CERTAIN CLASSES OF ANALYTIC AND MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

We introduce a subclass $K_{n+p-1}^*(A,B)$ of analytic and p-valent functions with negative coefficients. Coefficient estimates, some properties, distortion theorems and closure theorems of functions belonging to the class $K_{n+p-1}^*(A,B)$ are determined. Also we obtain radii of close-to-convexity, starlikeness and convexity for the class $K_{n+p-1}^*(A,B)$. We also obtain class preserving integral operator of the form

$$F(z) = \frac{c + p}{z^c} \int_0^z t^{c-1} f(t) dt, c > -p$$

for the class $K_{n+p-1}^*(A,B)$ Conversely when $F(z) \in K_{n+p-1}^*(A,B)$ radius of p-valence of $f(z)$ defined by the above equation is obtained.

1. Introduction

Let $S(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}z^{p+k} (p \in \mathbb{N} = \{1, 2, \cdots\}),$$  \hspace{1cm} (1.1)

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. Let $f(z)$ be in $S(p)$ and $g(z)$ be in $S(p)$. Then we denote by $f \ast g(z)$ the Hadamard product of $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k}z^{p+k} (p \in \mathbb{N}),$$  \hspace{1cm} (1.2)

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then

\[ f \ast g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \]  

(1.3)

The \((n+p-1)\)-th order Ruscheweyh derivative \(D^{n+p-1}f(z)\) of a function \(f(z)\) of \(S(p)\) is defined by

\[ D^{n+p-1}f(z) = \frac{z^p z^{n-1}f(z)_{n+p-1}}{(n+p-1)!} \]  

(1.4)

where \(n\) is any integer such that \(n > -p\). It is easy to see that

\[ D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} \ast f(z) \]  

(1.5)

\[ = z^p + \sum_{k=1}^{\infty} \delta(n,k) a_{p+k} z^{p+k}, \]  

(1.6)

where

\[ \delta(n,k) = \binom{n + p + k - 1}{n + p - 1}. \]  

(1.7)

Particularly, the symbol \(D^nf(z)\) was named the \(n\)-th order Ruscheweyh derivative of \(f(z) \in S(1)\) by Al-Amiri [1].

Let \(T(p)\) denote the subclass of \(S(p)\) consisting of functions of the form

\[ f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (a_{p+k} \geq 0; p \in \mathbb{N}). \]  

(1.8)

Also let \(K^*_{n+p-1}(A,B)\) denote the class of functions \(f(z) \in T(p)\) such that

\[ \left| \frac{2 \left( \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - 1 \right)}{2B \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - (A + B)} \right| < 1, (z \in U), \]  

(1.9)

where \(-1 \leq A < B \leq 1, 0 < B \leq 1\), and \(n > -p\).

We note that:

(i) \(K^*_{n+p-1}(-1,1) = K^*_{n+p-1}\) (Owa [3]);
(ii) \(K^*_{\gamma}((4\gamma - 3)\beta, \beta) = T^*(\gamma, \beta)(0 \leq \gamma < 1, 0 < \beta \leq 1\) (Gupta and Jain [2]);
(iii) \(K^*_{\gamma}(4\gamma - 3, 1) = T^*(\gamma)(0 \leq \gamma < 1\) (Silverman [4]).
2. Coefficient Estimates

Theorem 1. Let the function \( f(z) \) be defined by (1.8). Then \( f(z) \) is in the class \( K_{n+p-1}(A, B) \) if and only if

\[
\sum_{k=1}^{\infty} D_k \delta(n, K) a_{p+k} \leq (B - A)(n + p),
\]

where

\[
D_k = [2k(B + 1) + (B - A)(n + p)].
\]

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let \( |z| = 1 \). Then we get

\[
\left| 2(D^{n+p}f(z) - D^{n+p-1}f(z)) - 2BD^{n+p}f(z) - (A + B)D^{n+p-1}f(z) \right|
\]

\[
= \left| -2 \sum_{k=1}^{\infty} \left( \frac{k}{n + p} \right) \delta(n, k) a_{p+k} z^{p+k} \right| - |(B - A)z^p - \sum_{k=1}^{\infty} \left[ 2B \left( \frac{k}{n + p} \right) + (B - A) \right] \delta(n, k) a_{p+k} z^{p+k} |
\]

\[
\leq \sum_{k=1}^{\infty} \left( \frac{k}{n + p} \right) \delta(n, k) a_{p+k} - (B - A) + \sum_{k=1}^{\infty} \left[ 2B \left( \frac{k}{n + p} \right) + (B - A) \right] \delta(n, k) a_{p+k}
\]

\[
= \sum_{k=1}^{\infty} \left[ \frac{2k}{(n + p)} (B + 1) + (B - A) \right] \delta(n, k) a_{p+k} - (B - A)
\]

\[
\leq 0, \text{ by hypotheses.}
\]

Hence by the maximum modulus theorem \( f(z) \in K_{n+p-1}(A, B) \).

Conversely, suppose that

\[
\left| \frac{2 \left( \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - 1 \right)}{2B \left( \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - (A + B) \right)} \right| \leq 1, z \in U.
\]

\[
= \left| \frac{-2 \sum_{k=1}^{\infty} \left( \frac{k}{n + p} \right) \delta(n, k) a_{p+k} z^k}{(B - A) - \sum_{k=1}^{\infty} \left[ 2B \left( \frac{k}{n + p} \right) + (B - A) \right] \delta(n, k) a_{p+k} z^k} \right| \leq 1, z \in U. \tag{2.3}
\]
Since $|Re(z)| \leq |z|$ for all $z$, we have

$$
Re\left\{ \frac{-2\sum_{k=1}^{\infty} \frac{k}{n+p} \delta(n, k)a_{p+k}z^k}{(B - A) - \sum_{k=1}^{\infty} \left[ 2B \left( \frac{k}{n+p} \right) + (B - A) \right] \delta(n, k)a_{p+k}z^k} \right\} < 1. \quad (2.4)
$$

Choose values of $z$ on the real axis so that $\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}$ is real. Upon clearing the denominator in (2.4) and letting $z \to 1^-$ through real values, we obtain

$$
2\sum_{k=1}^{\infty} \left( \frac{k}{n+p} \right) \delta(n, k)a_{p+k} \leq (B - A)
$$

$$
-\sum_{k=1}^{\infty} \left[ 2B \left( \frac{k}{n+p} \right) + (B - A) \right] \delta(n, k)a_{p+k}.
$$

This gives the required condition.

Finally, the function

$$
f(z) = z^p - \frac{(B - A)(n + p)}{D_k \delta(n, k)} z^{p+k} \quad (k \geq 1) \quad (2.5)
$$

is an extremal function for the theorem.

\[ \square \]

**Corollary 1.** Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p-1}^*(A, B)$. Then

$$
a_{p+k} \leq \frac{(B - A)(n + p)}{D_k \delta(n, k)} \quad (k \geq 1). \quad (2.6)
$$

The result is sharp for the function $f(z)$ given by (2.5).

**3. Some Properties of the Class $K_{n+p-1}^*(A, B)$**

**Theorem 2.** $K_{n+p}^*(A, B) \subset K_{n+p-1}^*(A, B)$ for $p \in \mathbb{N}, n > -p, -1 \leq A < B \leq 1$, and $0 < B \leq 1$.

**Proof.** Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p}^*(A, B)$. Then, by Theorem 1, we have

$$
\sum_{k=1}^{\infty} \left( \frac{2k(B + 1)}{(n + p + 1)} + (B - A) \right) \delta(n + 1, k)a_{p+k} \leq (B - A) \quad (3.1)
$$
and since
\[
\left(\frac{2k(B + 1)}{(n + p)} + (B - A)\right)\delta(n, k) \leq \left(\frac{2k(B + 1)}{(n + p + 1)} + (B - A)\right)\delta(n + 1, k)
\]
for \(k \geq 1\), \hspace{1cm} (3.2)
we have
\[
\sum_{k=1}^{\infty} \left(\frac{2k(B + 1)}{(n + p)} + (B - A)\right)\delta(n, k)a_{p+k} \\
\leq \sum_{k=1}^{\infty} \left(\frac{2k(B + 1)}{(n + p + 1)} + (B - A)\right)\delta(n + 1, k)a_{p+k} \leq (B - A).
\]
(3.3)
The result follows from Theorem 1.

**Theorem 3.** Let \(-1 \leq A_1 \leq A_2 < B_1 \leq B_2 \leq 1\) and \(0 < B_1 \leq B_2 \leq 1\). Then we have
\[
K_{n+p-1}^*(A_1, B_2) \supseteq K_{n+p-1}^*(A_2, B_1).
\]
**Proof.** Theorem 3 is an immediate consequence of the definition of the class \(K_{n+p-1}^*(A, B)\).

4. Distortion Theorems

**Theorem 4.** Let the function \(f(z)\) defined by (1.8) be in the class \(K_{n+p-1}^*(A, B)\). Then we have
\[
|z|^p - \frac{(B - A)}{D_1}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{(B - A)}{D_1}|z|^{p+1}
\]
for \(z \in U\). The result is sharp.
**Proof.** Since \(f(z) \in K_{n+p-1}^*(A, B)\), in view of Theorem 1, we obtain
\[
D_1\delta(n, 1)\sum_{k=1}^{\infty} a_{p+k} \leq \sum_{k=1}^{\infty} D_k\delta(n, K)a_{p+k} \\
\leq (B - A)(n + p),
\]
(4.2)
which implies that

$$\sum_{k=1}^{\infty} a_{p+k} \leq \frac{(B - A)}{D_1}.$$  \hspace{1cm} (4.3)

Therefore we can show that

$$|f(z)| \geq |z|^p - |z|^{p+1} \sum_{k=1}^{\infty} a_{p+k}$$

$$\geq |z|^p - \frac{(B - A)}{D_1} |z|^{p+1}$$  \hspace{1cm} (4.4)

and

$$|f(z)| \leq |z|^p + |z|^{p+1} \sum_{k=1}^{\infty} a_{p+k}$$

$$\leq |z|^p + \frac{(B - A)}{D_1} |z|^{p+1}$$  \hspace{1cm} (4.5)

for $z \in U$. This completes the proof of Theorem 4. Finally, by taking the function

$$f(z) = z^p - \frac{(B - A)}{D_1} z^{p+1},$$  \hspace{1cm} (4.6)

we can show that the result of Theorem 4 is sharp.

\[\square\]

**Corollary 2.** Let the function $f(z)$ defined by (1.8) be in the class $K^*_{n+p-1}(A, B)$. Then $f(z)$ is included in a disc with its center at the origin and radius $r_1$ given by

$$r_1 = \frac{D_1 + (B - A)}{D_1}. \hspace{1cm} (4.7)$$

**Theorem 5.** Let the function $f(z)$ defined by (1.8) be in the class $K^*_{n+p-1}(A, B)$. Then we have

$$p|z|^{p-1} - \frac{(B - A)(p+1)}{D_1} |z|^p \leq |f'(z)| \leq p|z|^{p-1} + \frac{(B - A)(p+1)}{D_1} |z|^p \hspace{1cm} (4.8)$$

for $z \in U$. The result is sharp.
Proof. In view of Theorem 1, we have

\[
\frac{D_1 \delta(n, 1)}{(p + 1)} \sum_{k=1}^{\infty} (p + k) a_{p+k} \leq \sum_{k=1}^{\infty} D_k \delta(n, K) a_{p+k} \leq (B - A)(n + p)
\]

(4.9)

which implies that

\[
\sum_{k=1}^{\infty} (p + k) a_{p+k} \leq \frac{(B - A)(p + 1)}{D_1}.
\]

(4.10)

Hence, with the aid of (4.10), we have

\[
|f'(z)| \geq p|z|^{p-1} - |z|^p \sum_{k=1}^{\infty} (p + k) a_{p+k} \\
\geq p|z|^{p-1} - \frac{(B - A)(p + 1)}{D_1} |z|^p
\]

(4.11)

and

\[
|f'(z)| \leq p|z|^{p-1} + |z|^p \sum_{k=1}^{\infty} (p + k) a_{p+k} \\
\leq p|z|^{p-1} + \frac{(B - A)(p + 1)}{D_1} |z|^p
\]

(4.12)

for \(z \in U\). Further the results of Theorem 5 are sharp for the function \(f(z)\) given by (4.6). \(\square\)

Corollary 3. Let the function \(f(z)\) defined by (1.8) be in the class \(K^*_n+p-1(A, B)\). Then \(f'(z)\) is included in a disc with its center at the origin and radius \(r_2\) given by

\[
r_2 = \frac{PD_1 + (B - A)(p + 1)}{D_1}
\]

(4.13)
5. Closure Theorems

Let the functions \( f_i(z) \) be defined, for \( i = 1, 2, \cdots, m \), by

\[
f_i(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,i} z^{p+k} (a_{p+k,i} \geq 0)
\]  

(5.1)

for \( z \in U \).

We shall prove the following results for the closure of functions in the class \( K_{n+p-1}^*(A, B) \).

**Theorem 6.** Let the functions \( f_i(z) \) defined by (5.1) be in the class \( K_{n+p-1}^*(A, B) \) for every \( i = 1, 2, \cdots, m \). Then the function \( h(z) \) defined by

\[
h(z) = \sum_{i=1}^{m} c_i f_i(z) \quad (c_i \geq 0)
\]  

(5.2)

is also in the same class \( K_{n+p-1}^*(A, B) \), where

\[
\sum_{i=1}^{m} c_i = 1.
\]  

(5.3)

**Proof.** By means of the definition of \( h(z) \), we obtain

\[
h(z) = z^p - \sum_{k=1}^{\infty} \left( \sum_{i=1}^{m} c_i a_{p+k,i} \right) z^{p+k}.
\]  

(5.4)

Further, since \( f_i(z) \) are in \( K_{n+p-1}^*(A, B) \) for every \( i = 1, 2, \cdots, m \), we get

\[
\sum_{k=1}^{\infty} D_k \delta(n,k) a_{p+k,i} \leq (B - A)(n + p)
\]  

(5.5)

for every \( i = 1, 2, \cdots, m \). Hence we can see that

\[
\sum_{k=1}^{\infty} D_k \delta(n,k) \left( \sum_{i=1}^{m} c_i a_{p+k,i} \right) = \sum_{i=1}^{m} c_i \left( \sum_{k=1}^{\infty} D_k \delta(n,k) a_{p+k,i} \right) \leq \left( \sum_{i=1}^{m} c_i \right)(B - A)(n + p) = (B - A)(n + p)
\]  

(5.6)
with the aid of (5.5). This proves that the function \( h(z) \) is in the class \( K^*_{n+p-1}(A, B) \) by means of Theorem 1. Thus we have the theorem.

\[ \Box \]

**Theorem 7.** Let \( f_p(z) = z^p \) and

\[
f_{p+k}(z) = z^p - \frac{(B - A)(n + p)}{D_k \delta(n, k)} z^{p+k} \quad (k \geq 1)
\]

(5.7)

for \( p \in \mathbb{N}, n \geq -p, -1 \leq A < B \leq 1 \) and \( 0 < B \leq 1 \). Then \( f(z) \) is in the class \( K^*_{n+p-1}(A, B) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z),
\]

(5.8)

where \( \lambda_{p+k} \geq 0(k \geq 0) \) and \( \sum_{k=0}^{\infty} \lambda_{p+k} = 1 \).

**Proof.** Suppose that

\[
f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z)
\]

\[
= z^p - \sum_{k=1}^{\infty} \frac{(B - A)(n + p)}{D_k \delta(n, k)} \lambda_{p+k} z^{p+k}.
\]

(5.9)

Then it follows that

\[
\sum_{k=1}^{\infty} D_k \delta(n, k) \frac{(B - A)(n + p)}{D_k \delta(n, k)} \lambda_{p+k} = (B - A)(n + p) \sum_{k=1}^{\infty} \lambda_{p+k}
\]

\[
= (B - A)(n + p)(1 - \lambda_p) \leq (B - A)(n + p).
\]

(5.10)

So by Theorem 1, \( f(z) \in K^*_{n+p-1}(A, B) \).

Conversely, assume that the function \( f(z) \) defined by (1.8) belongs to the class \( K^*_{n+p-1}(A, B) \). Then

\[
a_{p+k} \leq \frac{(B - A)(n + p_k)}{D_k \delta(n, k)} \quad (k \geq 1).
\]

(5.11)
Setting
\[ \lambda_{p+k} = \frac{D_k\delta(n,k)}{(B - A)(n + p)} a_{p+k} \]  
(5.12)
and
\[ \lambda_p = 1 - \sum_{k=1}^{\infty} \lambda_{p+k}. \]  
(5.13)
We can see that \( f(z) \) can be expressed in the form (5.8). This completes the proof of Theorem 7.

**Corollary 4.** The extreme points of the class \( K_{n+p-1}^*(A, B) \) are the functions \( f_{p+k}(z)(k \geq 0) \) given by Theorem 7.

6. Integral Operators

**Theorem 8.** Let the function \( f(z) \) defined by (1.8) in the class \( K_{n+p-1}^*(A, B) \), and let \( c \) be a real number such that \( c > -p \). Then the function \( F(z) \) defined by
\[ F(z) = \frac{c + p}{z^c} \int_0^z t^{c-1} f(t)dt \]  
(6.1)
also belongs to the class \( K_{n+p-1}^*(A, B) \).

**Proof.** From the representation of \( F(z) \), it follows that
\[ F(z) = z^p - \sum_{k=1}^{\infty} b_{p+k}, \]
where
\[ b_{p+k} = \left( \frac{c + p}{c + p + k} \right) a_{p+k}. \]
Therefore,
\[ \sum_{k=1}^{\infty} D_k\delta(n,k)b_{p+k} = \sum_{k=1}^{\infty} D_k\delta(n,k) \left( \frac{c + p}{c + p + k} \right) a_{p+k} \leq \sum_{k=1}^{\infty} D_k\delta(n,k)a_{p+k} \leq (B - A)(n + p), \]
since \( f(z) \in K_{n+p-1}^*(A, B) \). Hence, by Theorem 1, \( F(z) \in K_{n+p-1}^*(A, B) \).

Putting \( c = 1 - p \) in Theorem 8, we get the following. 

\[ \square \]

**Corollary 5.** Let the function \( f(z) \) defined by (1.8) be in the class \( K_{n+p-1}^*(A, B) \) and let \( F(z) \) be defined by

\[ F(z) = \frac{1}{z^{1-p}} \int_0^z \frac{f(t)}{t^{1-p}} \, dt. \]  

(6.2)

Then \( F(z) \in K_{n+p-1}^*(A, B) \).

**Theorem 9.** Let \( c \) be a real number such that \( c > -p \). If \( F(z) \in K_{n+p-1}^*(A, B) \), then the function \( f(z) \) defined by (6.1) is \( p \)-valent in \( |z| < R_p^* \), where

\[ R_p^* = \inf_k \left\{ \frac{p(c + p)D_k\delta(n, k)}{(p + k)(c + p + k)(B - A)(n + p)} \right\}^{\frac{1}{k}} (k \geq 1). \]  

(6.3)

The result is sharp.

**Proof.** Let \( F(z) = z^p - \sum_{k=1}^{\infty} a_{p+k}z^{p+k} (a_{p+k} \geq 0) \). It follows from (6.1) that

\[ f(z) = z^{1-c}(z^cF(z))' a_{p+k}z^{p+k}, (c > -p) \]

\[ = z^p - \sum_{k=1}^{\infty} \left( \frac{c + p + k}{c + p} \right) a_{p+k}z^{p+k}. \]

To prove the result, it suffices to show that

\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p \text{ for } |z| < R_p^*. \]

Now

\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| = \left| -\sum_{k=1}^{\infty} (p + k) \left( \frac{c + p + k}{c + p} \right) a_{p+k}z^k \right| \]

\[ \leq \sum_{k=1}^{\infty} (p + k) \left( \frac{c + p + k}{c + p} \right) a_{p+k}|z|^k. \]

Thus \( \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p \) if

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\[ \sum_{k=1}^{\infty} \left( \frac{p + k}{p} \right) \left( \frac{c + p + k}{c + p} \right) a_{p+k} |z|^k \leq 1. \quad (6.4) \]

But Theorem 1 confirms that

\[ \sum_{k=1}^{\infty} \frac{D_k \delta(n, k)}{(B - A)(n + p)} a_{p+k} \leq 1. \quad (6.5) \]

Thus (6.4) will be satisfied if

\[ \left( \frac{p + k}{p} \right) \left( \frac{c + p + k}{c + p} \right) |z|^k \leq \frac{D_k \delta(n, k)}{(B - A)(n + p)} \quad (k \geq 1), \]

or if

\[ |z| \leq \left\{ \frac{p(c + p)D_k \delta(n, k)}{(p + k)(c + p + k)(B - A)(n + p)} \right\}^{\frac{1}{k}} \quad (k \geq 1). \quad (6.6) \]

The required result follows now from (6.6). The result is sharp for the function

\[ f(z) = z^p - \frac{D_k \delta(n, k)(c + p + k)}{(B - A)(n + p)(c + p)} z^{p+k} \quad (k \geq 1). \quad (6.7) \]

\[ \square \]

7. Radii of Close-to-Convexity, Starlikeness and Convexity

**Theorem 10.** Let the function \( f(z) \) defined by (1.8) be in the class \( K_{n+p-1}^\alpha(A, B) \), then \( f(z) \) is \( p \)-valently close-to-convex of order \( \alpha (0 \leq \alpha < p) \) in \( |z| < r_1(A, B, n, p, \alpha) \) where

\[ r_1(A, B, n, p, \alpha) = \inf_k \left[ \frac{(p - \alpha)D_k \delta(n, k)}{(p + k)(B - A)(n + p)} \right]^{\frac{1}{k}} \quad (k \geq 1). \quad (7.1) \]

The result is sharp with the extremal function \( f(z) \) given by (2.5).

**Proof.** We must show that \( \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \alpha \) for \( |z| < r_1(A, B, n, p, \alpha) \). We have

\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=1}^{\infty} (p + k)a_{p+k}|z|^k. \]

Thus \( \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \alpha \) if
\[
\sum_{k=1}^{\infty} \frac{(p + k)}{(p - \alpha)} a_{p+k} |z|^k \leq 1. \quad (7.2)
\]

Hence, by Theorem 1, (7.2) will be true if
\[
\frac{(p + k)}{(p - \alpha)} |z|^k \leq \frac{D_k \delta(n, k)}{(B - A)(n + p)}
\]
or if
\[
|z| \leq \left[ \frac{(p - \alpha)D_k \delta(n, k)}{(p + k)(B - A)(n + p)} \right]^\frac{1}{k}, (k \geq 1). \quad (7.3)
\]

The theorem follows easily from (7.3).

\[\square\]

**Theorem 11.** Let the function \( f(z) \) defined by (1.8) be in the class \( K_{n+p-1}^*(A, B) \), then \( f(z) \) is \( p \)-valently starlike of order \( \alpha \) \((0 \leq \alpha < p)\) in \(|z| < r_2(A, B, n, p, \alpha)\) where
\[
r_2(A, B, n, p, \alpha) = \inf_k \left[ \frac{(p - \alpha)D_k \delta(n, k)}{(p + k - \alpha)(B - A)(n + p)} \right]^\frac{1}{k}, (k \geq 1). \quad (7.4)
\]
The result is sharp with the extremal function \( f(z) \) given by (2.5).

**Proof.** It is sufficient to show that \( \left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \alpha \) for \(|z| < r_2(A, B, n, p, \alpha)\). We have
\[
\left| \frac{zf'(z)}{f(z)} - p \right| \leq \sum_{k=1}^{\infty} \frac{k a_{p+k} |z|^k}{1 - \sum_{k=1}^{\infty} a_{p+k} |z|^k}.
\]

Thus \( \left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \alpha \) if
\[
\sum_{k=1}^{\infty} \frac{(p + k - \alpha)a_{p+k} |z|^k}{(p - \alpha)} \leq 1. \quad (7.5)
\]

Hence, by Theorem 1, (7.5) will be true if
\[
\frac{(p + k - \alpha)|z|^k}{(p - \alpha)} \leq \frac{D_k \delta(n, k)}{(B - A)(n + p)}
\]
or if

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\[ |z| \leq \left[ \frac{(p - \alpha)D_k\delta(n,k)}{(p + k - \alpha)(B - A)(n + p)} \right]^{\frac{1}{2}} (k \geq 1). \] (7.6)

The theorem follows easily from (7.6).

**Corollary 6.** Let the function \( f(z) \) defined by (1.8) be in the class \( K^*_n+p-1(A, B) \), then \( f(z) \) is \( p \)-valently convex of order \( \alpha (0 \leq \alpha < p) \) in \( |z| < r_3(A, B, n, p, \alpha) \) where

\[ r_3(A, B, n, p, \alpha) = \inf_{k} \left[ \frac{p(p - \alpha)D_k\delta(n,k)}{(p + k)(p + k - \alpha)(B - A)(n + p)} \right]^{\frac{1}{2}} (k \geq 1). \] (7.7)

The result is sharp with the extremal function \( f(z) \) given by (2.5).

8. **Modified Hadamard Product**

Let the functions \( f_i(z) (i = 1, 2) \) be defined by (5.1). The modified Hadamard product of \( f_1(z) \) and \( f_2(z) \) is defined by

\[ f_1 \ast f_2(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,1}a_{p+k,2}z^{p+k}. \] (8.1)

**Theorem 12.** Let the function \( f_1(z) \) defined by (5.1) be in the class \( K^*_n+p-1(A, B) \) and the function \( f_2(z) \) defined by (5.1) be in the class \( K^*_n+p-1(C, E) \) \((-1 \leq C \leq E \leq 1, 0 < E \leq 1) \). Then the modified Hadamard product \( f_1 \ast f_2(z) \) belongs to the class

\[ K^*_n+p-1 \left( 1 - \frac{4(n+p)(B-A)(E-C)}{\delta(n,1)D_1G_1 - (n+p)^2(B-A)(E-C)} \right) \] (8.2)

where \( D_k(k \geq 1) \) is defined by (2.2) and \( G_k(k \geq 1) \) is given by

\[ G_k = [2k(E + 1) + (E - C)(n + p)]. \] (8.3)

The result is sharp.

**Proof.** From Theorem 1, we have

\[ \sum_{k=1}^{\infty} \frac{D_k\delta(n,k)}{(B - A)(n + p)}a_{p+k} \leq 1 \] (8.4)

and
\[\sum_{k=1}^{\infty} \frac{G_k \delta(n,k)}{(E-C)(n+p)} a_{p+k} \leq 1.\] (8.5)

We want to find the largest \( \beta = \beta(n,p,A,B,C,E) \) such that

\[\sum_{k=1}^{\infty} \frac{[4k + (1-\beta)(n+p)] \delta(n,k)}{(1-\beta)(n+p)} a_{p+k,1} a_{p+k,2} \leq 1.\] (8.6)

From (8.4) and (8.5) by means of Cauchy-Schwarz inequality we obtain

\[\sum_{k=1}^{\infty} \frac{D_k G_k}{(B-A)(E-C)(n+p)} \delta(n,k) \sqrt{a_{p+k,1} a_{p+k,2}} \leq 1.\] (8.7)

Hence (8.6) will be satisfied if

\[\sqrt{a_{p+k,1} a_{p+k,2}} \leq \frac{(1-\beta)}{[4k + (1-\beta)(n+p)]} \sqrt{n \frac{D_k G_k}{(B-A)(E-C)}} (k \geq 1).\] (8.8)

From (8.7) it follows that

\[\sqrt{a_{p+k,1} a_{p+k,2}} \leq \frac{(n+p)}{\delta(n,k)} \sqrt{n \frac{(B-A)(E-C)}{D_k G_k}} (k \geq 1).\] (8.9)

Therefore (8.6) will be satisfied if

\[\frac{(n+p)}{\delta(n,k)} \sqrt{n \frac{(B-A)(E-C)}{D_k G_k}} \leq \frac{(1-\beta)}{[4k + (1-\beta)(n+p)]} \sqrt{n \frac{D_k G_k}{(B-A)(E-C)}} (k \geq 1).\] (8.10)

that is, that

\[\beta \leq 1 - \frac{4k(n+p)(B-A)(E-C)}{\delta(n,k) D_k G_k - (n+p)^2(B-A)(E-C)}.\] (8.11)

The right-hand side of (8.11) is an increasing function of \( k \) \((k \geq 1)\). Therefore, setting \( k = 1 \) in (8.11) we get

\[\beta \leq 1 - \frac{4(n+p)(B-A)(E-C)}{\delta(n,1) D_1 G_1 - (n+p)^2(B-A)(E-C)}.\]

The result is sharp, with equality when

\[f_1(z) = z^p - \frac{(B-A)}{D_1} z^{p+1}\]

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and

\[ f_2(z) = z^p - \frac{(E - C)}{G_1} z^{p+1}. \]

References


NEGATİV KATSAYILI BAZI ANALİTİK VE ÇOK KATLI FONKSİYON SINİFLARI

Özet

Analitik, p-kath ve negativ katsayılı fonksiyonların bir \( K_{a+p-1}^*(A,B) \) alt sınıfı tanımlanıp, bu sınıf için katsayıyı kestirmeleri bozulma teoremleri, kapanış teoremleri kanıtlanmıştır. Ayrıca bu sınıfın özellikleri incelenmiş ve elemanlarının yıldızlık, konvekslik, yaklaşık konvekslik çapları hesaplanmıştır.

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