EXTENSIONS OF CARISTI- KIRK’S THEOREM

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Abstract

We give some extensions and/or improvements, to uniform spaces and to multi-valued mappings, of Caristi-Kirk’s theorem.

Key words and phrases: Uniform spaces, multi-valued mappings, fixed point theorem, maximal element, weak p-contraction mappings.

1. Introduction

It was observed that certain fixed point theorems can be deduced from the following result:

Let \((E, \leq)\) be an ordered set which admits a maximal element.

Let \(f : E \to E\) be a mapping such that \(x \leq f(x)\) for every \(x\) in \(E\).

Then \(f\) has a fixed point.

This result served as a basis for certain theorems about the existence of maximal elements ([1], [2], [3], [6]), and hence fixed point theorems. Considered spaces are often metric spaces endowed with an order defined via the distance ([2], [3], [4], [5]).

Ekeland’s variational principle, which concerns the existence of maximal elements ([2], [6]) and its generalizations allowed simple proof of Caristi-Kirk’s theorem ([2],[3],[8]).

Recently, V. Conserva ([5]) gave a slight improvement of this theorem in metric spaces.

In this paper we give some extensions and/or improvements to uniform spaces and to multi-valued mappings, of Caristi- Kirk’s theorem.

Let us notice that proofs we give here go along the lines of those given in the case of metric or topological vector spaces ([2], [3], [4], [5]).

In the following, for a uniform space \(E\), we consider a family \((d_i)_{i \in I}\) of semi-metrics which defines its uniform structure and such that \(\sup_{i \in I} d_i(x, y) < +\infty\), for all \(x, y\) in \(E\).

Let \(E\) be a uniform space and \(p : E \to \mathbb{R}_+\) a positive real functional on \(E\). Define a partial order on \(E\) as follows:

\(x \leq y\) if and only if \(d_i(x, y) \leq p(x) - p(y)\), for all \(i \in I\)

For an \(x\) in \(E\) we put \(S(x) = \{y \in E / x \leq y\}\).
Let \( A \) be a subset of \( E \), \( \operatorname{diam} (A) = \sup_{i \in I} \left( \sup_{x \in A} d_i(x, y) \right) \) will be called diameter of \( A \).

We denote by \( 2^E \), the set of all nonempty subsets of \( E \).

**I-Single Valued Mappings**

We will begin with the following result:

**Theorem I-1.** Let \( E \) be a uniform space and \( p : E \to \mathbb{R}_+ \) a real functional which is lower semi-continuous (l.s.c.). Let \( f : E \to E \) be an arbitrary self-mapping of \( E \).

(I-1): there exists an \( x \) in \( E \) such that, for all \( i \in I \),
\[
d_i(y, f(y)) \leq p(y) - p(f(y)), \text{ for all } y \in S(x).
\]

(I-2): any Cauchy sequence in \( S(x) \) converges in \( E \).

Then \( f \) has a fixed point which is maximal in \( (E, \leq) \).

**Proof.** Construct a sequence \((x_n)_n\) in \( E \) inductively as follows: \( x_1 = x \); when \( x_1, x_2, \ldots, x_n \) have been chosen, let \( a_n := \inf p(S(x_n)) \) and take \( x_{n+1} \) in \( S(x_n) \) such that \( p(x_{n+1}) \leq a_n + 1/n \).

Then \( x_n \leq x_{n+1} \) and for any \( y \) in \( S(x_n) \) we have \( a_n - 1 \leq a_n \leq p(y) \leq p(x_n) \leq a_n + 1/n - 1 \).

In particular, for \( n \leq m \) we have \( 0 \leq p(x_n) - p(x_m) \leq 1/n - 1 \).

This shows that \((x_n)_n\) is a Cauchy sequence and that, for all \( i \in I, d_i(x_n, x_m) \) converges to zero when \( n \) tends to infinity. Hence \( \operatorname{diam} (S(x_n)) \) converges to zero for all \( n \).

By hypothesis \((x_n)_n\) converges to an \( x_0 \) in \( E \). On the other hand, by the construction of \((x_n)_n\), we have, for all \( i \in I, d_i(x_n, x_{n+k}) \leq p(x_n) - p(x_{n+k}) \), for all \( k \geq 0 \).

Hence, allowing \( k \) tend to infinity we have \( d_i(x_n, x_0) \leq p(x_n) - p(x_0) \), for all \( n \) and for all \( i \in I \). This means that \( x_n \leq x_0 \) for all \( n \). Therefore \( x_0 \in S(x) \) and
\[
d_i(x_0, f(x_0)) \leq p(x_0) - p(f(x_0)), \text{ for all } i \in I \text{ i.e. } x_0 \leq f(x_0)
\]

Let now \((y_n)_n\) be a sequence such that \( x_n \leq y_n \) for all \( n \). Then \( \lim_n y_n = x_0 \), for \( \operatorname{diam} (S(x_n)) \) converges to zero for all \( n \).

Finally, suppose that \( y \) in \( E \) is such that \( x_0 \leq y \). Then we also have \( x_n \leq y \) for all \( n \) and it follows that \( y = x_0 \) (take \( y_n := y \) for all \( n \) in the preceding sequence), i.e. \( x_0 \) is maximal and then \( f(x_0) = x_0 \).

We have the following corollary:

**Corollary I-2.** Let \( E \) be a sequentially complete uniform space and \( p : E \to \mathbb{R}_+ \) a l.s.c. real functional. Let \( f \) be an arbitrary self-mapping of \( E \). Suppose that there exists an \( x \) in \( E \) such that \( d_i(y, f(y)) \leq p(y) - p(f(y)) \), for all \( y \) in \( S(x) \) and for all \( i \in I \).
Then \( f \) has a fixed point which is a maximal element in \((E, \leq)\).

As a consequence of this result we have

**Corollary I-3.** (Caristi-Kirk’s theorem). Let \((E, d)\) be a complete metric space and \(p : E \to \mathbb{R}_+\) a l.s.c. real functional. Let \(f\) be a self-mapping of \(E\) such that \(p(x, f(x)) \leq p(x) - p(f(x))\), for all \(x\) in \(E\). Then \(f\) has a fixed point.

Analyzing the proof of Theorem I-1, we can state the following

**Theorem I-4.** Let \(E\) be a uniform space and \(p : E \to \mathbb{R}_+\) a l.s.c. real functional. Let \(f\) be an arbitrary self-mapping of \(E\). Suppose that:

(I-3): there exists an \(x\) in \(E\) such that, for all \(i \in I\),
\[
d_i(y, f(y)) \leq p(y) - p(f(y)), \text{ for every } y \text{ in } S(x)
\]

(I-4): any nondecreasing sequence in \(S(x)\) is relatively compact.

Then \(f\) has at least one fixed point which is maximal in \((E, \leq)\).

**Proof.** Let \((x_n)_n\) be a sequence defined as follows: \(x_1 = x\); when \(x_1, x_2, \ldots, x_n\) have been chosen let \(a_n := \inf p(S(x_n))\) and take \(x_{n+1}\) in \(S(x_n)\) such that \(p(x_{n+1}) \leq a_n + 1/n\).

The sequence \((x_n)_n\) is increasing. One shows that, for all \(i \in I\),
\[
d_i(x_n, x_m) \leq p(x_n) - p(x_m) \leq 1/n - 1, \text{ for } n \leq m.
\]

Hence \((x_n)_n\) is a Cauchy sequence; moreover \(\text{diam } (S(x_n))\) converges to zero for all \(n\).

By hypothesis, \((x_n)_n\) is relatively compact. Therefore there exists a subsequence \((x_{n_k})_k\) of \((x_n)_n\) which converges to an \(x_0\) in \(E\). Since \((x_n)_n\) is a Cauchy sequence, it also converges to \(x_0\). By the same argument as in the proof of Theorem I-1 we get that \(x_0 \in S(x)\) and that \(x_0\) is maximal. Thus by hypothesis, for all \(i \in I\),
\[
d_i(x_0, f(x_0)) \leq p(x_0) - p(f(x_0)), \text{ i.e. } x_0 \leq f(x_0).
\]
Thus \(f(x_0) = x_0\).

\[\square\]

**Remark I-5.** Instead of condition (I-4) in Theorem I-4, if we suppose that \(S(x)\) is complete for each \(x\) in \(E\), the conclusion of the theorem still holds.

If we suppose likewise that \(E\) is sequentially complete, the condition (I-4) is no more needed. Corollary I-2 can again be obtained as a consequence.

**II-Multi-Valued Mappings**

Now we give an extension of Theorem I-1 to the case of certain multi-valued mapping, namely those which in some way are \(p\)-contractive. Thus we improve some of the results of M-H. Shih ([8]) which are of Caristi-Kirk type.

We slightly soften a definition of M-H. Shih ([8]).
Definition II-1. Let $A$ be a subset of $E$. A multi-valued mapping $f : E \to 2^E$ is said to be a weak $p$-contraction on $A$, if there exists a real functional $p : E \to \mathbb{R}_+$ such that for each $x$ in $A$ and $y \in f(x)$, $d_i(x, y) \leq p(x) - p(y)$, for all $i \in I$.

$f$ is said to be a $p$-contraction on $A$, if for each $x$ in $A$ and all $y$ in $f(x)$, $d_i(x, y) \leq p(x) - p(y)$, for all $i \in I$.

$f$ is said to be a weak $p$-contraction (respectively a $p$-contraction) in the sense of Shih, if $A = E$, $E$ being a metric space.

We have the following result:

Theorem II-2. Let $E$ be a uniform space and $f : E \to 2^E$ a closed multi-valued mapping. Suppose that:

(II-1): there exists an $x$ in $E$ such that $f$ is a weak $p$-contraction on $S(x)$;

(II-2): any Cauchy sequence in $S(x)$ converges in $E$.

Then $f$ has a fixed point.

Proof. Endow $E$ with the partial order corresponding to $p$ and construct a sequence $(x_n)_n$ as follows: $x_1 = x$ and for $n > 1$, take $x_{n+1}$ in $f(x_n)$ such that $x_n \leq x_{n+1}$ (this is possible for $f$ is a weak $p$-contraction on $S(x)$). One shows that $(x_n)_n$ is a Cauchy sequence. Therefore by hypothesis, there exists $x^* \in f(x^*)$, i.e. $x^*$ is a fixed point of $f$.

Moreover, we have, for all $i \in I$,

$$d_i(x_n, x_{n+k}) \leq p(x_n) - p(x_{n+k}), \text{ for } k \geq 0.$$ 

Tending $k$ to infinity we obtain

$$d_i(x_n, x^*) \leq p(x_n), \text{ for all } i \in I, n = 0, 1, 2 \cdots$$

As a consequence we get the following: \(\square\)

Corollary II-3. Let $E$ be a sequentially complete uniform space and $f : E \to 2^*$ a closed multi-valued mapping. Suppose that $f$ is a weak $p$-contraction. Then $f$ has a fixed point.

Corollary II-4 (M.-H. Shih ([8])). Let $(E, d)$ be a complete metric space and $f : E \to 2^E$ a closed multi-valued mapping. Suppose that $f$ is a weak $p$-contraction. Then $f$ has a fixed point.

We will need the following statement, in uniform spaces, of Ekeland's variational principle. A. Brondsted ([3]) stated it differently (in uniform spaces). Here we give a statement directly applicable to our case.
Theorem II-5. Let $E$ be a sequentially complete uniform space and $p : E \to \mathbb{R}$ a l.s.c. real functional which is bounded below. Then there exists an $x$ in $E$ such that:

$$(II - 3): \forall \ y \neq x, \exists \ i_o \in I : p(y) > p(x) - d_{i_o}(x, y).$$

Now we can state the following

**Theorem II-6** Let $E$ be a uniform space and $f : E \to 2^E$ a multi-valued mapping. Suppose that:

$$(II-4):$$ there exists an $x$ in $E$ such that $f$ is a weak $p$-contraction on $S(x)$ with $p$ being I.s.c. and $S(x)$ complete. Then $f$ has a fixed point.

**Proof.** By Theorem II-5, there exists $v \in S(x)$ such that for every $w \neq v$, there exists $i_o \in I$ such that $p(w) - p(v) > -d_{i_o}(w, v)$. We assert that $v \in f(v)$. Indeed, if not, then $p(w) - p(v) > d_{i_o}(w, v)$, for each $w$ in $f(v)$. Whence a contradiction to the p-contractness of $f$ on $S(x)$. \hfill \Box

**Corollary II-7.** Let $E$ be a sequentially complete uniform space and $f : E \to 2^E$ a multi-valued mapping. Suppose that $f$ is a weak $p$-contraction with $p$ being I.s.c. Then $f$ has a fixed point.

From the previous corollary we deduce the following result:

**Corollary II-8** (M-H. Shih ([8]). Let $(E, d)$ be a complete metric space and $f : E \to 2^E$ a multi-valued mapping. Suppose that $f$ is a weak $p$-contraction with $p$ being I.s.c. Then $f$ has a fixed point.

**Remark II-9.** Replacing weak $p$-contractness by $p$-contractness we get special cases of the results above and in particular some results of M-H. Shih ([8]).

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CARISTI- KIRK THEOREMİNİN BİR GENELLEŞTİRİLMESİ

Özet

Bu makalede Caristi- Kirk teoreminin düzgün uzaylara ve çok-değerli temsillere genişletilmeleri verilmiştir.

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