FIBONACCI SEQUENCES IN FINITE NILPOTENT GROUPS

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Abstract
We have proved that, for the 3-step Fibonacci recurrence and any finite $p$-group of exponent $p$ and nilpotency class 3, the length of a fundamental period of any loop satisfying the recurrence must divide the period of the ordinary 3-step Fibonacci sequence in the field $GF(p)$.

1. Introduction
We shall be interested in the shortest period of the 3-step Fibonacci sequence the entries of which are taken in any finite $p$-group of exponent $p$ and nilpotency class 3. This problem has already been the subject of investigation. It seems to have first been addressed by Wall [9] and then Vinson [8] for cyclic groups. This theory has been generalized in [4] to cover the 3-step Fibonacci case. Campbell, Doostie and Robertson [2] have attacked the problem of recurrences in the case of non-abelian finite simple groups. Pinch [6] has studied the relationship between the period of a general linear recurrence modulo a rational prime $p$ and the period modulo a power of that prime. He does this via examining the algebraic number theory of certain finite extensions of the $p$-adic numbers.

Wall distinguishes the special loop $s = (s_i)$ defined by the recurrence $s_{i+2} = s_i + s_{i+1}$ and the initial data $s_0 = 0$ and $s_1 = 1$ in $\mathbb{Z}/p^n\mathbb{Z}$. Let $k(s, p^n)$ denote the fundamental period of $s$.

Theorem 1.1: (D.D. Wall [9]) The number $k(s, p^n)$ divides $k(s, p)p^{n-1}$, and the two quantities are equal provided $k(s, p) \neq k(s, p^2)$.

Wall goes on to conjecture that for all primes $p$, we always have $k(s, p) \neq k(s, p^2)$. He announced that he had verified this result for all primes $p < 10^4$. We know by [1] that this is indeed the case for all primes $p < 10^8$. This work has also been a recent one in this area and proves that short loops must be geometric for the 3-step Fibonacci recurrences in $H$, the additive group of the finite field $GF(p^n)$. 

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Let $s = (s_i)$ denote the ordinary 3-step Fibonacci sequence in $GF(p)$ defined by the recurrence $s_{i+3} = s_i + s_{i+1} + s_{i+2}$ and the initial data $s_0 = 0$, $s_1 = 0$ and $s_2 = 1$. This is a bi-infinite periodic sequence or loop indexed by the integers. The shortest period of this sequence is called the fundamental period and it will be denoted by $k$. We sometimes refer to this quantity as Wall’s number [9].

2. The Main Theorem

We consider a 3-step Fibonacci sequence $r = (r_i)$ in a finite $p$-group $G$, given some initial data $r_0$, $r_1$ and $r_2$. Such a sequence or loop must be periodic and we denote the shortest period of this sequence sometimes called the fundamental period by $k(r, G)$. From now on $k$ denotes the fundamental period of the standard 3-step Fibonacci sequence $0, 0, 1, 1, 2\ldots$ taken modulo a distinguished prime $p$.

**Theorem 2.1:** Let $p > 3$ be a prime number, then if $G$ is a non-trivial finite $p$-group of exponent $p$ and nilpotency class 3 then $k(r, G) = k$. Of course if $G$ is the trivial group then $k(r, G) = 1$.

3. Some Lemmas Concerning 3-Step Fibonacci Sequence

Although the proofs of all the following lemmas are not intricate they are omitted here and can be found in [4]. The notation $\sum_{i<i,j}$ indicates that we are dealing with a double sum, taken over all $i$ and $j$ subject to the constraint that $0 \leq i < j \leq k - 1$.

**Lemma 3.1:** For all integers $\alpha$ and $\beta$ we have
\[
\sum_{i<j} s_{j+\alpha}s_{i+\beta} = 0.
\]

**Lemma 3.2:** For all integers $\alpha$, $\beta$ and $c$ we have
\[
\sum_{i<j} s_{j-i+\beta}s_{i-c}s_{i+\beta} = 0.
\]

**Lemma 3.3:** For all integers $\alpha$, $\beta$ and $\gamma$ we have
\[
\sum_{j=0}^{k-1} s_{j+\alpha}s_{j+\beta}s_{j-\gamma}s_j = 0.
\]

**Lemma 3.4:** For all integers $\alpha$, $\beta$, $c$, $d$, and $e$ we have
\[
\sum_{i<j} s_{j+\alpha}s_{j+\beta}s_{j-i-d}s_{i+e}s_{i+c} = 0.
\]
4. The Proof

We do our preliminary investigations, not with the relatively free group on three generators, but with a carefully selected group \( H \) which we now describe. \( H \) has two generators \( x \) and \( y \). A presentation of \( H \) is

\[
H = \langle h_1, h_2, h_3, h_4 : (h_2, h_1) = h_3, (h_3, h_1) = h_4, \explaw = p \rangle
\]

where pairs of generators with unspecified commutator are implicitly deemed to commute. Thus \( H \) is a copy of \( C_p^3 \) extended by a cyclic group of order \( p \).

Let \( G \) be the 3-generator relatively free exponent \( p \) class 3 group on \( g_1, g_2 \) and \( g_3 \). Thus \( G \) has order \( p^{14} \) and a power commutator presentation of \( G \) is given by

\[
\begin{align*}
(g_2, g_1) &= g_4 \\
(g_3, g_1) &= g_5 \\
(g_3, g_2) &= g_6 \\
(g_4, g_1) &= g_7 \\
(g_4, g_2) &= g_8 \\
(g_4, g_3) &= g_9 \\
(g_5, g_1) &= g_{10} \\
(g_5, g_2) &= g_{11} \\
(g_5, g_3) &= g_{12} \\
(g_6, g_1) &= g_{13} \\
(g_6, g_2) &= g_{14}
\end{align*}
\]

Once again we have the convention that pairs of generators with unspecified commutator are implicitly deemed to commute.

In \( GF(p) \)-vector notation, we put \( g_i = (\delta_{ij}) \in G \), where \( \delta_{ij} \) in the Kronecker symbol and \( j \) ranges from 1 to 14.

The group \( G \) is relatively free and so admits an automorphism \( \phi \), which we call the 3-step Fibonacci automorphism, defined by \( g_1 \phi = g_2, g_2 \phi = g_3 \) and \( g_3 \phi = g_1 g_2 g_3 \).

We define two maps \( \pi_1 : G \rightarrow H \) via

\[
g_1 \pi_1 = 1, \ g_2 \pi_1 = h_1 \text{ and } g_3 \pi_1 = h_2
\]

and

\[
g_1 \pi_2 = h_1, \ g_2 \pi_2 = h_2 \text{ and } g_3 \pi_2 = 1.
\]

Let \( g_i = (\delta_{ij}) \in G \), where \( \delta_{ij} \) in the Kronecker symbol and \( j \) ranges from 1 to 14.

\[
\begin{align*}
\Ker \pi_1 &= K_1 = (*, 0, 0, *, *, 0, *, *, *, 0, *, 0, *, 0); \\
\Ker \pi_2 &= K_2 = (0, 0, *, 0, *, 0, *, *, *, 0, *, 0, *, 0).
\end{align*}
\]

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Let $M = \text{Ker} \pi_1 \cap \text{Ker} \pi_2$, so that

$$M = (0, 0, 0, 0, *, 0, 0, *, *, *, 0, *),$$

in the sense that each * can independently be any element of $GF(p)$. Now $M$ is an elementary abelian group of order $p^7$, and is therefore a $GF(p)$-space of dimension 7. A basis of $M$ is $(g_5, g_8, g_9, g_{10}, g_{11}, g_{12}, g_{14})$.

Computer aided calculations [3] yield that

$$M \cap M\phi = (g_5g_{11}, g_9, g_{10}g_{11}g_{12}, g_{14}),$$

$$M \cap M\phi \cap M\phi^2 = (g_9g_{14}^{-2}, g_{10}g_{11}, g_{12})$$

and

$$M \cap M\phi \cap M\phi^2 \cap M\phi^3 = 1.$$

Thus we have a monomorphism

$$\pi : G \longrightarrow G/K_1 \times G/K_2 \times G/K_1\phi \times G/K_2\phi \times G/K_1\phi^2 \times G/K_2\phi^2 \times G/K_1\phi^3 \times G/K_2\phi^3,$$

where the codomain is isomorphic to $\times_{i=1}^8 H$. The automorphism $\phi^{-1}$ and its powers induce isomorphisms $G/K_i\phi^i \longrightarrow G/K_i$ which can be composed co-ordinatewise with $\pi$ to form a group monomorphism

$$\tilde{\pi} : G \longrightarrow x_{j=1}^4 (G/K_1 \times G/K_2)$$

defined by

$$x \longrightarrow (K_1x, K_2x, K_1(x\phi^{-1}), K_2(x\phi^{-1}), K_1(x\phi^{-2}), K_2(x\phi^{-2})K_1(x\phi^{-3}), K_2(x\phi^{-3})).$$

Now let us examine the image of the loop, $r = (r_i)$ beginning $r_0 = g_1, r_1 = g_2, r_2 = g_3$ under $\pi$. We have

$$\tilde{\pi} : r_i \longrightarrow (r_i\pi_1, r_i\pi_2, r_{i-1}\pi_1, r_{i-1}\pi_2, r_{i-2}\pi_1, r_{i-2}\pi_2, r_{i-3}\pi_1, r_{i-3}\pi_2).$$

The sequences in the odd positions are just rotations of $(r_i\pi_1)$ and the sequences in the even positions are rotations of $(r_i\pi_2)$. Thus, if we can show that $(r_i\pi_1)$ and $(r_i\pi_2)$ both have Wall Number $k$, it will follow that $r$ has Wall Number $k$ and will be done.

In $H$ the elements can be regarded as vectors and triple multiplication is determined by the following rules;

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\[(a_0, b_0, c_0, d_0), (a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) = (a_3, b_3, c_3, d_3)\]

where
\[
a_3 = a_0 + a_1 + a_2, \\
b_3 = b_0 + b_1 + b_2, \\
c_3 = c_0 + c_1 + c_2 + a_1 b_0 + a_2 (b_0 + b_1),
\]

and finally
\[
d_3 = d_0 + d_1 + d_2 + a_1 c_0 + a_2 (c_0 + c_1 + a_1 b_0) +\left(\frac{a_2}{2}\right)(b_0 + b_1) + \left(\frac{a_1}{2}\right) + b_0.
\]

We must consider two types of initial data for loops in \(H\). We have a loop \(v\) of type I with initial data
\[
v_0 = (0, 0, 0, 0) \\
v_1 = (1, 0, 0, 0) \\
v_2 = (0, 1, 0, 0)
\]
and another \(w\) of type II with initial data
\[
w_0 = (1, 0, 0, 0) \\
w_1 = (0, 1, 0, 0) \\
w_2 = (0, 0, 0, 0).
\]

The analysis of the type II loop is entirely similar to that of type I. Thus the type I loop begins
\[
v_0 = (t_0, s_0, 0, 0) \\
v_1 = (t_1, s_1, 0, 0) \\
v_2 = (t_2, s_2, 0, 0).
\]

We focus on the type I loop \((v_i) = (t_i, s_i, c_i, d_i)\), where
\[
(s_0, s_1, s_2) = (0, 0, 1)
\]
and
\[
(t_0, t_1, t_2) = (0, 1, 0).
\]

It can be easily seen that the sequence \(t_i\) can be written in terms of \(s_i\) as \(t_i = s_{i+1} - s_i\). Now, it follows from [5] that \(c_k = c_{k+1} = c_{k+2} = 0\) which corresponds to prove the similar theorem where the nilpotency class of the group reduces to 2. To conclude, we must demonstrate \(d_k = d_{k+1} = d_{k+2} = 0\) and begin with \(d_k = 0\).
We shall need a formula for \( c_\alpha \) in order to work out the formula for \( d_\alpha \). By induction it is

\[
c_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(s_i t_{i+1} + t_{i+2}(s_i + s_{i+1}))
\]

for \( \alpha \geq 0 \). This enables us, via a similar process, to describe \( d_\alpha \) for \( \alpha \geq 0 \) as

\[
d_\alpha = \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} t_{i+1} c_i + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(t_{i+1})^2 s_i + \sum_{i=0}^{\alpha-1} s_{\alpha-i-1} t_{i+2}(c_i + c_{i+1} + t_{i+1}s_i)
\]

\[+ \sum_{i=0}^{\alpha-1} s_{\alpha-i-1}(t_{i+1})^2 (s_i + s_{i+1}).\]

We can break up the expression for \( d_k \) as \( d_k = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 \), where

\[
\Delta_1 = \sum s_k s_{k-i-1} t_{i+1} c_i,
\]

\[
\Delta_2 = \sum s_k (2^{t_{i+1}}) s_i,
\]

\[
\Delta_3 = \sum s_k s_{k-i-1} t_{i+2}(c_i + c_{i+1} + t_{i+1}s_i)
\]

and

\[
\Delta_4 = \sum s_k s_{k-i-1}(2^{t_{i+1}})(s_i + s_{i+1}),
\]

and we shall attempt to show that each of these four expressions \( \Delta_i \) actually vanishes. To this end, we break these expressions up still further.

Now we have

\[
\Delta_1 = \sum s_k s_{k-i-1} t_{i+1} c_i = \sum_{j=0}^{k-1} s_k s_{k-j-1} t_{j+1} c_j
\]

\[= \sum_{j=0}^{k-1} s_k s_{k-j-1} t_{j+1} (\sum_{i=0}^{j-1} s_{j-i-1} t_{i+1} + \sum_{i=0}^{j-1} s_{j-i-1} t_{i+2}(s_i + s_{i+1})),\]

and so \( \Delta = \Delta_{11} \Delta_{12} \Delta_{13} \), where

\[
\Delta_{11} = \sum_{j=0}^{k-1} s_k s_{k-j-1} t_{j+1} s_{j-i-1} t_{i+1},
\]

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\[ \Delta_{12} = \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{k-j-1} t_{j+1} s_{j-i-1} s_{i+1} t_{i+2} \]

and

\[ \Delta_{13} = \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} s_{k-j-1} t_{j+1} s_{j-i-1} s_{i+1} t_{i+2}. \]

Moving to \( \Delta_2 \), we find that

\[ \Delta_2 = \sum_{j=0}^{k-1} s_{k-j-1} \left( \frac{t_{i+1}}{2} \right) s_{j} = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} t_{j+1} (t_{j+1}-1) s_{j}, \]

so that

\[ \Delta_2 = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} t_{j+1}^2 s_{j} - \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} t_{j+1} s_{j}. \]

Next we tackle \( \Delta_3 \). We have

\[ \Delta_3 = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} (c_j + c_{j+1} + t_{j+1} s_{j}), \]

so that

\[ \Delta_3 = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} \left( \sum_{i=0}^{j-1} s_{j-i-1} (s_{i+1} t_{i+1} + t_{i+2} (s_i + s_{i+1})) \right) \]

\[ + \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} \left( \sum_{i=0}^{j-1} s_{j-i} (s_{i+1} t_{i+1} + t_{i+2} (s_i + s_{i+1})) \right) \]

\[ + \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} t_{j+1} s_{j}. \]

Thus \( \Delta_3 = \Delta_{31} + \Delta_{32} + \Delta_{33} + \Delta_{34} + \Delta_{35} + \Delta_{36} + \Delta_{37} \), where

\[ \Delta_{31} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-i-1} s_{i+1} t_{i+1}. \]
\[ \Delta_{32} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-1} s_{i} t_{i+2}, \]
\[ \Delta_{33} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-i-1} s_{i+1} t_{i+2}, \]
\[ \Delta_{34} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-i-1} s_{i+1} t_{i+1}, \]
\[ \Delta_{35} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-i-1} s_{i+2} t_{i+2}, \]
\[ \Delta_{36} = \sum_{i<j} s_{k-j-1} t_{j+2} s_{j-i} s_{i+1} t_{i+2}, \]

and

\[ \Delta_{37} = \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} s_{j}. \]

Also we see that

\[ \Delta_{4} = \sum_{j=0}^{k-1} s_{k-j-1} \left( \frac{t_{j+2}}{2} \right) (s_{j} + s_{j+1}), \]

so that

\[ \Delta_{4} = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} (t_{j+2} - 1)(s_{j} + s_{j+1}); \]

but \( \Delta_{4} = \Delta_{41} - \Delta_{42} \), where

\[ \Delta_{41} = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2}^{2} (s_{j} + s_{j+1}) \]

and

\[ \Delta_{42} = \frac{1}{2} \sum_{j=0}^{k-1} s_{k-j-1} t_{j+2} (s_{j} + s_{j+1}). \]

We want to show all sums of type \( \Delta \) actually vanish. In fact, this is simply the upshot of lemmas given in section 2. Thus we have shown \( d_{k} = 0 \) for the type I sequence. It is
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a matter of algebraic manipulation to show that $d_{k+1} = d_{k+2} = 0$. The analysis of the type II sequence is extremely similar to that of the type I sequence.

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References


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Sonlu Nilpotent Gruplarda Fibonacci Dizileri

Özet

Gözönüne alınan 3-basamak Fibonacci dizisi ve nilpotent sınıfı 3, exponenti p olan herhangi bir sonlu p-grup için, bu grubun elemanlarıyla oluşturulan herhangi bir döngünün esas periyodunun uzunluğunun GF(p) cisminde adı 3- basamak Fibonacci dizisinin periyodunu böldüğü ispatlandı.

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