SUBORBITAL GRAPHS FOR THE NORMALIZER OF \( \Gamma_0(N) \)

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Abstract

In this paper we examine some properties of suborbital graphs for the normalizer \( \mathcal{N} \) of \( \Gamma_0(N) \) in \( \text{PSL}(2,\mathbb{R}) \) and show that, if \( \mathcal{N}/\Gamma_0(N) \) and the set of orbit representatives are denoted by \( B \) and \( \Omega \) respectively, the permutation group \( (B, \Omega) \) is regular and \( m \)-regular where \( m \) is an odd natural number.

Introduction

Let \( \text{PSL}(2,\mathbb{R}) \) denote the group of all linear fractional transformations

\[ T : z \mapsto \frac{az + b}{cz + d}, \text{ where, } a, b, c, d \text{ are real and } ad - bc = 1. \]

This is the automorphism group of the upper half plane \( \mathcal{U} = \{ z \in \mathbb{C} | \text{Im} z > 0 \} \).

\( \Gamma \), the modular group, is the subgroup of \( \text{PSL}(2,\mathbb{R}) \) such that \( a, b, c \) and \( d \) are rational integers. \( \Gamma_0(N) \) is the subgroup of \( \Gamma \) with \( N | c \). As a matrix representation the elements of \( \text{PSL}(2,\mathbb{R}) \) are the pairs of matrices

\[ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (a, b, c, d \in \mathbb{R}, ad - bc = 1) \quad (1.1) \]

We will omit the symbol \( \pm \), and identify each matrix with its negative.

Let \( \mathcal{N} \) denote the normalizer of \( \Gamma_0(N) \) in \( \text{PSL}(2,\mathbb{R}) \). The normalizer is studied by Lehner and Newman [7] in connection with the Weierstrass points of \( \Gamma_0(N) \). Lehner and Newman calculated the normalizer directly. In [4] Conway and Norton gave a more elegant description derived from [7] in connection with the Monster Simple group. The normalizer consists exactly of the matrices

\[ \begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix} \quad (1.2) \]

where \( e \parallel N/h^2 \) and \( h \) is the largest divisor of 24 for which \( h^2 | N \) with the understandings that the determinant of the matrix is \( e > 0 \), and that \( r \parallel s \) means that \( r \parallel s \) and

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\((r,s/r) = 1\) \((r\text{ is called an exact divisor of } s)\). From now on, unless otherwise stated explicitly, \(N\) will denote a square-free integer which means that every divisor of \(N\) is exact. In this case it is seen that \(h=1\).

2. The Action of \(N\) on \(\tilde{Q}\)

Every element of \(\tilde{Q}\) can be represented as a reduced fraction \(x/y\), with \(x, y \in \mathbb{Z}\) and \((x, y) = 1\). Since \(x/y = -x/-y\) this representation is not unique. We represent \(\infty\) as \(1/0 = -1/0\). As in \(\S 1\) the action of the matrix \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) on \(x/y\) is

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} : x/y \to \frac{ax + by}{cx + dy}.
\]

It is easily seen that if \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\) and if \(x/y \in \tilde{Q}\) is a reduced fraction then, since \(c(ax + by) - a(cx + dy) = -y\) and \(d(ax + by) - b(cx + dy) = x\),

\[(ax + by, cx + dy) = 1 \quad (2.1)\]

The action of a matrix on \(x/y\) and on \(-x/-y\) is identical.

**Lemma 2.1** (i) The action of the normalizer \(N\) on \(\tilde{Q}\) is transitive.

(ii) The stabilizer of a point is an infinite cyclic group

**Proof.** Before we prove this let us give the following theorem from [2].

**Theorem 2.1** Let \(N\) be any integer and \(N = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot p_3^{\alpha_3} \ldots p_r^{\alpha_n}\), the prime power decomposition of \(N\). Then \(N\) is transitive on \(\tilde{Q}\) if and only if \(\alpha_1 \leq 7, \alpha_2 \leq 3\) and \(\alpha_i \leq 1\), where \(i = 3, \ldots, r\).

The proof of the Lemma 2.1 (i) Since \(N\) is square-free, the \(\alpha_i \leq 1\), \(i = 1, 1, \ldots, r\), so we conclude that the action is transitive.

(ii) Since the action is transitive, the stabilizer of any two points in \(\tilde{Q}\) are conjugate in \(N\). So it is sufficient to consider the stabilizer \(N_\infty\) of \(\infty\). This consists of the elements of the form

\[
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \text{ with } b \in \mathbb{Z}.
\]

So \(N_\infty\) is the infinite cyclic group generated by the element \(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\)

We now consider the imprimitivity of the action of \(N\) on \(\tilde{Q}\). This will be a special case of the following:
Let \((G, \Omega)\) be a transitive permutation group, consisting of a group \(G\) acting on a set \(\Omega\) transitively. An equivalence relation \(\equiv\) on \(\Omega\) is called \(G\-invariant\) if, whenever \(\alpha, \beta \in \Omega\) satisfy \(\alpha \equiv \beta\) then \(g(\alpha) \equiv g(\beta)\) for all \(g \in G\). The equivalence classes are called blocks. We call \((G, \Omega)\) imprimitive if \(\Omega\) admits some \(G\-invariant\) equivalence relation different from

(i) the identity relation, \(\alpha \equiv \beta\) if and only if \(\alpha = \beta\);

(ii) the universal relation, \(\alpha \equiv \beta\) for all \(\alpha, \beta \in \Omega\).

Otherwise \((G, \Omega)\) is called primitive.

We give the above notion in a different way as follows. The set \(\Delta\) of \(\Omega\) is called a set imprimitivity of \((G, \Omega)\) if for every \(g \in G\) either \(g(\Delta) = \Delta\) or \(g(\Delta) \cup \Delta = \emptyset\).

Therefore the empty set, the one point subsets and \(\Omega\) itself are sets of imprimitivity, called the trivial sets of imprimitivity. If \((G, \Omega)\) has a non-trivial set of imprimitivity, the \((G, \Omega)\) is called imprimitive, otherwise primitive.

In fact the above defined blocks are sets of imprimitivity. Conversely if \(\{\Delta_i\}_{i \in I}\), where \(I\) is an indexing set, denote the different elements of the set \(\{g(\Delta)|g \in G\}\), where \(\Delta\) is a non-empty set of imprimitivity, then \(\Omega\) can be written as a direct sum: \(\Omega = \bigcup_{i \in I} \Delta_i \cdot \{\Delta_i\}_{i \in I}\) is called a system of sets of imprimitivity of \((G, \Omega)\). Therefore if we are given a system \(\{\Delta_i\}_{i \in I}\). of course, we can define a \(G\-invariant\) equivalence relation on \(\Omega\).

**Lemma 2.2.** \([3]\) Let \((G, \Omega)\) be transitive. The \((G, \Omega)\) is primitive if and only if \(G_\alpha\), the stabilizer of an point \(\alpha \in \Omega\), is a maximal subgroup of \(G\) for each \(\alpha \in \Omega\).

What the lemma is saying is whenever \(G_\alpha < H < G\), then \(\Omega\) admits some \(G\-invariant\) equivalence relation other than the trivial cases. In fact, since \(G\) acts transitively, every element of \(\Omega\) has the form \(g(\alpha)\) for some \(g \in G\). If we define the relation \(\equiv\) on \(\Omega\) as

\[ g(\alpha) \equiv g'(\alpha) \text{ if and only if } g' \in gH. \]

Then it is easily seen that it is non-trivial \(G\-invariant\) equivalence relation. That is \((G, \Omega)\) is imprimitive.

From the above we see that the number of blocks is equal to the index \(|G : H|\) \([6]\).

We now apply these ideas to the case where \(G\) is the normalizer \(N\), and \(\Omega\) is \(\hat{Q}\).

An obvious choice for \(H\) is \(\Gamma_0\) \((N)\). Clearly \(\Gamma_\infty < \Gamma_0\) \((N)\) \(< N\), if \(N > 1\).

So, from the above discussion, the normalizer \(N\) acts imprimitively on \(\hat{Q}\). Let \(\approx\) denote the \(N\-invariant\) equivalence relation induced an \(\hat{Q}\) by \(\Gamma_0\) \((N)\). And let \(v = r/s\) and \(w = x/y\) be elements of \(\hat{Q}\) such that \((s, N) = e_1, (y, N) = e'_1\) and \(s = s_1 e_1, y = y_1 e'_1\). If \(e_2 = N/e_1\) and \(e'_2 = N/e'_1\) then it is easily verified that there exist elements
$g = \left( \begin{array}{cc} r e_2 & * \\ s_1 N & d_1 e_2 \end{array} \right)$, $det = e_2$ and $g' = \left( \begin{array}{cc} y e'_2 \\ y_1 N & d_2 \cdot e'_2 \end{array} \right)$, $det = e'_2$.

belonging to $\mathcal{N}$ and send $\infty$ to $v$ and to $w$, respectively. If $v$ and $w$ are of the above form then we get that

$$v_e \approx v_f \text{ if and only if } e = f.$$ 

By our general discussion of imprimitivity, the number $\Psi(N)$ of blocks (equivalence classes) under $\approx$ is given by $\Psi(N) = |\mathcal{N} : \Gamma(N)|$.

The following formula for $\Psi(N)$ is known [1], but for completeness we will sketch a proof here.

**Lemma 2.3.** $\Psi(N) = 2^r$, where $r$ is the number of prime factors of $N$.

**Proof.** We will count equivalence classes under $\approx$. From the above we know that $v_e \approx v_f$ if and only if $e = f$. So counting the blocks is equivalent to counting the number of divisors of $N$. This means that the number of blocks is just $2^r$, where $r$ the number of primes dividing $N$. 

**3. Suborbital Graphs For $\mathcal{N}$ on $\mathbb{Q}$**

Let $(G, \Omega)$ denote a transitive permutation group. For $(\alpha, \beta) \in \Omega^2$ and $g \in G$, we define $g(\alpha, \beta) = (g(\alpha), g(\beta))$. Therefore $(G, \Omega^2)$ becomes a permutation group. The orbits of this action are called suborbitals of $G$, that containing $(\alpha, \beta)$ being denoted by $0(\alpha, \beta)$. From $0(\alpha, \beta)$ we form a suborbital graph $\Delta(\alpha, \beta)$: its vertices are the elements of $\Omega$ and there is a directed edge from $\gamma$ to $\delta$ if $(\gamma, \delta) \in 0(\alpha, \beta)$.

$0(\alpha, \beta)$ is also a suborbital, and it is either equal to or disjoint from $0(\alpha, \beta)$. In the latter case $\Delta(\beta, \alpha)$ is just $\Delta(\alpha, \beta)$ with the arrows reserved, and we call, in this case, $\Delta(\alpha, \beta)$ and $\Delta(\beta, \alpha)$ paired suborbital graphs.

In the former case, $\delta(\alpha, \beta) = \Delta(\beta, \alpha)$ and the graph consists of pairs of oppositely directed edges; it is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call self-paired.

The above ideas were first introduced by Sims [11], and are also described in a paper by Neumann [9] and in books by Tsuzuku [13] and by Biggs and white [3], the emphasis being on applications to finite groups.

We now apply the above to the normalizer $\mathcal{N}$ an $\mathbb{Q}$. Since $\mathcal{N}$ acts transitively on $\mathbb{Q}$, each suborbital contains a pair $(\infty, v)$ for some $v \in \mathbb{Q}$; writing $v = u/n$, with $n \geq 0$ and $(u, n) = 1$, we denote this suborbital by $O_{u,n}$, and corresponding suborbital graph by $\Delta_{u,n}$.
If \( v = \infty = 1/10 = -1/0 \), then this is the trivial suborbital graph \( \Delta_{1,0} = \Delta_{-1,0} \), so assume that \( v \in \hat{Q} \) (we are not interested in trivial suborbital graphs). If \( v' \in \hat{Q} \), then \( 0(\infty, v) = 0(\infty, v') \) if and only if \( v \) and \( v' \) are in the same orbit of \( \mathcal{N}_\infty \); since \( \mathcal{N}_\infty \) is generated by \( z : v \to v+1 \), this is equivalent to \( v' = u' \mod n \) where \( u = u' \mod n \). Therefore
\[
\Delta_{u,n} = \Delta_{u',n'} \text{ if and only if } n = n' \text{ and } u = u' \mod n.
\]
We will write \( r/s \to x/y \) in \( \Delta_{u,n} \) if \( (r/s, x/y) \in O_{u,n} \).

**Theorem 3.1** \( r/s \to x/y \) in \( \Delta_{u,n} \) if and only it \( \exists e \in \mathbb{Z} \) with \( e|N, N/e|s \) and if \( (n,e) = e_0, n = n_1 e_0, e = e_1 e_0 \) then either
\[
a) \ ry - sx = n_1 \text{ and } x = re_1 u \mod n_1, y = e_1 s \mod e_1 n \text{ or}
\]
\[
b) \ ry - sx = n_1 \text{ and } x = -re_1 u \mod n_1, y = -e_1 s \mod e_1 n.
\]

**Proof.** Let \( r/s \to x/y \) in \( \Delta_{u,n} \). Then there is an element \( \left( \begin{array}{cc} a \ e \\ cN \ de \end{array} \right) \in \mathcal{N} \) sending \( \infty \) to \( r/s \), and \( u/n \) to \( x/y \) and therefore \( ae/cN = r/s \) and \( (aeu/bn)/(cNu/dn) = x/y \).

Since the determinant \( ade^2 \cdot bcN = cN \) we get \( (a,cN/e) = 1 \). So \( a = r \) and \( s = cN/e \), that is \( N/e|s \). Let \( (n,e) = e_0, n = n_1 e_0 \) and \( e = e_1 e_0 \) since \( \left( \begin{array}{cc} a \ b \\ cN \ de \end{array} \right) \) has determinant 1, then using (2.1) we see that \( \langle aeu + bn, cNu/e + dn \rangle = 1 \). Hence we will have the following matrix equation:
\[
\left( \begin{array}{cc} a & b \\ cN & de \end{array} \right) \left( \begin{array}{cc} 1 & u \\ 0 & n \end{array} \right) = \left( \begin{array}{cc} ae & aeu + bn \\ cN & cNu + dn \end{array} \right) = \\
\left( \begin{array}{cc} ae & e_n(aeu + bn_1) \\ cN & e_n(cNu + e + dn) \end{array} \right) \left( \begin{array}{cc} (-1)^i e_r \\ (-1)^j e_s \end{array} \right) = \\
\left( \begin{array}{cc} (-1)^i e_n x \\ (-1)^j e_n y \end{array} \right),
\]
where \( i, j = 0, 1 \). If \( i = j = 0 \) then \( ae = e_r, en(aeu + bn_1) = e_n x, cN = es, e_ne_1(cNu/e + dn) = e_n y \). That is, \( x = re_1 u \mod n_1 \) and \( y = cNu e_1/e + dne_1 \). So \( x = re_1 u \mod n_1 \) and \( y = e_1 s \mod n_1 \) and taking determinants in (3.1) we see that \( ry - sx = n_1 \) and so (a) holds. Similarly if \( i=1 \) and \( j=0 \) we obtain (b). If \( i = j = -1 \), then again (a) holds.

If, finally, \( i=0 \) and \( j=1 \), then (b) holds.

Conversely, if (a) holds, then there exist integers \( b,d \) such that \( x = re_1 u + bn_1 \) and \( y = e_1 su + de_1 n \). We now show that the element \( \left( \begin{array}{cc} re \ b \\ se \ de \end{array} \right) \) belongs to \( \mathcal{N} \) and sends \( \infty \) to \( r/s \), and \( u/n \) to \( x/y \).

In fact, using \( ry - sx = n_1 \) and \( N/e|s \) we get \( rde^2 - sbe = e \), that is, the above element is in \( \mathcal{N} \). Finally \( re/\infty = r/s \) and \( (re + bn)/(se + dne) = e_n(re_1 u + bn_1)/e_n(se_1 u + de_1 n) = x/y \).

As above \( (re_1 u + bn_1, se_1 u + de_1 n) = 1 \). If (b) holds the proof follows similarly. \( \square \)

**Notation** Let "\( r/s \to x/y \) in \( \Delta_{u,n} \)" be denoted by "\( r/s \sim_{\Delta_{u,n}} x/y \), where \( e_1 \) is as in Theorem 3.1. The set of \( e_1 \)'s occurred in \( \Delta_{u,n} \) will be denoted by \( E_{u,n} \)."
Corollary 3.2 Let \( E_{u,n} = E_{v,n} = \{1\} \) and let \( uv = -1 \mod n \), then the suborbital graph \( \Delta_{u,n} \) is paired with \( \Delta_{v,n} \).

**Proof.** We will observe that \( r/s \rightarrow x/y \) in \( \Delta_{u,n} \) if and only \( x/y \rightarrow r/s \) in \( \Delta_{v,n} \). Since \( r/s \rightarrow x/y \) in \( \Delta_{u,n} \), using the hypothesis and Theorem 3.2, we have that \( \exists e \in \mathbb{Z}/n \mathbb{Z}, (n, e) = e \), \( n = n_1 e \) such that either \( x = ru \mod n_1 \), \( y = su \mod n_1 \) and \( ry - sx = n_1 \), or \( x = -ru \mod n_1 \), \( y = -su \mod n_1 \) and \( ry - sx = -n_1 \).

Suppose that the former holds. Then \( xs - yr = -n_1 \) and \( vx = ruv \mod n_1 \), \( vy = suv \mod n_1 \). Since \( vx = -vy \mod n_1 \), we have \( xs - yr = -n_1 \) and \( r = -vx \mod n_1 \), \( s = -vy \mod n_1 \), that is, \( x/s \rightarrow r/s \) in \( \Delta_{v,n} \).

Corollary 3.3 \( \Delta_{u,n} \) is self-paired if and only if \( \exists e \in \mathbb{N} \) such that \( n - ne \) and \( u^2 e = -1 \mod n \).

**Proof.** Suppose \( \Delta_{u,n} \) is self-paired. So the pair \( (\infty, u/n) \) is sent to \( (u/n, \infty) \) by \( \mathcal{N} \). It is easily seen that such elements of \( \mathcal{N} \) must be of the form \( \begin{pmatrix} ue & b \\ ne & -ue \end{pmatrix} \), where determinant is \( e \). Therefore \( e \in \mathbb{N} \) and \( n - ne \) and \( u^2 e = -1 \mod n \).

Conversely, let \( e \in \mathbb{N} \) such that \( n - ne \) and \( u^2 e = -1 \mod n \). Since \( u^2 e = -1 \mod n \), then there exists an integer \( b \) such that \( -u^2 e - bn = 1 \), that is, \( -u^2 e^2 - bne = e \). Therefore the element \( \begin{pmatrix} ue & b \\ ne & -ue \end{pmatrix} \) is in \( \mathcal{N} \) and satisfies the required properties.

4. The Quotient Group \( B = \mathcal{N} / \Gamma_0(N) \)

In this final section we do some calculations about the representatives of orbit of \( \Gamma_0(N) \). Then we show that the permutation group \( (B, \Omega) \) is regular and \( m \)-regular where \( m \) is an odd natural number.

Theorem 4.1 Given an arbitrary rational number \( k/s \) with \( (k, s) = 1 \), then there exist an element \( A \in \Gamma_0(N) \) such that \( A(k/s) = (k_1/s_1) \) with \( s_1 | N \).

**Proof.**

\[
\begin{pmatrix} a \\ cN \\ d \end{pmatrix} \begin{pmatrix} k \\ s \end{pmatrix} = \begin{pmatrix} ak + bs \\ Nck + ds \end{pmatrix}
\]

we find some pairs \( \{c, d\} \) for which the equation

\[
Nck + ds = (N, s)
\]

holds, for \( (N, s) | N \), so \( s_1 = (N, s) \) works.

Since \( (Nk/(N, s), s/(N, s)) = 1 \) there exists a pair \( \{c_0, d_0\} \) so that the equation (4.1) is satisfied. Therefore, as we know, the general solution of (4.1) is
\[ c = c_0 + sn/(N, s) \]
\[ d = d_0 + Nkn/(N, s), \text{ where } n \in \mathbb{Z} \] (4.2)

Let \( N = q_0^{a_0}q_1^{a_1}\ldots q_k^{a_k} \) be the prime power decomposition of \( N \). We must show that there exists a pair \((c_*, d_*)\) obeying (4.2) such that

\[ (Nc_*, d_*) = 1. \]

If \((d_0, N) = 1\), there is nothing to prove. If \((d_0, N) > 1\) then \(d_0\) does have a common factor with \(N, q_0\) say. Using (4.1), \((q_0, Nk/(N, s)) = 1\) therefore taking \(n=1\) in (4.2) we get an integer \(d_1\) such that \(q_0|d_1\).

If \((d_1, N) > 1\) then \(d_1\) has a common factor with \(N, q_1\) say. Let \(d_2 = d_1 - q_0Nk/(N, s)\) then \(d_2\) does not have \(q_1\) as a factor. If \((d_2, N) > 1\), \(d_2\) has a common factor with \(N, q_2\) say. Eventually we arrive at

\[ d_3 = d_2 - q_0q_1Nk/(N, s), \text{ and so } d_3 \text{ has no } q_0, q_1, q_2 \text{ as factors} \]
\[ d_{k+1} = d_{k+1} - q_0q_1\ldots q_{k+1}Nk/(N, s), \text{ and so } d_{k+1} \text{ has no } q_0, q_1, \ldots, q_k \text{ as factors.} \]

Hence \((d_{k+1}, N) = 1\). Let \(d_* = d_{k+1}\) and the corresponding \(c_*, c_* \) say, and so \((Nc_*, d_*) = 1\). This implies that there exists an element \(A \in \Gamma_0(N)\) such that \(A(k/s) = k_1/s_1\) with \(s_1|N\).

Therefore we have

**Corollary 4.2.** Let \(d_1|N\) and for some \(A \in \Gamma_0(N)\) \(A(a_1/d_1) = (a_2/d_1)\) with \((a_1, d_1) = (a_2, d_1) = 1\). Then \(a_1 = a_2 \mod t\), where \(t = (d_1, N/d_1)\).

**Corollary 4.3.** Let \(d|N\) and let \((a_1, d) = (a_2, d) = 1\). Then \(\left(\begin{array}{c} a_1 \\ d \end{array}\right) \text{ and } \left(\begin{array}{c} a_2 \\ d \end{array}\right)\) are conjugate under \(\Gamma_0(N)\) if and only if \(a_1 = a_2 \mod t\), where \(t = (d, N/d)\).

**Proof.** Using the above and a theorem from [10] the result follows.

From the above lemma and corollaries we can write down the set of orbits of \(\Gamma_0(N)\) as \(O = \left\{\left[\frac{1}{d}\right] | d|N\right\}\) and it can be easily seen that the number of them is just \(2^r\), where \(r\) is the number of primes dividing \(N\). So we take \(\Omega\) as the set \(\left\{\frac{1}{d} : d|N\right\}\), as the set of representatives of \(O\).

We see that \(W_e\) of all matrices of the form \(\begin{pmatrix} ae & b \\ cN & de \end{pmatrix}\) is a single coset of \(\Gamma_0(N)\), where \(e|N\) and the determinant is \(e\). We have the relation \(W^2_e = 1, W_e W_f = W_f W_e = 385\)
AKBAŞ & BAŞKAN

\[ W_\varphi(\text{mod} \Gamma_\varphi(N)), \text{ where } g = \begin{pmatrix} e_j \ell_j \\ 0 \end{pmatrix}, \begin{pmatrix} e_j \ell_j \\ 0 \end{pmatrix}, \begin{pmatrix} e_j \ell_j \\ 0 \end{pmatrix}, \begin{pmatrix} e_j \ell_j \\ 0 \end{pmatrix}. \] This means that any element (except the identity) of \( B \) has order 2. since \( N \) acts transitively on \( \hat{Q} \), then \( B \) acts transitively on \( \Omega \). Therefore \( (B, \Omega) \) is a transitive permutation group.

Furthermore, we have the following results

**Corollary 4.4.** \((B, \Omega)\) is a regular permutation group.

**Proof.** As we see above the number \( |\Omega| \) is equal to \( 2^r \), and on the other hand \( |B| = 2^r \). So the stabilizer \( B_x \) of any element \( x \) is just the identity. Hence the action is regular. \( \square \)

**Corollary 4.5** Let \( m \) be an odd natural number. Then the group \( B \) is \( m \)-regular.

**Proof.** Since the abelian group \( B \) is finite then is it a torsion group. On the other hand, the order of any element of \( B \) is relatively prime to \( m \). so \( B \) is \( m \)-regular. \( \square \)

**References**


386
Γₙ(N) nin NORMALLEŞTİRİCİSİ İÇİN ALT ÇEVRESEL GRAFİKLER

Özet

Bu çalışmada Γₙ(N) nin PSL(2,R) deki η normalleştiricişi için altyöngüsel grafiklerin bazı özellikleri belirtildi ve eğer N/Γₙ(N) ve yörünge temsilciler kümesi sırası ile B ve Ω ile gösterilirse, (B, Ω) parmutasyon grubunun regular ve ayrıca m bir tek doğal sayı ise m-reguler olduğu gösterildi.