nabla g)^T \subset T_x M$ and $(\nabla g)^\perp \perp T_x M$. Using this fact, for $V, W \in T_x M$, at $x$,

$$Hess f (V, W) = < \nabla_V \nabla f, W >$$

$$= < \nabla_V \nabla f - B(\nabla f, V), W >$$

$$= < \nabla_V \nabla f, W > - < B(\nabla f, V), W >$$

$$= < \nabla_V \nabla f, W > - < \nabla_V \nabla f, W >$$

$$< V < \nabla f, W > - < \nabla f, \nabla_V W >$$

$$= V < \nabla f, + \nabla g^\perp, W > - < \nabla f, \nabla_V W >$$

$$= V < \nabla g, W > - < \nabla f, \nabla_V W >$$

$$+ < \nabla g^\perp, \nabla_V W > + < \nabla g^\perp, \nabla_V W >$$

$$= < \nabla_V \nabla g, W > + < \nabla g^\perp, \nabla_V W >$$

Not, let $\alpha(t) = x + tV$ be a curve in $R^{n+1}$. Clearly, $\alpha(0) = x$ and $\alpha'(0) = V$. We can restrict $g$ to the curve $\alpha$ and the directional derivative with respect to the vector $V$ as

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\[ dg(x)V = \frac{d(g\circ\alpha)}{dt} \big|_{t=0}. \]

But, since
\[ g\circ\alpha(t) = \langle \alpha(t), \alpha(t) \rangle = \langle x + tV, x + tV \rangle = \langle x, x \rangle + 2t < x, V > + t^2 < V, V >, \]
we have
\[ \langle \nabla g, V \rangle (x) = dg(x)V \frac{d(g\circ\alpha(t))}{dt} \big|_{t=0} = 2 < x, V > + \langle 2x, V \rangle \]
and we obtain
\[ \nabla g(x) = 2x. \]

Hence, at \( x \)
\[ \text{Hess } f(V, W) = \text{Hess } g(V, W) + 2 < x, B(V, W) > \quad (2) \]

Now let us compute the Hessian of \( g \) at \( x \in \overline{R^{n+1}} \) in direction unit vector \( V \in T_x \overline{R^{n+1}} \).
\[ \text{Hess } g(V, V) = < \nabla^2 g, V > \]
\[ = V < 2x, V > = < \nabla g, \nabla V > \]
Let \( \beta(x) = x + tV \) in \( \overline{R^{n+1}} \). Then
\[ V < 2x, V > = V < \nabla g, V > = \frac{d}{dt} < \nabla g, V > (\beta(t)) \big|_{t=0} = \frac{d}{dt} < 2\beta(t), V > \big|_{t=0} = 2 < \beta'(t), V > \big|_{t=0} = 2 \]
Moreover,
\[ < \nabla g, \nabla V > = x \]
because \( \beta(t) \) is a geodesic, \( V(\beta) = \beta' \). So, if we put in equation (2) these results, then lemma follows. \( \square \)
3. An Alternative Proof of Otsuki’s Lemma

Let $M^n$ be a compact hypersurface in $R^{n+1}$ and $X_0$ be a unit vector in $T_pM$ such that $|B(V,V)|$ attains its minimum value for all unit vectors $V \in T_pM$. Since the normal space $N_pM$ has dimension 1 has $B(X_0, X_0) \neq 0$ by lemma for compact manifold $M^n$, therefore the kernel of $B(X_0, \cdot) : T_pM \to N_pM$ has dimension $n-1$. Hence we can write tangent space $T_pM$, at $p$ of $M$ as

$$T_pM = \text{Ker}B(X_0,0) \oplus <X_0>.$$  

Let $\eta \in N_pM, ||\eta|| = 1$. The linear operator $S_\eta : T_pM \to M_pM$ given by

$$<S_\eta(x),Y> = H_\eta(X,Y) = <B(X,Y),\eta> \text{ for } X, Y \in T_pM$$

is symmetric. Moreover,

$$S_\eta(\text{Ker}B(X_0,\cdot)) \subset \text{Ker}B(X_0,\cdot)$$

because, if $Y \in \text{Ker}B(X_0,0)$ then $<S_\eta(Y),X_0> = 0$, i.e. $S_\eta(Y) \perp X_0$. This implies $S_\eta(Y) \in \text{Ker}B(X_0,0)$.

Let $Y_1, ..., Y_{n-1}$ be an orthonormal basis in $\text{Ker} B(X_0, \cdot)$ diagonalizing $S_\eta|_{\text{Ker}B(X_0, \cdot)}$. Since $<S_\eta(X_0), X_0> \neq 0$ we have $S_\eta(X_0) = \lambda_0 X_0$. Therefore $X_0, Y_1, ..., Y_{n-1}$ is an orthonormal basis composed of principle directions of $B$ in $T_pM$.

Let $\eta$ be unit normal vector in $T_pM$. For the orthonormal basis $X_0, Y_1, ..., Y_{n-1}$ of principal directions of $B$ in $T_pM$ with $||X_0|| = ||Y_i|| = 1, i = 1, 2, ..., n-1$ and the principal curvature $\lambda_0, ..., \lambda_{n-1}$ at $p$, we have $B(X_0, Y_i) = 0, i = 1, 2, ..., n-1$ and

$$<B(X_0, X_0), \eta> = <S_\eta(X_0), X_0>$$

$$= \lambda_0 X_0, X_0$$

$$= \lambda_0.$$  

Hence

$$B(X_0, X_0) = \lambda_0 \eta.$$  

Similarly, for $i = 1, 2, ..., n-1$ we find

$$B(Y_i, Y_i) = \lambda_i \eta.$$  

If we suppose $||B(X_0, X_0)|| \leq ||B(X, X)||$ for all $X \in T_pM$ then we observe that it is possible to give an alternative proof to Otsuki’s lemma as a theorem.
Theorem 3.1. If \( B(X_0, X_0) \neq 0 \) then

(i) \( X_0 \perp \text{Ker}B(X_0, \cdot) \)

(ii) for any \( Y_i \in \text{Ker}B(X_0, \cdot) \) we have

\[
\sum_{i=1}^{n-1} <B(X_0, X_0), B(Y_i, Y_i)> \geq \sum_{i=1}^{n-1} ||B(X_0, X_0)||^2.
\]

Proof. (i) We saw this in the above.

(ii) Since \( M^n \) is compact, there is a maximum point \( p \) of \( f \). Hence for any unit vector \( V \in T_p M \) we have \( \text{Hess} f(p)(V, V) \leq 0 \). First we show that \( \lambda_0 \lambda_i > 0, i = 1, ..., n - 1 \). By Lemma 2.1,

\[
\text{Hess} f(p)(X_0, X_0) = 2 <B(X_0, X_0), p> + 2 \leq 0
\]

\[
\Rightarrow \lambda_0 < \eta, p > = <B(X_0, X_0), p> \leq -1 \quad \text{by (3)}
\]

\[
\Rightarrow \lambda_0 < \eta, p > \leq -1.
\]

Similarly, \( \text{Hess} f(p)(Y_i, Y_i) \leq 0 \) with (4) implies that

\[
\lambda_i < \eta, p > \leq -1 \quad 1 \leq i \leq n - 1.
\]

Then we have

\[
\lambda_0 \lambda_i < \eta, p >^2 > 0 \quad 1 \leq i \leq n - 1.
\]

Hence we obtain for \( i = 1, ..., n - 1 \)

\[
\lambda_0 \lambda_i > 0. \quad (5)
\]

Using our assumption and (5), for some \( i \), we have

\[
<B(X_0, X_0), B(Y_i, Y_i)> = \lambda_0 \lambda_i = |\lambda_0| |\lambda_i| = |\lambda_0|^2 |\lambda_i|^2 
\]

\[
\geq |\lambda_0| |\lambda_0| = \lambda_0^2 = ||B(X_0, X_0)||^2.
\]

The proof is verified. \( \square \)

This theorem with maximum principle leads to an important result due to Leung [4]. Let \( M^n \) be a n-dimensional compact connected hypersurface in \( \mathbb{R}^{n+1} \) such that \( M^n \subset B(r) \), where \( B(r) \) denotes a closed ball centered at the origin with radius \( r \) in \( \mathbb{R}^{n+1} \). If for any \( p \in M^n \) and for any unit vector \( V \in T_p M \) we have \( \text{Ric}(V, V) \leq \)
(n − 1)/r², then \( M^n \) must be boundary of \( B(r) \). The reason of this claim is as follows. We take a point \( p \) in \( M^n \) realizing the maximum of the distance function \( f \) to the origin. The upper bound on the Ricci curvature implies that \( f(p) = r \) (radius of the ball \( B(r) \)) and \( p \) is umbilic. Then we show that \( f \) is subharmonic in a small neighbourhood of \( p \). By the maximum principle, \( f \) is constant in this neighbourhood. By connectiveness of \( M \), \( f \) is constant in \( M \) and the proof of claim is done.

References


Öklid Uzayında Bir Hiperyüzey Üzerinde Hessian Tensörü ve Otsuki Yardımcı Teoremi

Özet

Bu makalenin amacı, Öklid uzayında bir hiperyüzey için Hessian Tensörü’ne ait bir koşul elde etmek ve bu koşulu kullanarak Otsuki yardımcı teoreminin değişik bir ispatını vermektır.