(σ, τ)- LIE IDEALS IN PRIME RINGS WITH DERIVATION

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Abstract

Let R be a prime ring, char R ≠ 2,3, σ, τ : R → R two automorphisms, U a nonzero (σ, τ)- Lie ideal of R and o ≠ d : R → R a derivation such that σd = dσ, τd = dτ. In this paper we have proved the following results. (1) If d(U) ⊂ Z then U ⊂ Z (2) If d(U) ⊂ U and d²(U) ⊂ Z then U ⊂ Z.

Introduction

Let R be a ring and U an additive subgroup of R, and σ, τ : R → R two mappings. We set [x, y]σ,τ = xσ(y) − τ(y)x. The definition of (σ, τ)-Lie ideal was given in [4] as follows: (i) U is called a (σ, τ)-right Lie ideal of R if [U, R]σ,τ ⊂ U. (ii) U is called (σ, τ)-left Lie ideal of R if [R, U]σ,τ ⊂ U. (iii) U is called a (σ, τ)-Lie ideal of R if U is both a (σ, τ)-right Lie ideal and (σ, τ)-left Lie ideal of R. Every Lie ideal of R is a (1,1)-right(left) Lie ideal of R, where 1 : R → R is the identity map. Let

\[ R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d, e \in I \right\}, \quad U = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in I \right\} \subset R, \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \quad \tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ c & d \end{pmatrix}. \]

Then σ and τ are automorphisms of R, and U is a (σ, τ)-left Lie ideal or R but not Lie ideal of R.

Let d be a nonzero derivation of R. The following results have been proved for Lie ideals in [1] and [3]. (i) If d(U) ⊂ Z then U ⊂ Z. (ii) If d²(U) ⊂ Z then U ⊂ Z. In this paper we generalized the above results in prime rings with (σ, τ)-Lie ideals.

Throughout R will represent a prime ring σ and τ automorphisms of R, d a nonzero derivation of R such that dσ = σd, dτ = τd and Z the center of R. Further, we shall often use the relations:

\[ [x, y]σ,τ = x[y, z]σ,τ + [x, τ(z)]y = x[y, σ(z)] + [x, z]σ,τy \quad \text{and} \quad [x, yz]σ,τ = τ(y[x, z]σ,τ + [x, y]σ,τσ(z)]. \]
Results

Lemma Let $R$ be a prime ring, $U$ a nonzero $(\sigma, \tau)$-left Lie ideal. If $[R,U]_{\sigma,\tau} \subset Z$ then $U \subset Z$.

Proof. For all $x \in R, u \in U, [x,u]_{\sigma,\tau} \in [R,U]_{\sigma,\tau}$. We replace $x$ by $x\sigma(u)$, then we get $[x,u]_{\sigma,\tau}\sigma(u) \in [R,U]_{\sigma,\tau}$ and so $[x,u]_{\sigma,\tau}\sigma(u) \in Z$ for all $x \in R, u \in U$. Thus, since $[x,u]_{\sigma,\tau} \in Z$ we get $[x,u]_{\sigma,\tau} = 0$ for all $x \in R, u \sigma(u) \in Z$. If $\sigma(u) \in Z$, then $u \in Z$. If $[x,u]_{\sigma,\tau} = 0$ for all $x \in R$, then replacing $x$ by $xy, y \in R$ we get $\sigma = R[R,\sigma(u)]$ and so by the primeness of $R$ we obtain $u \in Z$. Thus $U \subset Z$ is obtained.

Lemma Let $R$ be a prime ring with char $R \neq 2, U$ a nonzero $(\sigma, \tau)$-Lie ideal of $R$, $d$ a nonzero derivation of $R$. If $d(U) \subset Z$ then $U \subset Z$.

Proof. Since $d(U) \subset Z$, for all $x \in R, u \in U$ we have,

$$Z \ni d([x,u]_{\sigma,\tau}) = [d(x), u]_{\sigma,\tau} + [x, d(u)]_{\sigma,\tau}$$

(1)

In (1), if we replace $x$ by $xd(v), v \in U$ and using $d(U) \subset Z$ and (1) we get $[x,u]_{\sigma,\tau}d^2(v) \in Z$ for all $x \in R, u, v \in U$. Thus, since $d^2(v) \in Z$, for all $x \in R, u, v \in U$, we get $[x,u]_{\sigma,\tau} \in Z$ or $d^2(v) = 0$. That is $d^2(U) = 0$ or $[R,U]_{\sigma,\tau} \subset Z$. If $d^2(U) = 0$ then by [2, Theorem 2] we get $U \subset Z$. If $[R,U]_{\sigma,\tau} \subset Z$, then by lemma 1 we have $U \subset Z$.

Lemma Let $R$ be a prime ring with char $R \neq 2, 3$ U a nonzero $(\sigma, \tau)$-Lie ideal of $R$, $d$ a nonzero derivation of $R$ such that $d(U) \subset U, d^2(U) \subset Z$. If $d^3(U) = 0$ then $U \subset Z$.

Proof. Assume that $U \notin Z$. Since $[x,u]_{\sigma,\tau} \in U$, replacing $x$ by $\tau(u)x$ we have $\tau(u)[x,u]_{\sigma,\tau} \in U$. For all $x \in R$ and $u \in U, 0 = d^3(\tau(u)[x,u]_{\sigma,\tau}) = 3\tau(d^2(u))d([x,u]_{\sigma,\tau}) + \tau(d(u))d^2([x,u]_{\sigma,\tau}))$ and so,

$$\tau(d^2(u))d([x,u]_{\sigma,\tau}) + \tau(d(u))d^2([x,u]_{\sigma,\tau}) = 0$$

(2)

In (2), we replace $u$ by $d(u)$, then for all $x \in R, u \in U$ we get $0 = \tau(d^2(u))d^2([x,d(u)]_{\sigma,\tau})$. Thus, since $d^2(u) \in Z$ we have that

$$d^2(u) = 0$$

(3)

If $d^2([x,d(u)]_{\sigma,\tau}) = 0$ for all $x \in R$; replacing $x$ by $x\sigma(d(u))$ in the last equation and using char $R \neq 2$ we obtain

$$d([x,d(u)]_{\sigma,\tau}) = 0$$

for all $x \in R$. or $d^2(u) = 0$.

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If \( d([x, d(u)]_{\sigma, \tau}) = 0 \) for all \( x \in R \); then from the relation \( 0 = d([x \sigma(d(u)), d(u)]_{\sigma, \tau}) \) we have, \( [x, d(u)]_{\sigma, \tau} = 0 \) for all \( x \in R \) or \( d^2(u) = 0 \). Let \( [x, d(u)]_{\sigma, \tau} = 0 \), for all \( x \in R \). If we replace \( x \) by \( xy, y \in R \) then we see that \( d(u) \in Z \). Thus in the second case (3), we have \( d(u) \in Z \). Consequently by (3) we obtain that \( d^2(u) = 0 \) or \( d(u) \in Z \). Let \( K = \{ u \in U | d^2(u) = 0 \} \) and \( L = \{ u \in U | d(u) \in Z \} \). \( K \) and \( L \) are additive subgroups of \( U \) and \( U = K \cup L \). Since \( d \neq 0 \) and \( U \nsubseteq Z \), by [2, Theorem 2], \( U \neq K \). Thus, by Brauer Trick we have \( U = L \). But, if \( U = L \) then \( d(U) \subset Z \) and so \( U \subset Z \) by Lemma 2. This is a contradiction. Therefore we have \( U \subset Z \). \( \square \)

**Theorem** Let \( R \) be a prime ring with \( \text{char } R \neq 2, 3, U \) a nonzero \((\sigma, \tau)\)-Lie ideal of \( R \), \( d \) a nonzero derivation of \( R \) such that \( d(U) \subset U \) and \( d^2(U) \subset Z \) then \( U \subset Z \).

**Proof.**
Let \( d(U) \subset U \) and \( d^2(U) \subset Z \). Then, for all \( x \in R, u \in U \)

\[
Z \ni d^2([x, u]_{\sigma, \tau}) = [d^2(x), u]_{\sigma, \tau} + 2[d(x), d(u)]_{\sigma, \tau} + [x, d^2(u)]_{\sigma, \tau}
\]

(4)

In (4), replacing \( x \) by \( xd^2(v), v \in U \) and using \( d^3(U) \subset Z \) we get

\[
2d^3(v) d([x, u]_{\sigma, \tau}) + [x, u]_{\sigma, \tau} d^4(v) \in Z, \text{ for all } x \in R, u, v \in U
\]

(5)

In the last equation, if we take \( xd^2(w), w \in U \) instead of \( x \) and use that char \( R \neq 2 \), then we have \( d^3(v) d^3(w) [x, u]_{\sigma, \tau} \in Z \), for all \( x \in R, u, v, w \in U \). Since \( d^2(U) \subset Z \), thus we have \( d^3(U) = 0 \) or \( [R, U]_{\sigma, \tau} \subset Z \). If \( d^3(U) = 0 \), then \( U \subset Z \) by Lemma 3. If \( [R, U]_{\sigma, \tau} \subset Z \), then \( U \subset Z \), by Lemma 1. Consequently we obtain that \( U \subset Z \). \( \square \)

**References**


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TÜREVLİ ASAL HALKALARDA $(\sigma, \tau)$- LE İDEALLER

Özet

Bu makalede aşağıdaki sonuçlar ispatlanmıştır. $R$, char $R \neq 2, 3$ olan bir asal halka, $U, R$ nin sıfırdan farklı bir ideali, $\sigma$ ve $\tau$ $R$ nin iki otomorfizmi ve $0 \neq d : R \to R, d\sigma = \sigma d, d\tau = dt$ olacak şekilde $R$ nin bir türevi olsun 1) $Z$, $R$ nin merkezi olmak üzere $d(U) \subset Z$ ise $U \subset Z$. 2) $d(U) \subset U$ ve $d(U) \subset Z$ ise $U \subset Z$ dir.

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Received 28.11.1994