DIFFERENTIABLE FUNCTIONS AND THE GENERATORS ON A HILBERT-LIE GROUP

Erdal Coşkun

Abstract
A convolution semigroup plays an important role in the theory of probability measure on Lie groups. The basic problem is that one wants to express a semigroup as a Lévy-Khinchine formula. If \((\mu_t)_{t \in \mathbb{R}_+^*}\) is a continuous semigroup of probability measures on a Hilbert-Lie group \(G\), then we define

\[ T_{\mu}f := \int f_{a, \mu}(da) \quad (f \in C_u(G), t > 0). \]

It is apparent that \((T_{\mu_t})_{t \in \mathbb{R}_+^*}\) is a continuous operator semigroup on the space \(C_u(G)\) with the infinitesimal generator \(N\). The generating functional \(A\) of this semigroup is defined by \(Af := \lim_{t \to 0} \frac{1}{t} (T_{\mu_t}f(e) - f(e))\). We have the problem of constructing a subspace \(C_{(2)}(G)\) of \(C_u(G)\) such that the generating functional \(A\) on \(C_{(2)}(G)\) exists. This result will be used later to show that the Lévy-Khinchine formula holds for Hilbert-Lie groups.

Key words: Continuous convolution semigroup, operator semigroup, Hilbert-Lie group, Lévy measure, infinitesimal generator, generating functional

Introduction
Let \((\mu_t)_{t \in \mathbb{R}_+^*}\) be a continuous convolution semigroup of probability measures on a Hilbert-Lie group \(G\) and \(C_u(G)\) the Banach space of all bounded left uniformly continuous real-valued functions on \(G\). Then there is associated a strongly continuous semigroup \((T_{\mu_t})_{t \in \mathbb{R}_+^*}\) of contraction operators on \(C_u(G)\) with the infinitesimal generator \((N, D(N))\).

The generating functional \((A, D(A))\) of the convolution semigroup \((\mu_t)_{t \in \mathbb{R}_+^*}\) is defined by

\[ Af := \lim_{t \to 0} \frac{1}{t} (T_{\mu_t}f(e) - f(e)) \]

for all \(f\) in its domain \(D(A)\). For finite dimensional Lie groups, infinite dimensional Hilbert spaces and Banach spaces of cotype 2, we have
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\[ C_2(G) \subset D(A) \]

(cf. [4], [6] and [8] resp.). In this paper we shall prove that the above result is also true for a class of infinite dimensional Hilbert-Lie groups. At several points we shall use ideas and techniques used in [4]. We first obtain the Taylor expansion for the functions \( f \in C_2(G) \). In Lemma 2.1 we prove that, for every neighborhood of \( e \) in any Hilbert-Lie group \( G \), the supremum \( \sup_{t > 0} \frac{1}{t} \mu_t(U^c) \) is finite. Using this result and Banach-Steinhaus Theorem, we prove Theorem 2.9.

1. Preliminaries

\( \mathbb{N} \) and \( \mathbb{R} \) denote the sets of positive integers and real numbers, respectively. Moreover let \( \mathbb{R}_+ := \{ r : r \geq 0 \}, \mathbb{R}_+^* := \{ r : r > 0 \} \).

Let \( A \) be a set and \( B \) a subset of \( A \). Then by \( 1_B \) we denote the indicator function of \( B \). Let \( I \) be a nonvoid set. \( \delta_{ij} \) is the Kronecker delta \((i,j \in I)\).

By \( G \) we denote a topological Hausdorff group with identity \( e \). \( G \) is called Polish group, if \( G \) is a topological group with a countable basis of its topology and with a complete left invariant metric \( d \) which induces the topology.

For every function \( f : G \rightarrow \mathbb{R} \) and \( a \in G \) the functions \( f^*, R_a f = f_a \) and \( L_a f = a f \) are defined by \( f^*(b) = f(b^{-1}) \), \( f_a(b) = f(ba) \) and \( a f(b) = f(ab) \) for all \( b \in G \), respectively. Moreover let \( \text{supp}(f) = \{ a \in G : f(a) \neq 0 \} \) denote the support of \( f \).

By \( C_u(G) \) we denote the Banach space of all real-valued bounded left uniformly (or \( d \)-uniformly) continuous functions on \( G \) furnished with the supremum norm \( \| \cdot \| \). A Hilbert-Lie group is a separable analytic manifold modeled on a separable Hilbert space, whose group operations are analytic. It is we known that the Hilbert-Lie groups are Polish (cf. [2]).

For the exponential mapping \( \exp : T_e \rightarrow G \) there exists an inverse mapping log from a neighborhood \( U_e \) of \( e \) onto a neighborhood \( N_e \) of zero in \( T_e \), where \( T_e \) is the tangential space in \( e \in G \) ([5]).

By \( \mathcal{B}(G) \) we denote the \( \sigma \)-field of Borel subsets of \( G \). Moreover, \( \mathcal{V}(e) \) denotes the system of neighborhoods of the identity \( e \) of \( G \) which are in \( \mathcal{B}(G) \).

\( \mathcal{M}(G) \) denotes the vector space of real-valued (signed) measures on \( \mathcal{B}(G) \). As it is well known, \( \mathcal{M}(G) \) is a Banach algebra with respect to convolution \( * \) and the norm \( \| \cdot \| \) of total variation. \( \mathcal{M}_+(G) \) is the set of positive measures in \( \mathcal{M}(G) \) and \( \mathcal{M}^1(G) = \{ \mu \in \mathcal{M}_+(G) : \mu(G) = 1 \} \) is the set of probability measures on \( G \).

Now let \( \gamma_X(t) := \exp(tX) \) for \( X \in H \) and \( t \in \mathbb{R}^* := \mathbb{R} \setminus \{0\} \).

**Definition 1.1** Let \( f \in C_u(G), X \in H \) and \( a \in G \).

\( f \) is called left differentiable at \( a \in G \) with respect to \( X \) ("\( Xf(a) \) exists" for short), if

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\[ Xf(a) := \lim_{t \to 0} \frac{1}{t} [L_{\gamma x}(t)f(a) - f(a)] \]

exists. \( f \) is called continuously left differentiable, if \( Xf(a) \) exists for all \( a \in G \) and \( X \in H \), and if the mappings \( a \mapsto Xf(a) \), \( X \mapsto Xf(a) \) are continuous.

Derivatives of higher orders are defined inductively. Differentiability from the right is defined in replacing \( L_{\gamma x}(t) \) by \( R_{\gamma x}(t) \).

The following properties of the derivatives are well known for continuously left differentiable functions (cf. [1]).

**Remark 1.2** Let \( f, g \in C_u(G), X \in H \) and \( a \in G \).

(i) If \( Xf(a) \) exists, then the mapping \( X \mapsto Xf(a) \) is linear.

(ii) If \( Xf(a) \) and \( Xg(a) \) exists, then also \( X(f \cdot g)(a) \) exists and \( X(f \cdot g)(a) = Xf(a) \cdot g(a) + f(a) \cdot Xg(a) \).

Now let \( f \in C_u(G) \) be twice continuously left differentiable function. Then the mapping

\[ Df(a) : X \mapsto Xf(a) \quad (D^2f(a) : (X, Y) \mapsto XYf(a)) \]

is continuous and linear (resp. symmetric, continuous and bilinear) functional on \( H \) (resp. \( H \times H \)) for all \( a \in G \). There hold

\[ < Df(a), X >= Xf(a) \text{ and } < D^2f(a)(X), Y >= XYf(a) \]

for all \( a \in G \) and \( X, Y \in H \).

We define by \( C_2(G) \) the space of all twice continuously left differentiable functions \( f \in C_u(G) \) such that the mapping \( a \mapsto D^2f(a) \) is \( d \)-uniformly continuous and \( \|Df\| := \sup_{a \in G} \|Df(a)\| < \infty, \quad \|D^2f\| := \sup_{a \in G} \|D^2f(a)\| < \infty \). It is easy to see that the space \( C_2(G) \) is a Banach space with respect to the norm

\[ \|f\|_2 := \|f\| + \|Df\| + \|D^2f\|, \quad f \in C_2(G) \]

and

\[ R_aC_2(G) \subset C_2(G) \]

is satisfied for all \( a \in G \). However \( C_2(G) \) is not dense in \( C_u(G) \) (cf. [6]). By \( a_i(a) := \log(a), X_i > (i \in \mathbb{N}) \) we define maps \( a_i \) from the canonical neighborhood \( U_c \) in \( \mathbb{R} \). Now we call the system \( (a_i)_{i \in \mathbb{N}} \) of maps from \( U_c \) in \( \mathbb{R} \) a system of canonical coordinates of \( G \) with respect to the orthonormal basis \( (X_i)_{i \in \mathbb{N}} \), if for all \( a \in U_c \) the property \( a = \exp(\sum_{i=1}^{\infty} a_i(a)X_i) \) is satisfied.
Lemma 1.3 Let $f \in C_2(G)$. Then

(i) $(\sum_{i=1}^{\infty} a_i(a)X_i)f = \sum_{i=1}^{\infty} a_i(a)X_i f$ for all $a \in U_e$.

(ii) $(\sum_{i=1}^{\infty} a_i(a)X_i)((\sum_{j=1}^{\infty} a_j(c)X_i)f) = \sum_{i=1,j=1}^{\infty} a_i(a)a_j(c)X_iX_j f$ for all $a, c \in U_e$.

Proof. (i) For any $a \in U_e$ there exists an $X \in H$ with $X = \log(a)$. Then we have $X = \sum_{i=1}^{\infty} <X_i,X_i>X_i = \sum_{i=1}^{\infty} a_i(a)X_i$. Thus

\[
Xf(e) = \frac{d}{dt}|_{t=0}f(\gamma_X(t)) = <Df(e),X>
= \sum_{i=1}^{\infty} a_i(a) <Df(e),X_i> = \sum_{i=1}^{\infty} a_i(a)X_i f(e).
\]

Now let $b \in G$ be an arbitrary point. Then $R_b f \in C_2(G)$, whence the assertion. The proof of (ii) can be carried out similarly. 

In the following we give the Taylor expansion for the functions $f \in C_2(G)$.

Proposition 1.4 Let $f \in C_2(G)$. Then the Taylor-expansion of the second order for $f$ at $e \in G$ is given by

\[
f(a) = f(e) + \sum_{i=1}^{\infty} a_i(a)X_i f(e) + \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i(a)a_j(a)X_iX_j f(\bar{a})
\]

for all $a \in U_e$, where $\bar{a}$ is a point of $U_e$.

Proof. Let $f \in C_2(G)$ and $X \in H$. Then the function $\xi : t \mapsto f(\gamma_X(t))$ is twice differentiable on $\mathbb{R}$ and therefore admits a Taylor-expansion valid up to the second order:

\[
\xi(t) = \xi(0) + \xi'(0) \cdot t + \frac{1}{2} \xi''(\bar{t}) \cdot t^2
\]

for some $\bar{t} \in [-|t|,|t|]$. Since $\xi'(0) = Xf(e)$ and $\xi''(\bar{t}) = XXf(\gamma_X(\bar{t}))$, it follows from Lemma 1.3 that

\[
f(\gamma_X(t)) = f(e) + \sum_{i=1}^{\infty} <tX_i,X_i>X_i f(e)
+ \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} <tX_i,X_i><tX_i,X_i>X_iX_j f(\gamma_X(\bar{t}))
\]

for some $\bar{t} \in [-|t|,|t|]$. This yields the assertion. 

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Remark 1.5 The Taylor-expansion of \( f \in C_2(G) \) can be written in a closed form, i.e.

\[
f(a) = f(e) + \langle Df(e), \log(a) \rangle + \frac{1}{2} \langle D^2f(\bar{a})(\log(\bar{a})), \log(a) \rangle
\]

for all \( a \in U_e \) and for some \( \bar{a} \) in the canonical neighborhood \( U_e \).

2. Convolution Semigroups of Probability Measures and the Generators

For any probability measure \( \mu \) on \( G \), we define the operator \( T_\mu \) on \( C_u(G) \) by

\[
T_\mu f := \int f_a \mu(da) \quad \text{(Bochner-Integral)}.
\]

It is easy to see that \( T_\mu C_u(G) \subset C_u(G) \) and \( T_{\mu * \nu} = T_\mu \circ T_\nu \).

A convolution semigroup is a family \( (\mu_t)_{t \in \mathbb{R}_+^*} \) in \( \mathcal{M}_1(G) \) such that \( \mu_0 = \varepsilon_e \) and \( \mu_s * \mu_t = \mu_{s+t} \) for all \( s, t \in \mathbb{R}_+^* \).

\( (\mu_t)_{t \in \mathbb{R}_+^*} \) is called continuous if \( \lim_{t \to 0} \mu_t = \varepsilon_e \) (weakly). It is well known that the convolution semigroup \( (\mu_t)_{t \in \mathbb{R}_+^*} \) is continuous iff the corresponding operator semigroup \( (T_\mu_t)_{t \in \mathbb{R}_+^*} \) is (strongly) continuous. The Hille-Yosida theorem establishes a bijection between (strongly) continuous operator semigroups \( (T_\mu_t)_{t \in \mathbb{R}_+^*} \) and their infinitesimal generators. \( N \) is defined on its domain \( D(N) \) which is dense in \( C_u(G) \). It is clear that \( N \) commutes with the left translations, i.e.

\[
L_a D(N) \subset D(N) \quad \text{and} \quad L_a \circ N = N \circ L_a \quad \text{for all} \quad a \in G.
\]

A continuous convolution semigroup \( (\mu_t)_{t \in \mathbb{R}_+^*} \) in \( \mathcal{M}_1(G) \) admits a Lévy measure \( \eta \), i.e. \( \eta \) is a \( \sigma \)-finite positive measure on \( \mathcal{B}(G) \) such that \( \eta(\{e\}) = 0 \) and such that

\[
\lim_{t \to 0} \frac{1}{t} \int f d\mu_t = \int f d\eta,
\]

for all \( f \in C_u(G) \) with \( e \not\in \text{supp}(f) \) (cf. [7]).

Lemma 2.1 Let \( (\mu_t)_{t \in \mathbb{R}_+^*} \) be a continuous convolution semigroup in \( \mathcal{M}_1(G) \). Then for every \( U \in \mathcal{V}(e) \)

\[
\sup_{t \in \mathbb{R}_+^*} \frac{1}{t} \mu_t(U^c) < \infty.
\]

Proof. Let \( U \) and \( V \) be two neighborhoods of \( e \in G \) with \( V \subset U \). Since \( G \) is a normal group, there exists a function \( f \in C_u(G) \) such that
Then we have \( \frac{1}{t} \mu_t(U_c) \leq \frac{1}{t} \int f d\mu_t \) for all \( t \in \mathbb{R}^*_+ \). \( f \in C_u(G) \) with \( e \not\in \text{supp}(f) \) implies that

\[
\lim_{t \to 0} \frac{1}{t} \int f d\mu_t = \int f d\eta < \infty.
\]

Hence the assertion. \( \square \)

Let \( H \) be a separable Hilbert space with a complete orthonormal system \( (X_i)_{i \in \mathbb{N}} \) and \( G \) a Hilbert-Lie group on \( H \). Moreover, let

\[ H_n := \langle \{x_1, x_2, \ldots, x_n\} \rangle \]

be the space of all linear combinations of \( X_1, X_2, \ldots, X_n \) and \( H_n^\perp \) the orthogonal complement of \( H_n \) in \( H \) (for all \( n \in \mathbb{N} \)). Then \( H/H_n^\perp \) and \( H_n \) are isomorphic. Clearly

\[ G_n := \text{Exp}(H_n^\perp) \]

is a closed subgroup of \( G \) for all \( n \in \mathbb{N} \). The quotient spaces \( G/G_n \) are finite-dimensional Hilbert-Lie groups. Now let \( p_n \) be the canonical projection from \( G \) onto \( G/G_n \) and \( \{b^n_i : i = 1, 2, \ldots, n\} \) a system of canonical coordinates with respect to \( \{X_1, X_2, \ldots, X_n\} \).

We now define the functions \( d^n_i := b^n_i \circ p_n \in C_2(G) \); then \( X_j d^n_i \) exist and

\[
X_j d^n_i = X_j (b^n_i \circ p_n) = X_j b^n_i \circ p_n = 0
\]

hold for all \( j > n \) and \( i = 1, 2, \ldots, n \).

**Definition 2.2** Let \( G \) be a Hilbert-Lie group on \( H \), and \( (X_i)_{i \in \mathbb{R}} \) an orthonormal basis in \( H \). For any \( n \in \mathbb{N} \) we define

\[
C_{(2), n}(G) := \{ f \in C_2(G) : \ X_i f = 0 \text{ for all } i > n \text{ and } \ X_i X_j f = 0 \text{ for all } i > n \text{ or } j > n \}.
\]

**Remark 2.3** Let \( f \in C_u(G) \) be a left uniformly differentiable function with respect to \( X \) which satisfies \( X_i f = 0 \) for all \( i > n \) \((n \in \mathbb{N})\). Let \( \pi_n \) be the orthogonal projection from \( H \) onto \( H_n \). Then we have

\[
X f = \pi_n(X) f \text{ for all } X \in H.
\]
Hence \( f \) is continuously left differentiable and clearly \( (C_{(2),n}(G))_{n\in\mathbb{N}} \) is a strictly increasing sequence of Banach subalgebra of Banach algebra \( C_2(G) \).

Further properties of \( C_{(2),n}(G) (n \in \mathbb{N}) \):

(i) \( C_{(2),n}(G) \) are \( \| \cdot \|_2 \)-closed in \( C_2(G) \)

and

(ii) For any probability measure \( \mu \in \mathcal{M}^1(G) \), we have

\[
T_\mu C_{(2),n}(G) \subset C_{(2),n}(G) \text{ for all } n \in \mathbb{N}.
\]

Thus \( C_{(2),n}(G) \cap D(N)^{\| \cdot \|_2} = C_{(2),n}(G) \). Now consider the subspace

\[
C_{(2)}(G) := \bigcup_{n \in \mathbb{N}} C_{(2),n}(G).
\]

\( C_{(2)}(G) \) is obviously an linear subspace of \( C_2(G) \) with \( T_\mu C_{(2)}(G) \subset C_{(2)}(G) \) for probability measures \( \mu \in \mathcal{M}^1(G) \). Especially \( C_{(2)}(G)^{\| \cdot \|_2} \) is a Banach space with \( T_\mu C_{(2)}(G)^{\| \cdot \|_2} \subset \overline{C_{(2)}(G)^{\| \cdot \|_2}} \).

**Definition 2.4** For \( n \in \mathbb{N} \) let \( \{b_i^n : i = 1, 2, \ldots, n\} \) be a system of extended canonical coordinates with respect to \( \{X_1, X_2, \ldots, X_n\} \). Then we say that the Hilbert-Lie group \( G \) has the property \((K)\), if

\[
b_i^n \in C_{(2)}(G) \text{ for all } i = 1, 2, \ldots, n, \ n \geq n_0
\]

and for any \( n_0 \in \mathbb{N} \).

Every commutative Hilbert-Lie group and every finite dimensional Lie group have clearly the property \((K)\). In the finite dimensional case we have \( n_0 = \dim(G) \). Since \( C_{(2),n}(G) \subset C_{(2),n+1}(G) \), a system \( \{b_i^n, b_{n+1}^{n+1} : i = 1, 2, \ldots, n\} \subset C_{(2),n+1}(G) \) of canonical coordinates exists with respect to \( \{X_1, X_2, \ldots, X_{n+1}\} \). We also have the following Proposition:

**Proposition 2.5** Let \( G \) be a Hilbert-Lie group with the property \((K)\). Then a system \( (d_n)_{n \in \mathbb{N}} \) of functions in \( C_{(2)}(G) \) exists with

\[
d_i = b_i^{n_0} \text{ for all } i = 1, 2, \ldots, n_0
\]

and

\[
d_n = b_n^n \text{ for all } n > n_0.
\]

This system \( (d_n)_{n \in \mathbb{N}} \) is called a system of local canonical coordinates with respect to \( (X_i)_{i \in \mathbb{N}} \).

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Now let $G$ be a Hilbert-Lie group with the property $(K)$. We define for any $n \in \mathbb{N}$ the functions
\[
\Phi_n(a) := \sum_{i=1}^{n} d_i(a)^2, \quad a \in G,
\]
where $(d_i)_{i=1,2,\ldots,n}$ is a system of local canonical coordinates with respect to $\{X_1, X_2, \ldots, X_n\}$. Then $\Phi_n \in C(2)_n(G)$ and $\Phi_n(a) > 0$ for all $a \in G \setminus \{\Phi_n = 0\}$. Therefore
\[
X_i\Phi_n(e) = 0, \quad X_i X_j \Phi_n(e) = 2\delta_{ij}, \quad i, j = 1, 2, \ldots, n
\]
(cf. [3], Lemma 4.1.9 and 4.1.10).

**Remark 2.6**

(a) For $f \in C(2)_n(G), n \in \mathbb{N}$ and $i, j = 1, 2, \ldots, n$ we denote the numbers $X_i f(e)$ and $X_i X_j f(e)$ by $A_if$ and $A_ij f$, resp. Obviously $f \mapsto A_if$ and $f \mapsto A_ij f$ are continuous linear functionals on $C(2)_n(G)$ for $i, j = 1, 2, \ldots, n$.

(b) Let $E$ be a locally convex vector space and $E_1$ a dense subspace of $E$. Moreover, let $F$ be a subspace of $E$ of finite codimension, $y \in E$ and $M := y + F$. Then $M_1 := M \cap E_1$ is dense in $M$ ([3], Lemma 4.1.11).

**Lemma 2.7**

For every $f \in C(2)_n(G)$ and every $\varepsilon > 0$ there exists a $g := g_\varepsilon \in C(2)_n(G) \cap D(N)$ such that $\|f - g\|_2 < \varepsilon$, $f(e) = g(e), X_i f(e) = X_i g(e)$ and $X_i X_j e = X_i X_j g(e)$ for $i, j = 1, 2, \ldots, n$.

**Proof.** Let $K_n$ be a map from $C(2)_n(G)$ to $\ell^2(n^2)$ with
\[
f \mapsto K_n(f) := (X_i X_j f(e))_{i,j=1,2,\ldots,n} = (A_ij f)_{i,j=1,2,\ldots,n}, \quad n \in \mathbb{N}.
\]
Then $K_n$ is linear and continuous, where $\ell^2(n)$ is a finite-dimensional subspace of the Hilbert space $\ell^2$.

Similarly, let $L_n$ be a continuous linear map from $C(2)_n(G)$ to $\ell^2(n+1)$ with
\[
f \mapsto L_n(f) := (f(e), X_1 f(e), \ldots, X_n f(e)) = (f(e), A_1 f, \ldots, A_n f).
\]
Moreover, let
\[
F := \text{Kern}(L_n) \cap \text{Kern}(K_n),
\]
then $F$ is a closed subspace of $C(2)_n(G)$ of finite codimension. From Remark 2.6 b)
\[
[f + F] \cap [C(2)_n(G) \cap D(N)] = f + F
\]
the assertions follow. \qed
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Proposition 2.8 Let $G$ be a Hilbert-Lie group with the property $(K)$, $(\mu_t)_{t \in \mathbb{R}^*_+}$ a convolution semigroup in $\mathcal{M}^1(G)$ and $\Phi_n(n \in \mathbb{N})$ be as above. Then the suprema

$$\sup_{t \in \mathbb{R}^*_+} \frac{1}{t} \int \Phi_n d\mu_t$$

are finite for every $n \in \mathbb{N}$.

Proof. Application of Lemma 2.7 to the function $\Phi_n \in C_{(2),n}(G)$ yields the existence of a function $\Psi_n \in C_{(2),n}(G) \cap D(N)$ with the property

$$\|\Phi_n - \Psi_n\|_2 < \varepsilon, \Psi_n(e) = \Phi_n(e) = 0, \quad X_i \Psi_n(e) = X_i \Phi_n(e) = 0$$

and $X_i X_j \Psi_n(e) = X_i X_j \Phi_n(e) = 2 \delta_{ij}, i, j = 1, 2, \ldots, n$.

Taylor expansion of $\Psi_n \in C_{(2),n}(G) \cap D(N)$ in a neighborhood $W_1$ of $e$ with $W_1 \subset U_\varepsilon$ gives

$$\Psi_n(a) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n d_i(a) d_j(a) X_i X_j \Psi_n(\bar{a}),$$

for all $a \in W_1$ and for some $\bar{a} \in W_1$. Since $\|\Phi_n - \Psi_n\|_2 < \varepsilon$ and $X_i X_j \Psi_n(e) = 2 \delta_{ij}, i, j = 1, 2, \ldots, n$ there exists a neighborhood $W_2$ of $e$ with the properties

$$-\varepsilon \leq X_i X_j \Psi_n(a) \leq \varepsilon \quad \text{for all } i, j = 1, 2, \ldots, n, \quad i \neq j,$$

$$2 - \varepsilon \leq X_i X_j \Psi_n(a) \leq 2 + \varepsilon \quad \text{for all } i = 1, 2, \ldots, n,$$

whenever $a \in W_2$. Putting $\delta_n := \delta_n(e) := \frac{1}{2} (2 - \varepsilon - \varepsilon(n - 1))$ and $W := W_1 \cap W_2$, we obtain

$$\Psi_n(a) \geq \delta_n \cdot \sum_{i=1}^n d_i(a)^2 \quad \text{for all } a \in W.$$

Since $\Psi_n \in C_{(2),n}(G) \cap D(N)$, we obtain $\sup_{t \in \mathbb{R}^*_+} \frac{1}{t} \int_W \Psi_n d\mu_t < \infty$ from Lemma 2.1. Thus $\sup_{t \in \mathbb{R}^*_+} \frac{1}{t} \int_W \Phi_n d\mu_t < \infty$, and since $\Phi_n$ is bounded, the assertion follows from Lemma 2.1. \qed

Now let $G$ be a Hilbert-Lie group with the property $(K)$ and $(d_i)_{i \in \mathbb{N}}$ a system of local canonical coordinates with respect to $(X_i)_{i \in \mathbb{N}}$. By Lemma 2.7 there exist functions $z_i \in C_{(2),n}(G) \cap D(N), (n \in \mathbb{N})$ with the property

$$z_i(e) = d_i(e) = 0, \quad X_j z_i(e) = X_j d_i(e) = \delta_{ij}, \quad i, j = 1, 2, \ldots, n.$$
Theorem 2.9 Let \( G \) be a Hilbert-Lie group with the property \((K)\) and \((\mu_t)_{t \in \mathbb{R}^*_+}\) a convolution semigroup in \( \mathcal{M}^1(G) \). Then the generating functional \( A \) of \((\mu_t)_{t \in \mathbb{R}^*_+}\) on \( C_{(2)}(G) \) exists, i.e.

\[
C_{(2)}(G) \subset D(A).
\]

Proof. Let \( f \in C_{(2),n}(G) \) \((n \in \mathbb{N})\) and set

\[
g(a) := f(a) - f(e) - \sum_{i=1}^{n} z_i(a) \cdot X_i f(e) \quad \text{for all } a \in G,
\]

where the functions \( z_i, i = 1, 2, \ldots, n \) are as above. Then \( g \in C_{(2),n}(G) \) with \( g(e) = 0, X_j g(e) = X_j f(e) - \sum_{i=1}^{n} X_j z_i(e) \cdot X_i f(e) = X_j f(e) - \sum_{i=1}^{n} \delta_{ij} \cdot X_i f(e) = 0. \) The Taylor expansion of \( g \) in a neighborhood \( W \subset U_n \) gives

\[
g(a) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_i(a) d_j(a) X_i X_j g(a), \quad a \in W.
\]

Thus there is a constant \( k_1 \in \mathbb{R}^*_+ \) such that

\[
|g(a)| \leq k_1 \cdot \|g\|_2 \cdot \Phi_n(a) \quad \text{for all } a \in W.
\]

It follows from Proposition 2.8 that

\[
\sup_{t \in \mathbb{R}^*_+} \frac{1}{t} \int_{W} g d\mu_t \leq k_1 \cdot \|g\|_2 \cdot \sup_{t \in \mathbb{R}^*_+} \int \Phi_n d\mu_t < \infty. \quad (1)
\]

Clearly, \( \frac{1}{t} \int_{W^c} g d\mu_t \leq \|g\|_2 \cdot \frac{1}{t} \mu_t(W^c) \), and \( \sup_{t \in \mathbb{R}^*_+} \frac{1}{t} \int_{W^c} g d\mu_t \) \( < \infty \). Hence, there exists a constant \( k_2 \in \mathbb{R}^*_+ \) independent of \( t \) such that

\[
\frac{1}{t} \int_{W^c} g d\mu_t \leq k_2 \cdot \|g\|_2 \quad \text{for all } t \in \mathbb{R}^*_+. \quad (2)
\]

Adding the inequalities (1) and (2) we get

\[
\frac{1}{t} [T_{\mu_t} f(e) - f(e)] - \sum_{i=1}^{n} X_i f(e) \cdot T_{\mu_t} z_i(e) \leq k_3 \cdot \|f\|_2, \quad \text{for all } t \in \mathbb{R}^*_+.
\]

where \( k_3 \) is a constant (independent of \( t \)). Since \( z_i \in D(N) \) and \( z_i(e) = 0 \), we have \( \sup_{t \in \mathbb{R}^*_+} \frac{1}{t} T_{\mu_t} z_i(e) \leq \infty \) for all \( i = 1, 2, \ldots, n \).

Hence we obtain a constant \( k(n) \in \mathbb{R}^*_+ \) depending only on \( n \) such that

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\[ \frac{1}{t} (T_{\mu_t} f(e) - f(e)) \leq k(n) \cdot \|f\|_2 \]

for all \( t \in \mathbb{R}_+^* \) and \( f \in C_{(2),n}(G) \). By the Banach-Steinhaus Theorem the limit

\[ \lim_{t \to 0} \frac{1}{t} [T_{\mu_t} f(e) - f(e)] \]

exists for every \( f \in C_{(2)}(G) \).

\[ \square \]

**Remark 2.10** Let \( G \) be commutative Hilbert-Lie group and \((\mu_t)_{t \in \mathbb{R}_+^*}\) a convolution semigroup in \( \mathcal{M}^1(G) \). As in the proof of Theorem 2.9, we can find a constant \( k(n) \in \mathbb{R}_+^* \) (independent of \( a \in G \) and \( t \in \mathbb{R}_+^* \)) such that

\[ \frac{1}{t} [T_{\mu_t} f(a) - f(a)] = \frac{1}{t} [T_{\mu_t} (L_a f)(e) - (L_a f)(e)] \]
\[ \leq k(n) \cdot \|L_a f\|_2 = k(n) \cdot \|f\|_2 \]

for all \( f \in C_{(2),n}(G) \) and \( a \in G \). The Banach-Steinhaus Theorem now yields the existence of the limit

\[ N f(a) = \lim_{t \to 0} \frac{1}{t} [T_{\mu_t} f(a) - f(a)] \]

uniformly in \( a \in G \). This implies existence of the infinitesimal generator \( N \) on \( C_{(2)}(G) \).

**Remark 2.11** Let \( G = H \) be a separable Hilbert space and \( C_u^{(2)}(H) \) the space of all twice Fréchet differentiable functions \( f \in C_u(H) \) such that \( \|f'\| := \sup_{x \in H} \|f'(x)\| < \infty \), \( \|f''\| := \sup_{x \in H} \|f''(x)\| < \infty \) and \( f'' \) is uniformly continuous in \( x \). Then we have \( C_u^{(2)}(H) \subset D(N) \) (cf. [6]) and \( C_2(H) = C_u^{(2)}(H) \).

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**References**

Hilbert-Lie Grubu Üzerinde Diferensiyellenebilir Fonksiyonlar ve Generatörler

Özet

Lie gruplarında oalasılık ölçümü teorisinde, konvolüsyon yarŋrupları önemli rol oynamaktadır. Temel problem, yarŋgrubu Lévy-Khinchine formülü olarak ifade etmektir. Hilbert-Lie grubu $G$ üzerinde oalasılık ölçümünün sürekli bir yarŋgrubu $(\mu_t)_{t \in \mathbb{R}^*_+}$ ise,

$$T_{\mu_t}f := \int f_\mu \mu_t(da)(f \in C_u(G), t > 0).$$

ile $C_u(G)$ uzayı üzerinde $N$ infinitesimal generatörüne sahip sürekli operatör yarŋgrubu $(T_{\mu_t})_{t \in \mathbb{R}^*_+}$ tanımlanır. Bu yarŋgrup için doğurucu fonksiyonel $A, Af := \lim_{t \to 0^+} \frac{1}{t}(T_{\mu_t}f(e) - f(e))$ biçiminde tanımlanır. Buna göre problem, $A$ doğurucu fonksiyonelinin tanımlı olmaması $C_u(G)$ nin bir $C_{(2)}(G)$ alt uzayını oluştururmaktdır. Bu sonuç, daha sonra Hilbert-Lie gruplarında Lévy-Khinchine formülüünün elde edilmesinde kullanılabılır.

Erdal COŞKUN

Hacettepe Üniversitesi,
Eğitim Fakültesi, Fen Bilimleri Bölümü,
06532 Beytepe, Ankara-TURKEY
E-mail: coskun@eti.cc.hun.edu.tr

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