PRODUCTS AND QUOTIENTS OF \((p, \sigma)\)-ABSOLUTELY CONTINUOUS OPERATOR IDEALS*

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Abstract

We obtain a generalization of the ideal \(\mathcal{M}_{(q,p)}\) of \((q,p)\)-mixing operators-in the sense of Pietsch- as a consequence of the study of the quotients of \((p, \sigma)\)-absolutely continuous operator ideals \(\mathcal{P}_{p,\sigma}\) -in the sense of Jarchow and Matter-. Inclusions \(\mathcal{P}_{p,\sigma}(E,F) \subset \mathcal{M}_{(q,\sigma)}(E,F)\) are also investigated, specially for the cases \(E = C(K)\) and \(E = L_1\).

The ideal \(\mathcal{P}_{p,\sigma}\) of \((p, \sigma)\)-absolutely continuous operators -where \(1 \leq p < \infty\) and \(0 \leq \sigma \leq 1\)- was defined by Matter [6] in order to give good characterizations of super-reflexivity and other properties of Banach spaces. It is closely related to the ideal of absolutely continuous operators defined by Niculescu [8], and it was introduced as an interpolated operator ideal between \(\mathcal{P}_p\) -the ideal of \(p\)-absolutely summing operators- and \(\mathcal{L}\) -the ideal of continuous operators-using an interpolative procedure ([4], [12]). This technique was motivated by the characterization of the uniform closure of the injective hull of an operator ideal proved by Jarchow and Pelczynski [3].

The ideal \(\mathcal{P}_{p,\sigma}\) satisfies intermediate properties between \(\mathcal{P}_p\) and \(\mathcal{P}_{(\frac{p}{1-\sigma}, p)}\) -the ideal of \((\frac{p}{1-\sigma}, p)\)-absolutely summing operators- and its description generalizes the case \(\mathcal{P}_p\). The aim of the first section of our work is to study those operators -that we call \((q, p, \sigma)\)-mixing operators and we denote \(\mathcal{M}_{(q, p, \sigma)}\) -that satisfy \(\mathcal{P}_{q,\sigma} \circ \mathcal{M}_{(q, p, \sigma)} \subseteq \mathcal{P}_{q,\sigma}\). We obtain in this way a generalization of \((q,p)\)-mixing operators. The second part of this paper is devoted to find inclusions between the ideals of \((p, \sigma)\)-absolutely continuous operators and the ideals of \((q,p)\)-mixing operators. Special attention is paid to operators from \(C(K)\)-spaces and \(L_1\)-spaces on arbitrary Banach spaces \(F\). In this study we obtain some properties of operators belonging to \(\mathcal{L}(L_1, F)\) that factorize through Lorentz function spaces and spaces of Schatten-Von Neumann classes, that are closely related to a theorem due to Carl and Defant (see [1] and [2]). In the third section we obtain some results about products of \((p, \sigma)\)-absolutely continuous operators.

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0. Background and Notation

Throughout this paper we employ standard Banach space notation. We shall consider only operators on Banach spaces. $E, F$ and $G$ are Banach spaces and $B_E$ is the unit ball of $E$. $W(B_E)$ is the set of all regular Borel probabilities on $B_E$. In the weak* topology, if $(x_i) \in l_p(E)$, we denote

$$W_p((x_i)) := \sup_{x' \in B_{E'}} \left( \sum_{i=1}^{\infty} |<x_i, x'>|^p \right)^{1/p}, \quad 1_p((x_i)) := \left( \sum_{i=1}^{\infty} \|x_i\|^p \right)^{1/p}$$

and

$$\delta_{p,\sigma}((x_i)) := \sup_{x' \in B_{E'}} \left( \sum_{i=1}^{\infty} \left( |<x_i, x'>|^1 - \sigma \|x_i\|^\sigma \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}.$$

The following definition is due to Matter.

**Definition 0.1.** [6]. Let $\mathcal{U}$ be an operator ideal and let $0 \leq \sigma < 1$. An operator $T : E \to F$ belongs to $\mathcal{U}_{\sigma}$ if there exist a Banach space $G$ and an operator $S \in \mathcal{U}(E, G)$ such that $\|Tx\| \leq \|x\| \|Sx\|$ for all $x \in E$. If $\mathcal{U}$ is a normed operator ideal and $\alpha$ is its norm, $\mathcal{U}_{\sigma}$ is a normed operator ideal with norm $\inf \alpha(S)^{1-\sigma}$.

For the particular case $\mathcal{U} = P_p$, the following theorem holds.

**Theorem 0.2** [6]. For every operator $T : E \to F$, the following are equivalent:

(i) $T \in P_{p,\sigma}(E, F)$.

(ii) There is a constant $C > 0$ and a probability measure $\mu$ on $B_{E'}$ such that

$$\|Tx\| \leq C \left( \int_{B_{E'}} \left( |<x, x'>|^1 - \sigma \|x\|^\sigma \right)^{\frac{p}{1-\sigma}} d\mu(x') \right)^{\frac{1-\sigma}{p}} \quad \forall x \in E.$$

(iii) There exist a constant $C > 0$ such that for every finite sequence $x_1, \ldots, x_n$ in $E$

$$1_{\bigcap \mathcal{U}_p((Tx_i))} \leq Cb_{p,\sigma}(\xi(x_i)).$$

In addition, the operator norm $\pi_{p,\sigma}(T)$ on $P_{p,\sigma}(E, F)$ is the smallest number $C$ for which (ii) and (iii) hold.

Let $E$ be a Banach space and consider $\mu$ a probability defined on $B_{E'}$. We denote by $J_p$ the map $E \to L_p(B_{E'}, \mu)$ given by $J_p(x) = \langle x, \cdot \rangle$, and by $N(J_p)$ the kernel of $J_p$. We write $E_\mu$ for the quotient space $E/N(J_p)$, and $\|n\|$ for the quotient norm.

Consider an interpolation couple $(E_0, E_1)_{1-\sigma, 1}$. The norm restricted to $E_0$ is equivalent to

$$\inf \left\{ \sum_{i=1}^{n} \|x_i\|_1^{1-\sigma} \|x_i\|_{E_0}^\sigma : \sum_{i=1}^{n} x_i = x, \quad x_i \in E_0 \quad \forall 1 \leq i \leq n \right\}.$$
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(see [7]). Throughout this paper we use this expression for the interpolation norm, since we only need its explicit formula for the elements \( x \in E_0 \).

1. \((q,p,\sigma)\)-Mixing Operators

Definition 1.1. Let \( T \) be an operator. We say that \( T \) is \((q,p,\sigma)\)-mixing if it belongs to the quotient operator ideal \( \mathcal{M}_{(q,p,\sigma)} := \mathcal{P}_q^{-1} o \mathcal{P}_{p,\sigma} \). We denote by \( \mathcal{M}_{(q,p,\sigma)} \) the quotient ideal norm \( \sup \{ \pi_{p,\sigma}(\text{SoT}) : \pi_{q,\sigma}(S) \leq 1 \} \).

Obviously, this definition and the following characterization can be adapted to the case \( \mathcal{P}_{p,\nu} \) and \( \mathcal{P}_{q,\sigma} \) when \( \sigma \neq \nu \). We restrict our attention to the case \( \sigma = \nu \).

Theorem 1.2. For every operator \( T : E \to F \) the following are equivalent:

(i) \( T \in \mathcal{M}_{(q,p,\sigma)}(E,F) \).

(ii) There is a constant \( C > 0 \) such that for each probability measure \( \mu \) on \( B_F \), there is a probability measure \( \nu \) on \( B_F \) such that

\[
\inf \left\{ \left. \sum_{i=1}^n \left( \int_{B_F} \left( |< y_i, y' | > \| y_i \| \| y' \|^{q-\sigma} \right)^{\frac{p}{q}} \, d\mu(y') \right)^{\frac{1}{q-\sigma}} : \sum_{i=1}^n y_i = Tx \right\} \leq C \inf \left\{ \left. \sum_{i=1}^n \left( \int_{B_F} \left( |< x_i, x' | > \| x_i \| \| x' \|^{q-\sigma} \right)^{\frac{p}{q}} \, d\nu(x') \right)^{\frac{1}{q-\sigma}} : \sum_{i=1}^n x_i = x \right\}
\]

\forall x \in E.

(iii) There exist a constant \( c > 0 \) such that for every finite sequence \( x_1, \ldots, x_n \) in \( E \)

\[
\left( \sum_{j=1}^m \inf \left\{ \sum_{i=1}^n \left( \frac{1}{|< y_i, y'_i | > \| y_i \| \| y'_i \|^{q-\sigma}} \right)^{\frac{p}{q-\sigma}} : \sum_{i=1}^n y_i = Tx_j \right\} \right)^{\frac{1}{q-\sigma}} \leq C \delta_{p,\sigma}((x_j))^{\frac{1}{q-\sigma}}(y'_k))
\]

In this case, \( \mathcal{M}_{(q,p,\sigma)} = \inf C \), where the infimum is taken over all \( C \) satisfying (ii) or all satisfying (iii).

Proof. (i)\(\to\)(ii) If \( T \in \mathcal{M}_{(q,p,\sigma)}(E,F) \) and \( \mu \) is a probability measure on \( B_F \), the canonical embedding \( I : F \to F_\mu \to (F_\mu, L_q(\mu))_{1-\sigma,1} \) is \((p,\sigma)\)-absolutely continuous [7] and hence \( I_\delta T \in \mathcal{P}_{p,\sigma}(E,(F_\mu, L_q(\mu))_{1-\sigma,1}) \). By theorem 0.2 there exists a probability measure \( \nu \) on \( B_F \) such that for all \( x \in E \)
\[ \| \text{IoT}_x \| \leq \pi_{p,\sigma}(\text{IoT}) \left( \int_{B_{F'}} \left( | < x, x' > | \leq 1 - \sigma \| x \|^{\sigma} \right)^{\frac{1 - \sigma}{\sigma}} d\nu(x') \right)^{\frac{1 - \sigma}{\sigma}} \] (1)

where \( \| \text{IoT}_x \| \) is the norm of the element \( \text{IoT}_x \) of the interpolated space \((F_\mu, L_q(\mu))_{1-\sigma,1}\), i.e.

\[ \| \text{IoT}_x \| = \inf \left\{ \sum_{i=1}^{n} \left( \int_{B_{F'}} \left( | < y_i, y' > | \leq 1 - \sigma \| y_i \|^{\sigma} \right)^{\frac{q}{\sigma}} d\mu(y') \right)^{\frac{1 - \sigma}{\sigma}} : \sum_{i=1}^{n} y_i = Tx \right\}. \]

Just by using the triangle inequality, we find that the second part of (1) can be replaced by

\[ \pi_{p,\sigma}(\text{IoT}) \inf \left\{ \sum_{i=1}^{n} \left( \int_{B_{F'}} \left( | < x_i, x' > | \leq 1 - \sigma \| x_i \|^{\sigma} \right)^{\frac{q}{\sigma}} d\nu(x') \right)^{\frac{1 - \sigma}{\sigma}} : \sum_{i=1}^{n} x_i = x \right\}. \]

Now we claim that \( \| \|_\mu \) in (1) can also be replaced by \( \| \|_F \); consider a representation \( \sum_{i=1}^{n} y_i \) of \( Tx \) and suppose that \( \| y_1 \|_\mu \leq \| y_1 \| \). For each \( \epsilon > 0 \) there is an \( y_0 \in N(J_p) \) verifying \( \| y_0 + y_1 \| < (1 + \epsilon) \| y_1 \|_\mu \). Obviously,

\[
\left( \int_{B_{F'}} \left( | < y_0 + y_1, y' > | \leq 1 - \sigma \| y_0 + y_1 \|^{\sigma} \right)^{\frac{q}{\sigma}} d\mu(y') \right)^{\frac{1 - \sigma}{\sigma}} + \\
\left( \int_{B_{F'}} \left( | < y_0, y' > | \leq 1 - \sigma \| y_0 \|^{\sigma} \right)^{\frac{q}{\sigma}} d\mu(y') \right)^{\frac{1 - \sigma}{\sigma}} \leq \\
\left( 1 + \epsilon \right)^{\sigma} \left( \int_{B_{F'}} \left( | < y_1, y' > | \leq 1 - \sigma \| y_1 \|^{\sigma} \right)^{\frac{q}{\sigma}} d\mu(y') \right)^{\frac{1 - \sigma}{\sigma}}.
\]

Thus, it is enough to consider the new representation \( \sum_{i=2}^{n} y_i + (y_1 + y_0) - y_0 = Tx \). The result is obtained by repeating the argument for all \( 2 \leq i \leq n \) and let \( \epsilon \to 0 \). Finally, since \( \pi_{q,\sigma}(I) \leq 1, \pi_{p,\sigma}(\text{IoT}) \leq M_{(q,p,\sigma)}(T) \).

(ii) → (iii) Let \( (y_k')_{k=1}^{n} \subset F'\) and consider the probability measure on \( B_{F'} \) given by \( \mu = \left( \sum_{k=1}^{n} \| y_k' \|^{q} \delta_k \right) \left( \sum_{k=1}^{n} \| y_k' \|^{q} \right)^{-1} \), where \( \delta_k \) is the Dirac measure \( \delta \) at the point \( \frac{1}{\| y_k' \|} y_k' \). Then

\[
\left( \sum_{j=1}^{m} \inf \left\{ \sum_{i=1}^{s_j} \sum_{k=1}^{n} | < y_i', y_k' > | \leq \| y_i' \|^{q} \right\}^{\frac{1 - \sigma}{q}} : \sum_{i=1}^{n} y_j = Tx_j \right)^{\frac{1 - \sigma}{\sigma}} =
\]

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\[
= 1_q^{-\sigma}(y_k) \left( \sum_{j=1}^{m} \inf \left\{ \sum_{i=1}^{s_j} \left( \int_{B_{p'}} \left| < y_i', y' > \right|^q \left\| y_i' \right\|^{\frac{q}{1-q}} \, d\mu(y') \right) \right\}^{\frac{1-q}{q}} : \sum_{i=1}^{n} y_j^i = Tx_j \right\}^{\frac{p}{q} \cdot \frac{1-q}{q}} \leq C \delta_{p,\sigma}(x_j) 1_q^{-\sigma}(y_k).
\]

(iii) \rightarrow (i) Condition (iii) means that all discrete probability measures \( \mu \) on \( B_{p'} \) satisfy for all \( x_1, \ldots, x_n \in E \)

\[
\left( \sum_{j=1}^{m} \inf \left\{ \sum_{i=1}^{s_j} \left( \int_{B_{p'}} \left| < y_i', y' > \right|^q \left\| y_i' \right\|^{\frac{q}{1-q}} \, d\mu(y') \right) \right\}^{\frac{1-q}{q}} : \sum_{i=1}^{n} y_j^i = Tx_j \right\}^{\frac{p}{q} \cdot \frac{1-q}{q}} \leq C \delta_{p,\sigma}(x_j)
\]

(2)

Since the set of all discrete probabilities is dense in \( W(B_{p'}) \) with respect to the weak \( C(B_{p'}) \)-topology, we only need to verify that the function \( f(\lambda) \) defined on \( B_{p'} \) by

\[
f(\lambda) := \left( \sum_{j=1}^{m} \inf \left\{ \sum_{i=1}^{s_j} \left( \int_{B_{p'}} \left| < y_i', y' > \right|^q \left\| y_i' \right\|^{\frac{q}{1-q}} \, d\lambda(y') \right) \right\}^{\frac{1-q}{q}} : \sum_{i=1}^{n} y_j^i = Tx_j \right\}^{\frac{p}{q} \cdot \frac{1-q}{q}}
\]

is continuous with respect to this topology to see that inequality (2) holds for every \( \lambda \in W(B_{p'}) \). But this holds since \( \int_{B_{p'}} \left| < y, y' > \right|^q \left\| y \right\|^{\frac{q}{1-q}} \, d\lambda(y') \) is continuous for each \( y \in F \).

Hence if \( S \in \mathcal{P}_{q,\sigma}(F) \), theorem 0.2 gives

\[
1_{\frac{p}{q} \cdot \frac{1-q}{q}} ((SoTx_j)) \leq \pi_{p,\sigma}(S) \left( \sum_{j=1}^{m} \inf \left\{ \sum_{i=1}^{s_j} \left( \int_{B_{p'}} \left| < y_i', y' > \right|^q \left\| y_i' \right\|^{\frac{q}{1-q}} \, d\mu(y') \right) \right\}^{\frac{1-q}{q}} : \sum_{i=1}^{n} y_j^i = Tx_j \right\}^{\frac{p}{q} \cdot \frac{1-q}{q}} \leq C \pi_{p,\sigma}(S) \delta_{p,\sigma}(x_j).
\]

This means that \( \pi_{p,\sigma}(SoT) \leq C \pi_{q,\sigma}(T) \) which completes the proof. \( \square \)

**Definition 1.3.** Consider \( 1 \leq p \leq q \leq \infty \) such that \( \frac{1}{r} + \frac{1}{q} = \frac{1}{p} \). For any finite collection of vectors \( x_1, \ldots, x_n \in E \) we set

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\[ m_{(q,p,\sigma)}((x_i)) := \inf \left\{ 1_{\frac{r}{1-\delta}}((\tau_i)) \delta_{q,\sigma}((x_i^0)) : \forall i, x_i = \tau_i x_i^0, x_i^0 \in E \right\} \]

We are going to use this expression to characterize when a Banach space operator belongs to \( \mathcal{M}_{(q,p,\sigma)} \). We need the following lemma.

**Lemma 1.4.** For every \( (x_i)_{i=1}^n \subset E \),

\[ m_{(q,p,\sigma)}((x_i)) = \sup \left\{ \left( \sum_{i=1}^n \left( \frac{1}{\tau_i} \right) \right)^{\frac{1-\delta}{\delta}} \right\}^{\frac{1-\sigma}{\sigma}} : \mu \in W(B_{E'}) \right\} \]

**Proof.** For every set of factorizations \( x_i = \tau_i x_i^0, \ 1 \leq i \leq n \), and every \( \mu \in W(B_{E'}) \) the following inequalities hold, just by applying Hölder’s inequality with indexes \( r/p \) and \( q/p \).

\[
\phi_{\mu,\epsilon}(\xi) := \sum_{i=1}^n (\xi_i + \epsilon)^{-\frac{q}{p}} \int_{B_{E'}} -\frac{q}{r} x_i x' > |^q \parallel x_i \parallel^{\frac{q}{p}} d\mu(x')
\]

defined on

\[ \mathcal{K} := \left\{ (\xi_i) \in \mathbb{K}^n : \sum_{i=1}^n \xi_i^\frac{q}{p} \leq \theta^{\frac{q}{p}} \right\} \]

where

\[ \theta := \sup \left( \sum_{i=1}^n \left( \frac{1}{\tau_i} \right) \right)^{\frac{1-\delta}{\delta}} \left( \sum_{i=1}^n \left( \frac{1}{\tau_i} \right) \right)^{\frac{1-\sigma}{\sigma}} \]

for some \( \epsilon > 0 \). Taking \( \xi_i = \left( \int_{B_{E'}} -\frac{q}{r} x_i x' > |^q \parallel x_i \parallel^{\frac{q}{p}} d\mu(x') \right)^{\frac{q}{p}} \) for each \( 1 \leq i \leq n \), we obtain \( \phi_{\mu,\epsilon}(\xi_i) \leq \theta^{\frac{q}{p}} \) and \( \sum_{i=1}^n \xi_i^\frac{q}{p} \leq \theta^{\frac{q}{p}} \), since \( \frac{r+q}{r} = \frac{q}{p} \) and
\( \frac{p}{r+q} \frac{q}{p} = \frac{q}{q} \). Since the set \( \mathcal{F} \) is concave and for each function \( \phi_{\mu, \epsilon} \) there is an element \((\xi_i) \in \mathcal{K}\) such that \( \phi_{\mu, \epsilon}((\xi_i)) \leq \theta^{\frac{p}{r+q}} \), we can apply Ky Fan's lemma (see for example E.4.9) in order to obtain an element \((\xi_0^n) \in \mathcal{K}\) verifying \( \phi_{\mu, \epsilon}((\xi_0^n)) \leq \theta^{\frac{p}{r+q}} \) for all \( \phi_{\mu, \epsilon} \) simultaneously. Now, if we define \( \tau_i = \|\xi_0^n\|^{\frac{1}{r}} \) and \( x_i^0 = \tau_i^{-1} x_i \), the inequality \( 1_{\frac{1}{r+q}((\tau_i))} \delta_{\phi, \epsilon}((x_0^n)) \leq \theta \) holds, using the fact that

\[
\left( \sum_{i=1}^{n} <x_i^0, x'> \| x_i^0 \|^{\frac{r}{r+q}} d\mu(x') \right)^{\frac{r}{r+q}} = \lim_{\epsilon \to 0} \left( \sum_{i=1}^{n} (\xi_i + \epsilon)^{-\frac{r}{r+q}} | <x_i, x'> \| x_i \|^{\frac{r}{r+q}} \right)^{\frac{r}{r+q}} \leq \theta^{\frac{1}{r+q}}
\]

for each \( x' \in E' \) verifying \( \| x' \| \leq 1 \) as can be deduced from \( \delta_{\phi, \epsilon}((\xi_0^n)) \leq \theta^{\frac{p}{r+q}} \), where \( \delta_{x'} \) is the Dirac measure at the point \( x' \). This proves the lemma. \( \square \)

**Proposition 1.5.** Let \( T \in \mathcal{L}(E, F) \). The following two statements are equivalent:

(i) There is a \( C_1 > 0 \) such that for every \( (x_i)_{i=1}^{n} \subset E \),

\[
m_{(q,p,\sigma)}((Tx_i)) \leq C_1 \delta_{\phi, \epsilon}((x_i)).
\]

(ii) There is a \( C_2 > 0 \) such that for every \( (x_i)_{i=1}^{n} \subset E \) and \( (y_k)_{k=1}^{m} \subset F' \) the following inequality holds

\[
\left( \sum_{i=1}^{n} \left( \sum_{k=1}^{m} | <Tx_i, y_k'> \| Tx_i \| \frac{r}{r+q} \right)^{\frac{r}{r+q}} \right)^{\frac{r}{r+q}} \leq C_2 \delta_{\phi, \epsilon}((x_i)) 1_{\frac{1}{q} - \sigma}((y_k)).
\]

Moreover, if \( T \) verifies these conditions, \( \inf C_1 = \inf C_2 \).

**Proof.** (i) \(\rightarrow\) (ii) Let \( T \in \mathcal{L}(E, F) \). Given \( y_1', \cdots, y_k' \in F' \) we define the discrete probability \( \mu \) as in theorem 1.2((iii) \(\rightarrow\) (iii)). We obtain in this way an integral expression of

\[
\left( \sum_{i=1}^{n} \left( \sum_{k=1}^{m} | <Tx_i, y_k'> \| Tx_i \| \frac{r}{r+q} \right)^{\frac{r}{r+q}} \right)^{\frac{r}{r+q}}
\]

for every \( (x_i)_{i=1}^{n} \). Using the previous lemma the result holds. (ii) \(\rightarrow\) (i) Take \( (x_i)_{i=1}^{n} \subset E \). As in the case (iii) \(\rightarrow\) (i) of theorem 1.2, \( \left( \sum_{i=1}^{n} \left( \int_{B_{q'}} | <Tx_i, y_k'> \| Tx_i \| \frac{r}{r+q} \right)^{\frac{r}{r+q}} \right)^{\frac{r}{r+q}} \)

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The fact that these probabilities are dense in \( W(B_{F'}) \) and lemma 1.4 complete the proof. \( \square \)

**Corollary 1.6** Let \( T \in \mathcal{L}(E,F) \). If \( T \) verifies (i) (and (ii)) of proposition 1.5, then \( T \in \mathcal{M}_{(q,p,q)} \).

As immediate consequences of the characterization given in theorem 1.2, the following corollaries hold.

**Corollary 1.7.** Let \( 1 \leq p \leq q \leq r \leq \infty \) and \( 0 \leq \sigma < 1 \). Then \( \mathcal{M}_{(r,q,q)} \cap \mathcal{M}_{(q,p,q)} \subset \mathcal{M}_{(r,p,q)} \).

**Corollary 1.8.** Let \( 1 \leq p_1 \leq p_2 \leq q_2 \leq q_1 \leq \infty \) and \( 0 \leq \sigma < 1 \). Then \( \mathcal{M}_{(q_2,p_1,q)} \subset \mathcal{M}_{(q_1,p_2,q)} \).

**Remark 1.9.** Let \( 1 \leq p \leq q \leq r \leq \infty \) and \( \mathcal{U} \) an operator ideal such that \( \mathcal{P}_q \mathcal{U} \subset \mathcal{P}_p \). Consider an operator \( T \in \mathcal{U}(E,F) \), \( 0 \leq \sigma < 1 \) and \( S \in \mathcal{P}_{q,q}(F,G) \). By definition 0.1, there exists an \( S_0 \in \mathcal{P}_q \) satisfying \( \| S_0 y \| \leq \| y \| \| S_0 y \|^{\frac{r-\sigma}{q-\sigma}} \) for every \( y \in F \). Thus

\[
\| STx \| \leq \| Tx \|^{\sigma} \| S_0Tx \|^{1-\sigma} \leq \| T \|^{\sigma} \| x \|^{\sigma} \| S_0Tx \|^{1-\sigma}
\]

for every \( x \in E \) and \( S_0T \in \mathcal{P}_p \). This means that \( ST \in \mathcal{P}_{p,q} \). Hence

\[
\mathcal{U} \subset \mathcal{M}_{(q,p)} \implies \mathcal{U} \subset \mathcal{M}(q,p,\sigma)
\]

As an immediate consequence, if \( \frac{1}{r} = \frac{1}{p} - \frac{1}{q} \), then

\[
\mathcal{P}_r \subset \mathcal{M}_{(q,p)} \subset \mathcal{M}_{(q,p,q)}
\]

2. **The Inclusion** \( \mathcal{P}_{p,q}(E,F) \subset \mathcal{M}_{(q,p,q)} \).

The purpose of this section is to study sufficient conditions to assure \( \mathcal{P}_{p,q}(E,F) \subset \mathcal{M}_{(q,p,q)}(E,F) \). We obtain special results in this direction in the case \( E = C(K) \) and \( E = L_1 \). The following assertion gives the best \( q \) verifying \( \mathcal{P}_{p,q} \subset \mathcal{M}_{(q-\epsilon,p)} \) for every \( \epsilon > 0 \) and a fixed \( \sigma \).

**Proposition 2.1.** Let \( p \geq 1 \), \( 0 \leq \sigma < 1 \) and \( \epsilon > 0 \). Then \( \mathcal{P}_{p,q} \subset \mathcal{M}_{(p/(\sigma(1+\epsilon)),p)} \) for each \( \epsilon > 0 \).

Moreover, \( \frac{p}{\sigma} = \sup \{ q : \mathcal{P}_{p,q} \subset \mathcal{M}_{(q,p)} \} \).
Proof. By [6] the minimum $q$ satisfying $\mathcal{P}_{p,\sigma} \subset \mathcal{P}_{(q,p)}$ is $\frac{p}{1+\sigma}$. On the other hand, if $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$ then $\mathcal{P}_{(q,p)} \subset \mathcal{M}(s,\epsilon,p)$ for each $\epsilon > 0$. Thus $\mathcal{P}_{p,\sigma} \subset \mathcal{M}(p/(\sigma(1+\epsilon)),p)$ for each $\epsilon > 0$. Since $\mathcal{M}(s,p) \subset \mathcal{P}_{(q,p)}$ if $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$, if we take a $s > p/\sigma$ then $\mathcal{P}_{p,\sigma} \subset \mathcal{P}_{(\frac{p}{1+\sigma},\epsilon,p)}$, a contradiction. \hfill \Box

There are many examples of Banach spaces $F$ such that this inclusion is satisfied for $\epsilon = 0$. For example, if either $F \in \text{space } (\mathcal{M}(p/\sigma,p))$ or $E \in \text{space } (\mathcal{M}(p/\sigma,p))$ (see [9] for notation) then $\mathcal{P}_{p,\sigma}(E,F) \subset \mathcal{M}(p/\sigma,p)(E,F)$ is easily verified.

Even equality is available in certain cases:
Let $1 \leq p, q \leq \infty$ with $\frac{1}{q} = |\frac{1}{2} - \frac{1}{p}|$. Carl and Defant proved that $\mathcal{L}(l_1,l_p) = \mathcal{M}(q,1)(l_1,l_p)$ for these $p$ and $q$. This also means that $\mathcal{L}(l_1,L_p) = \mathcal{M}(q,2)(L_1,L_p)$, since $\mathcal{M}(q,1) \subset \mathcal{M}(q,2)$ [9]. On the other hand, Matter obtained the equality $\mathcal{L}(L_1,L_p) = \mathcal{P}_{2,\sigma}(L_1,L_p)$ for $\frac{2}{1+\sigma} \leq p \leq \frac{2}{1-\sigma}$ (Theorem 9.1.(i) [6]). In particular, this means that $\mathcal{P}_{2,\sigma}(L_1,L_p) = \mathcal{M}(2/\sigma,2)(L_1,L_p)$.

An application of proposition 2.1 to another results of Matter allows us to obtain the following corollary about operators on $L_1$ factoring through spaces of Schatten-Von Neumann classes $S_{p,q}$ and Lorentz spaces $L_{p,q}$. It holds just by applying 2.1 to theorem 9.2 of Matter [6] and Grothendieck’s theorem.

Corollary 2.2. Let $F$ a Banach space, $p, q \leq 1$ and $q \leq \sigma < 1$ such that $\frac{2}{1+\sigma} < p, q < \frac{2}{1-\sigma}$. Suppose that $T \in \mathcal{L}(L_1,F)$ admits a factorization $T = T_0T_0$ through $B = L_{p,q}$ or $S_{p,q}$ verifying $T_1 \in \mathcal{P}_{\frac{p}{1+\sigma}}(B,F)$. Then $T \in \mathcal{P}_1(L_1,F)$.

However, the inclusion $\mathcal{P}_{p,\sigma}(C(K),F) \subset \mathcal{M}(p/\sigma,p)$ does not hold for the general case. The following proposition characterizes those Banach those Banach spaces $F$ such that the inclusion

$\mathcal{P}_{1,\sigma}(C(K),F) \subset (\mathcal{M}_{1/\sigma,1}(C(K),F))$ holds.

Using this result we find a Banach space $F$ such that inclusion is not true.

Proposition 2.3 Let $F$ be a Banach space and $K$ compact set. The following assertions are equivalent for $0 < \sigma < 1$.

(i) $\mathcal{P}_{1,\sigma}(C(K),F) \subset \mathcal{M}_{1/\sigma,1}(C(K),F)$.

(ii) $\mathcal{P}_{1/\sigma,1}(C(K),F) = \mathcal{P}_{1/\sigma,1}(C(K),F)$.

(iii) For every Banach space $G$ and every $T \in \mathcal{L}(C(K),F)$, if $T$ verifies that there is a probability measure $\lambda$ on $K$ such that there exist a factorization $T = T_0T_0I$ where $I$ is the canonical injection $C(K) \rightarrow L_{1/\sigma,1}(\lambda)$, $\bar{T} \in \mathcal{L}(L_{1/\sigma,1}(\lambda),F)$ and $T_1 \in \mathcal{P}_{\frac{p}{1+\sigma}}(F,G)$, then $T \in \mathcal{P}_1(C(K),G)$. 

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Proof. (i) → (ii) For every Banach space $F$, $\mathcal{M}_{1/\sigma,1}(C(K), F) = \mathcal{P}_{1/\sigma,1}(C(K), F)$ holds (see [2, ex. 32.3]). If $\mathcal{P}_{1,\sigma}(C(K), F) \subset \mathcal{M}_{1/\sigma,1}(C(K), F)$ then (ii) holds, since for $C(K)$-spaces $\mathcal{P}_{1/\sigma,1}(C(K), F) = \mathcal{P}_{1,\sigma}(C(K), F)$ is satisfied (see [7] and [11]). (ii) → (i) The inclusion $\mathcal{P}_{1/\sigma,1}(C(K), F) \subset \mathcal{P}_{1,\sigma}(C(K), F)$ and (ii) imply (i). (iii) ↔ (i) By (iv) of theorem 2.4 of [11], $ToI \in \mathcal{P}_{1,\sigma}(C(K), F)$, and every $R \in \mathcal{P}_{1,\sigma}(C(K), F)$ can be factored in this way. On the other hand (iii) means that $R = ToI$ also belongs to $\mathcal{M}_{1/\sigma,1}(C(K), F)$.

Observe that (ii) → (i) holds for every space $E$, and not only for $E = C(K)$.

Counterexample 2.4 The equality (ii) is not valid for all $F$. Let $F = L_p([0,1])$ for $p = \frac{1}{1-\sigma} > 2$ and $E = L_\infty([0,1])$. Then by theorem 7 of [5], $\mathcal{L}(E, F) \neq \mathcal{P}_{1/\sigma,1}(E, F)$. However, by a theorem of Orlicz, $\mathcal{L}(E, F) = \mathcal{P}_{1/\sigma,1}(E, F)$ (see e.g. 22.6.2 [9]).

Remark 2.5. Consider the canonical map $J_{p,\sigma} : E_\mu \to (E_\mu, L_p(B_{E'}(\mu)))_{1-\sigma,1}$ for a given probability $\mu$ defined on $B_{E'}$. The canonical inclusion $J_p : E_\mu \to L_p(B_{E'}(\mu))$ is p-absolutely summing and thus $J_p \in \mathcal{M}_{1,\sigma}(E_\mu)$. Obviously the identity map of $E_\mu$ is continuous and thus belongs to $\mathcal{L}(E_\mu, E_\mu) = \mathcal{M}_{1,\sigma}(E_\mu, E_\mu)$. Taking $s = p/\sigma$, $s_1 = p$ and $s_1 = \infty$, and applying 20.1.13 [9] we obtain that

$J_{p,\sigma} \in \mathcal{M}_{p,\sigma}(E_\mu, L_p(B_{E'}(\mu)))_{1-\sigma,1}$.

This implies that for every couple of Banach spaces $(E, F)$, $\mathcal{P}_{p,\sigma}(E, F) \subset \mathcal{M}_{p,\sigma}(E, F)$, a contradiction. These arguments show that there is a flaw in [9] 20.1.13. However, there is no problem with the application of proposition 20.1.13 in the proof of theorem 20.1.15 [9], since 20.1.13 is only used there for the case $F_0 = F_1 = F$.

3. Products of $(p, \sigma)$-Absolutely Continuous Operators.

The following proposition extends in a certain sense the classical Pietsch result about products of p-absolutely summing operators [9]: $\mathcal{P}_{p,\sigma}(E, F) \subset \mathcal{P}_{p,\sigma}(E, F)$, for each $\epsilon > 0$.

Proposition 3.1. Let $0 \leq \sigma < 1$ and $1 \leq r, p, q \leq \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then

$\mathcal{P}_{q,\sigma} \circ \mathcal{P}_{1/r,\sigma} \circ \mathcal{P}_{1/r,\sigma}$ for each $\epsilon > 0$.

Moreover, this inclusion is also valid for $\epsilon = 0$ for couples $(E, F)$ of Banach spaces which satisfy $\mathcal{P}_{p,\sigma}(E, F) \subset \mathcal{M}_{p,\sigma}(E, F)$.

Proof. If $\sigma = 0$ then $\mathcal{P}_{1/(1+\epsilon)} = \mathcal{L}$ and nothing is to prove. If $\sigma > 0$, proposition 2.1 and remark 1.9 give the result.
Finally, we give some results for the case $E = C(K)$. □

Proposition 3.2. Let $1 \leq r, q \leq \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $\epsilon > 0$ and $\frac{1}{q} + \frac{1}{(1 - \sigma)} = 1$. The following inclusions hold for all compact $K$ and for all couple of Banach spaces $(F, G)$.

i) $P_{\frac{1}{q} - \sigma}(C(K), F) \circ P_{\epsilon, \sigma}(C(K), G) \subseteq P_{\frac{1}{q} - \epsilon, \sigma}(C(K), F) \subseteq P_{\frac{1}{q} - \epsilon, \sigma}(C(K), G)$

ii) $P_{\frac{1}{q} - \epsilon, \sigma}(F, G) \circ P_{\epsilon, \sigma}(C(K), F) \subseteq P_{\frac{1}{q} - \epsilon, \sigma}(C(K), G)$.

Proof. $P_{\frac{1}{q} - \sigma}(C(K), F) \subseteq P_{\epsilon, \sigma}(C(K), F) \subseteq \bigcap_{\epsilon > 0} P_{\frac{1}{q} - \epsilon, \sigma}(C(K), F)$ are satisfied (see [6] 5.2). This means that the inclusion

$P_{\frac{1}{q} - \sigma}(F, G) \circ P_{\epsilon, \sigma}(C(K), F) \subseteq P_{\frac{1}{q} - \epsilon, \sigma}(C(K), G)$

holds, and by remark 1.9, i) holds.

On the hand, since for each $\epsilon > 0$

$P_{\epsilon, \sigma}(C(K), G) = P_{\frac{1}{q} - \epsilon, 1}(C(K), G) \subseteq M_{(\frac{1}{q} - \epsilon, 1)}(C(K), G)$

(see [10], [11], [6] or [7], and [9]), we also have the inclusion ii) just by an application of remark 1.9. □

References


\text{(p, \sigma) MUTLAK SÜREKLİ OPERATÖR İDEALLERİN ÇARPIMLARI VE BÖLÜMLERİ ÜZERİNE}

\text{Özet}

Bu makalede A. Pietsch in tanımladığı $M_{(q,p)}$ operatör ideallerin bir genelleştirilmesi tanımlanmış, bu kavramın çeşitli uygulamaları ele alınmıştır.

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