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Spin TQFT and the Birman-Craggs Homomorphisms

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Introduction

Let Σ_g be a closed oriented surface of genus g, and Γ_g be its *mapping class group.* Recall that the *Torelli group* $\mathcal{T}_g \subset \Gamma_g$ is the kernel of the natural homomorphism $\Gamma_g \to Aut(H_1(\Sigma_g;\mathbf{Z})) = \mathrm{Sp}(2g,\mathbf{Z})$. The classical *Rochlin* μ-invariant gives rise to homomorphisms from the Torelli group $\mathcal{T}_g$ to $\mathbf{Z}/2$ first studied by Birman and Craggs [8]. Johnson [10] showed that the distinct homomorphisms thus obtained are naturally indexed by the spin structures on Σ_g with Arf invariant zero; it follows that there are exactly $2^{g-1}(2^g+1)$ of them.

The theory of (Jones-Witten-Reshetikhin-Turaev-...) quantum invariants and Topological Quantum Field Theory (TQFT) gives rise to many new projective representations of the mapping class group Γ_g, which lift to linear representations of a certain central extension $\widetilde{\Gamma}_g$ of Γ_g. In a special case, the quantum invariant of a closed oriented 3-manifold M is the sum of suitably modified Rochlin invariants of the different spin structures on M. The representation of $\widetilde{\Gamma}_g$ corresponding to this invariant has rank $2^{g-1}(2^g+1)$. In her Ph.D. thesis, G. Wright [23] has shown by explicit computation that the restriction of that representation to the Torelli group can be diagonalized, and is isomorphic to the direct sum of the Birman-Craggs homomorphisms. (This uses implicitly the fact that the restriction of the extension $\widetilde{\Gamma}_g$ to the Torelli group is trivial.)

The aim of this note is to explain G. Wright's theorem from the point of view of TQFT. Her result is a special case of the *transfer theorem* of [7] which describes the relationship between the Spin TQFT functors V^s_{8k} constructed by Blanchet and the author [7] and the 'unspun' TQFT functors V_{8k} of Blanchet, Habegger, Vogel, and the author [6]. The case $k = 1$ of the transfer theorem contains G. Wright's theorem. (The fact that the representation can be diagonalized when restricted to the Torelli group also follows from the direct sum decomposition of $V_{8k}(\Sigma)$ obtained in [6], see remark 3.6 below.) The cases $k \geq 2$ imply results similar to hers for the representations $V_{8k}(\Sigma_g)$ (corresponding to the 'higher μ-invariants' constructed independently by Blanchet [4], Turaev [21] and Kirby-Melvin [14]).

Reference for this note is [4] for the construction of the invariants, [6] for the TQFT-functors V_p (see [18] for a short introduction), and [7] for the Spin TQFT-functors V^s_{8k} (see [17] for a short introduction).

1. Invariants for Spin 3-Manifolds

The Rochlin invariant. (See for example [13].) The classical *Rochlin* μ-invariant associates to any closed oriented 3-manifold with spin structure, M, an element $\mu(M) \in \mathbf{Z}/16$. It is defined by the formula

$$\mu(M) = \text{signature}(W) \mod 16$$

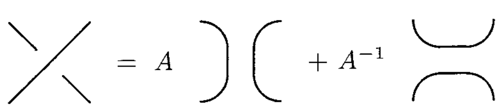
where W is a compact oriented 4-manifold with $\partial W = M$, admitting a spin structure inducing the given one on M. (This does not depend on the choice of W because of Rochlin's theorem.)

Remark 1.1. The name 'Rochlin invariant' is often used for a $\mathbf{Z}/2$-valued invariant of homology spheres, given by $M \mapsto \mu(M)/8 \in \mathbf{Z}/2$. (This makes sense because if M is a homology sphere, then the spin structure on M is unique, and the signature of a spin 4-manifold W as above is divisible by 8.)

The surgery picture. (See e.g. [13].) Assume that M is presented as the result of *surgery* on a framed link L in the 3-sphere S^3. Thus, $M = \partial W_L$ where W_L is the 4-ball B^4 with 2-handles added along the components L_i of $L \subset \partial B^4 = S^3$. The number signature$(W_L)$ is the signature of the linking matrix $(L_i \cdot L_j)$. (Here, as usual, $L_i \cdot L_i$ is the self-linking number of L_i, given by the framing). Then W_L is spin if and only if L is an *even* link (i.e. all $L_i \cdot L_i$ are even), and in this case the spin structure on W_L is unique. It is a fact that any spin structure on M arises as restriction of the unique spin structure on W_L for some even link L.

In the general situation, if a spin structure on M is given, the obstruction to extending it to W_L defines a sublink C of L, called the *characteristic* sublink. (This is because the obstruction lies in $H^2(W_L, M; \mathbf{Z}/2) \approx H_2(W_L; \mathbf{Z}/2)$ which is a $\mathbf{Z}/2$-vector space with basis vectors corresponding to the components of L.)

The Kauffman bracket [12]. The Kauffman bracket is an invariant for banded links[1] in the 3-sphere S^3, with values in the ring $\mathbf{Z}[A, A^{-1}]$ of Laurent polynomials in the indeterminate A. It is denoted by $L \mapsto \langle L \rangle$. (The Kauffman bracket is equivalent to Jones' original V-polynomial [11], and in the quantum group picture corresponds to the group $SU(2)$.) It is defined by the following skein relations[2]:



[1] A banded link is an embedding of a disjoint union of annuli $S^1 \times \mathbf{I}$. Alternatively, it is a framed link where one has forgotten the orientation of its core.

[2] We use the convention that any line is to represent a band parallel to the plane, with orientation compatible with that of the plane.

$$L \cup \bigcirc = (-A^2 - A^{-2})\, L$$

It is normalized so that the bracket of the empty link is 1.

(In the figures above, the brackets $\langle\ \rangle$ are omitted. The first equation means that a link, which locally (in a ball) is given by the left hand side, may be replaced by the linear combination given (locally) by the right hand side. The second equation means that a link having an un-knotted component, lying in a disk disjoint from the rest of the link, may be removed at the cost of the factor $-A^2 - A^{-2}$.)

Expressing the μ-invariant using the bracket. Let $A = \zeta_{16}$ be a primitive 16-th root of unity. It will be convenient to rewrite the Rochlin μ-invariant as follows: we put

$$\tilde{\mu}(M) = (-A^{-3})^{\mu(M)}$$

Observe that $\tilde{\mu}(M) = (-A^{-3})^{\mathrm{signature}(W)}$ where W is as in the definition of $\mu(M)$.

Proposition 1.2. *Let M be presented by the framed link L in S^3, and let the spin structure on M be given by the characteristic sublink C. Let $A = \zeta_{16}$. Then*

$$\tilde{\mu}(M) = (-A^{-3})^{\mathrm{signature}(W_L)} \frac{\langle C \rangle}{\delta^{\sharp C}}$$

where $\delta = -A^2 - A^{-2}$ and $\sharp C$ is the number of components of C.

Note that if C is empty, the formula holds by definition. Thus it suffices to show that the right hand side is an invariant of spin 3-manifolds. This can be done using the Kirby calculus and elementary properties of the Kauffman bracket. (See [4].)

The reason for this rewriting of the μ-invariant is that the formulation above was generalized by Blanchet to obtain 'higher μ-invariants'. To describe this, we need the concept of *cabling.*

Cabling. Let L be a framed link in an oriented 3-manifold M, and let K be a framed link in the standard solid torus. The result of cabling L by K, is a new link, denoted by $L(K)$, obtained as follows. For each component of L, we take out that component and replace it by K. (This makes sense, since the framing gives an identification of a tubular neighborhood of the component with the standard solid torus.)

For example, if $K = \emptyset$ is the empty link, then so is $L(K)$, for all L. Let us denote by z the link given by the *core* of the solid torus, then $L(z) = L$.

In this language, the formula of prop. 1.2 may be rewritten as follows:

$$\tilde{\mu}(M) = (-A^{-3})^{\mathrm{signature}(W_L)} \langle C(\omega_1) \cup (L - C)(\omega_0) \rangle$$

with $\omega_0 = \emptyset$, and $\omega_1 = \delta^{-1} z$.

Note that $\delta^{-1} z$ is not a link, but a formal multiple of one. Implicitly, we are using here the notion of *skein modules.* Let us make this explicit. The (Jones-Kauffman) skein module $\mathcal{K}(M)$ is the $\mathbf{Z}[A, A^{-1}]$-module freely generated by the set of isotopy classes of banded links in M, divided by the Kauffman skein relations described above. Elements of $\mathcal{K}(M)$ are called *skein elements* of M. For example, the skein module of the 3-sphere is free of rank one (this is just a rephrasing of the fact that the bracket is well defined). Another example is the skein module of the solid torus, which is free of infinite rank, a basis being given by the z^n, $n \geq 0$, where z^n means n parallel copies of z.

Given a framed link L in S^3, we may cable it by any skein element of the solid torus. The result will be a skein element of S^3, so that its bracket is well defined. For example, the bracket of $L(\delta^{-1}z)$ is $\delta^{-\sharp(L)}$ times the bracket of L.

With this preparation, here is Blanchet's theorem:

Theorem 1.3 ([4]). *For each root of unity ζ of order not congruent to 8 mod 16, one can find skein elements ω_0, ω_1 in the solid torus and a number λ such that*

$$\lambda^{\mathrm{signature}(W_L)} \langle C(\omega_1) \cup (L - C)(\omega_0) \rangle \Big|_{A=\zeta}$$

is an invariant of the 3−manifold with spin structure presented by the framed link L and characteristic sublink C.

Remark 1.4. 1) These invariants are strictly stronger than the Rochlin μ-invariant. For example, the two spin structures on the lens space $L(16, 1)$ have the same μ-invariant, but are distinguished by the above invariant at a 32-th root of unity.

2) Given a closed oriented 3-manifold M, one may associate to it an 'unspun' invariant by taking the sum over all spin structures on M of the spin invariants given by theorem 1.3 above. If the order of the root of unity ζ is divisible by 16, it turns out that this sum may be written

$$\lambda^{\mathrm{signature}(W_L)} \langle L(\omega) \rangle \Big|_{A=\zeta}$$

where $\omega = \omega_0 + \omega_1$. From this, one sees that this unspun invariant is a Witten-Reshetikhin-Turaev-invariant [22, 20], in the form obtained by Lickorish [15, 16] and Blanchet, Habegger, Vogel, and the author [5], using the Kauffman bracket. Conversely, Turaev [21] and Kirby-Melvin [14] have observed that at the right roots of unity, their invariants constructed using the quantum group approach decompose as a sum of invariants for closed 3-manifolds with spin structure.

2. The Functors V and V^s

The abbreviation 'TQFT' is short for *Topological Quantum Field Theory.* This terminology is used to describe certain gluing properties of certain invariants of 3-manifolds. It was introduced by Atiyah [1] following Witten's [22] interpretation, in terms of quantum field theory, of the Jones polynomial invariant of links in the 3-sphere.

In the TQFT language, gluing properties are described in terms of functors on cobordism categories. A *cobordism category* is a category together with an empty object $\emptyset$ and a notion of *disjoint union* (denoted by $\amalg$), *orientation reversal* (denoted by a minus sign), and *boundary* (denoted by ∂). These have to satisfy certain obvious axioms which are abstracted from the basic example where objects are closed n-dimensional manifolds, and morphisms are diffeomorphism classes (rel boundary) of $(n+1)$-dimensional cobordisms between them.[3]

In many interesting examples of cobordism categories, the objects and morphisms are manifolds equipped with some kind of extra *structure*. Here, we take $n = 2$, and we shall consider a spin cobordism category $\mathcal{C}^s$, and an 'unspun' cobordism category $\mathcal{C}$, with a forgetful functor from $\mathcal{C}^s$ to $\mathcal{C}$. The extra structure on objects of $\mathcal{C}$ is a *p_1-structure* [6], and (possibly) a colored banded link. Objects of $\mathcal{C}^s$ have in addition a *w_2-structure* (a precise version of spin structure, used to make gluing well defined).

The notions of w_2- and p_1-structure are defined as follows (see [7] for more details). A spin structure on a manifold N may be viewed as a homotopy class of lifts of the stable tangent bundle $\tau_N : N \to BSO$ to $BSpin$. An actual lift is called a w_2-structure. Consider the space X_{p_1} which is the homotopy fiber of the map $BSO \longrightarrow K(\mathbf{Z}, 4)$ corresponding to the first Pontryagin class p_1. A lift of the stable tangent bundle to X_{p_1} is called a p_1-structure.

The structure on morphisms (i.e. cobordisms) of $\mathcal{C}$ is a *relative p_1-structure,* i.e. a homotopy class of p_1-structures relative to the given p_1-structures on the ends. On cobordisms of $\mathcal{C}^s$, one has in addition a *relative spin structure,* i.e. a homotopy class of w_2-structures relative to the given w_2-structures on the ends.

(There may also be colored banded links or graphs but we'll suppress them from the discussion, as we don't need them explicitly in this paper.)

Remark 2.1. These additional structures lead naturally to *central extensions* of mapping class groups. These extensions come about as follows. Let Σ be a closed oriented surface, and $\Gamma = \pi_0(\mathrm{Diff}^+(\Sigma))$ be its mapping class group. Given $f \in \Gamma$, its mapping cylinder C_f is a cobordism from Σ to itself.[4] By fixing a p_1-structure on Σ, one obtains an object of $\mathcal{C}$. But there is a $\mathbf{Z}$ worth of choice of a relative p_1-structure on each component of the mapping cylinder C_f. Thus, there is an extended mapping class group $\widetilde{\Gamma}$ which is a central extension of Γ by $H^0(\Sigma; \mathbf{Z})$.

By fixing, in addition to the p_1-structure, a w_2-structure σ on Σ, one obtains an object of $\mathcal{C}^s$. Denote by $s = [\sigma]$ the induced spin structure (i.e. the homotopy class of σ), and let Γ_s be the subgroup of Γ fixing s. The mapping cylinder C_f admits a relative spin structure (relative to σ on both ends) if and only if $f \in \Gamma_s$, and in this case, the relative spin structures are parametrized by $H^0(\Sigma; \mathbf{Z}/2)$. Thus, there is an extended spin mapping class group which is a central extension of Γ_s by $H^0(\Sigma; \mathbf{Z}) \oplus H^0(\Sigma; \mathbf{Z}/2)$.

Recall that a cobordism category has a particular object: the empty manifold $\emptyset$. The morphisms from $\emptyset$ to itself are called closed bordisms. Fix $k \geq 1$ and let $A = \zeta_{16k}$ be a primitive $16k$-th root of unity. In [6], the invariant of [5] is modified to obtain an

[3] Taking equivalence classes ensures that composition is associative, and the product manifold, $[0,1] \times \Sigma$, plays the role of the identity morphism of Σ.

[4] For simplicity, we ignore here the distinction between a diffeomorphism and its isotopy class.

invariant, denoted by $< >$, of closed bordisms in $\mathcal{C}$. In [7], the invariant of [4] is modified to obtain an invariant, denoted by $< >^{spin}$, of closed bordisms in $\mathcal{C}^s$. The relationship between the two invariants is given by the formula

$$(1) \qquad < M >= 2^{-b_0(M)} \sum_{\widetilde{\sigma} \in Spin(M)} < (M, \widetilde{\sigma}) >^{spin}$$

(see remark 1.4.2).) Here, $Spin(M)$ is the set of spin structures on M, and $b_0(M)$ is the number of components of M.

These invariants take values in a certain ring Λ; one may take $\Lambda = \mathbf{C}$.

In [6], a *universal construction* is applied to the invariant $< >$. It produces a functor V from $\mathcal{C}$ to Λ-modules such that $V(\emptyset) = \Lambda$, and the image of a closed bordism $M \in \mathrm{Hom}_{\mathcal{C}}(\emptyset, \emptyset)$ is the endomorphism of $V(\emptyset) = \Lambda$ given by multiplication by $< M >$. Moreover, any M with $\partial M = \Sigma$ defines an element $Z(M)$ of $V(\Sigma)$. Similarly, in [7] the universal construction is applied to the invariant $< >^{spin}$ and produces a functor V^s from $\mathcal{C}^s$ to Λ-modules such that $V^s(\emptyset) = \Lambda$, and the image of a closed bordism is multiplication by its $< >^{spin}$-invariant. Moreover, any M with $\partial M = \Sigma$ (in the spin sense) defines an element $Z^s(M)$ of $V^s(\Sigma)$.

Remark 2.2. The elements $Z(M)$ (where $\partial M = \Sigma$) generate $V(\Sigma)$. Moreover, a linear combination $\sum \lambda_i Z(M_i)$ is zero in $V(\Sigma)$ if and only if $\sum \lambda_i < M_i \underset{\Sigma}{\cup} (-M') >$ is zero for all M' with $\partial M' = \Sigma$. (The same holds for V^s. In fact, these two properties are more or less the definition of the universal construction.)

Remark 2.3. In [7], the notations $< >$, $< >^{spin}$, V, V^s, are sometimes affected with a subscript $8k$, to indicate the dependence on k. We suppress this except when considering specifically the 'Rochlin' case $k = 1$, where everything has a subscript 8.

The functors V and V^s are quantization functors in the sense of [6]. By construction, they describe gluing properties of the invariants $< >$ and$< >^{spin}$. The main result about them is the following finiteness result:

Theorem 2.4 ([6]). *For all objects Σ of $\mathcal{C}$, the modules $V(\Sigma)$ are free of finite rank.*

Theorem 2.5 ([7]). *For all objects Σ of $\mathcal{C}^s$, the modules $V^s(\Sigma)$ are free of finite rank.*

Remark 2.6. 1) The ranks of the modules $V(\Sigma)$ are given by (a version of) the *Verlinde formula.* (See [6].)

2) Explicit formulae for the ranks of the modules $V^s(\Sigma)$ were computed in [7]. In the 'Rochlin' case (i.e. in the case of the 16-th root of unity), the rank is one, if the spin structure on Σ has Arf invariant zero, and the rank is zero otherwise.

3) The modules $V(\Sigma)$ satisfy a tensor product formula for disjoint unions, and the functor V is a TQFT-functor in the sense of [6]. The modules $V^s(\Sigma)$ satisfy a more general formula for disjoint unions. The functor V^s may be embedded into a $\mathbf{Z}/2$-graded TQFT-functor, see [7] or [17].

3. The Transfer Theorem

The *transfer theorem* [7] describes the relationship between the Spin TQFT functor V^s and the 'unspun' TQFT functor V. The question is: how to generalize the sum formula (1) on the level of TQFT-functors? A partial answer was given already in section 7 of [6] where it was shown that the V-module of a surface is naturally decomposed as an orthogonal direct sum of submodules associated to spin structures on the surface[5]:

$$V(\Sigma) = \bigoplus_{s \in Spin(\Sigma)} V(\Sigma, s) \tag{2}$$

Thus, one might expect that the unspun functor is the direct sum of the spin functors (the sum being over the different spin structures on the surface). However, this is not the case. Rather, the unspun theory is the direct sum of the 'zero graded parts' of the spin theories.

Let Σ be an object of $\mathcal{C}$, and let σ be a w_2-structure on Σ so that (Σ, σ) is an object of $\mathcal{C}^s$. It turns out that the module $V^s(\Sigma, \sigma)$ is naturally graded by the reduced homology group $\widetilde{H}_0(\Sigma; \mathbf{Z}/2)$. The submodule corresponding to grading zero is denoted by

$$V^s((\Sigma, \sigma), 0) \subset V^s(\Sigma, \sigma)$$

Here is a (not quite complete) statement of the transfer theorem.

Theorem 3.1 (Transfer theorem). a) *Let Σ be an object of $\mathcal{C}$, and let σ be a w_2-structure on Σ. There exists a well defined linear map*

$$\Phi_\sigma : V(\Sigma) \longrightarrow V^s(\Sigma, \sigma)$$

with the following property. For every bordism M in $\mathcal{C}$ such that $\partial M = \Sigma$

$$\Phi_\sigma(Z(M)) = 2^{-b_0(M,\Sigma)} \sum_{\tilde{\sigma} \in Spin(M|\sigma)} Z^s(M, \tilde{\sigma}) \tag{3}$$

where $Spin(M|\sigma)$ is the set of relative spin structures on M extending σ.

b) *Let $s = [\sigma]$ denote the induced spin structure. The morphism Φ_σ factors through the orthogonal projection $\pi_s : V(\Sigma) \to V(\Sigma, s)$, and induces an isomorphism*

$$\Phi_\sigma : V(\Sigma, s) \xrightarrow{\sim} V^s((\Sigma, \sigma), 0)$$

Thus, formula (2) expresses the unspun theory essentially as the sum of the 'zero graded parts' of the spin theories.

Remark 3.2. Observe that if Σ is a *connected* surface, i.e. $\widetilde{H}_0(\Sigma; \mathbf{Z}/2) = 0$, then the unspun theory of Σ is indeed the sum of the spin theories.

[5] One makes use here of the correspondence between spin structures on surfaces and quadratic forms on mod 2 homology (Atiyah [2], Johnson [9]).

Let us spell out what the transfer theorem means for the representations of mapping class groups. We recall how the extended mapping class group $\widetilde{\Gamma}$ acts on $V(\Sigma)$. Let an element $\tilde{f} \in \widetilde{\Gamma}$ be represented by a mapping cylinder C_f. Here $f \in \Gamma$, and C_f is equipped with a relative p_1-structure which is suppressed from the notation. (See remark 2.1.) The action of $\widetilde{\Gamma}$ on $V(\Sigma)$ is given by

$$Z(M) \mapsto \rho(\tilde{f})(Z(M)) = Z(M \underset{\Sigma}{\cup} C_f)$$

The action of the extended spin mapping class group is defined in the same way.

Recall that Γ_s denotes the subgroup of Γ fixing the spin structure s. Let $\widetilde{\Gamma}_s$ be the restriction of $\widetilde{\Gamma}$ to Γ_s. The extended spin mapping class group is a central extension of $\widetilde{\Gamma}_s$ by $H^0(\Sigma; \mathbf{Z}/2)$. (Elements of the kernel of the extension are represented geometrically by the distinct relative spin structures on the identity cobordism $\Sigma \times \mathbf{I}$ extending the w_2-structure σ on $\Sigma \times \partial\mathbf{I}$.) The submodule $V^s((\Sigma,\sigma),0)$ of $V^s(\Sigma,\sigma)$ is defined by the condition that the kernel $H^0(\Sigma; \mathbf{Z}/2)$ act trivially. Thus, there is a well-defined action ρ^{spin} of $\widetilde{\Gamma}_s$ on $V^s((\Sigma,\sigma),0)$. It is given by the assignment

$$Z^s(\sum \lambda_i Z^s(M_i,\tilde{\sigma}_i)) \mapsto \rho^{spin}(\tilde{f})(Z^s(\sum \lambda_i Z^s(M_i,\tilde{\sigma}_i))) = \sum \lambda_i Z^s((M_i,\tilde{\sigma}_i) \underset{\Sigma}{\cup} C_f)$$

Remark 3.3. One should think of an element of $V^s((\Sigma,\sigma),0)$ as some kind of 'average'. In particular, this module need not be generated by elements of the form $Z^s(M,\tilde{\sigma})$. This is why we consider elements of the form $\sum \lambda_i Z^s(M_i,\tilde{\sigma}_i)$ in the formula above. The new relative spin structure on $(M_i,\tilde{\sigma}_i) \underset{\Sigma}{\cup} C_f$ is obtained by gluing with *any* relative spin structure on C_f. (Of course, one must choose the same one for all M_i.)

Observe that if $\partial M = \Sigma$, there is a canonical bijection

$$Spin(M|\sigma) \approx Spin(M \underset{\Sigma}{\cup} C_f|\sigma)$$

Thus, formula (3) in the transfer theorem implies the following

Corollary 3.4. *The representation ρ of $\widetilde{\Gamma}$ on $V(\Sigma)$ has the following property: The subgroup $\widetilde{\Gamma}_s \subset \widetilde{\Gamma}$ leaves the submodule $V(\Sigma,s)$ of $V(\Sigma)$ fixed, and the isomorphism $V(\Sigma,s) \approx V^s((\Sigma,\sigma),0)$ in part* b) *of the transfer theorem is an isomorphism of representations of $\widetilde{\Gamma}_s$.*

Observe that Γ_s contains the Torelli group $\mathcal{T}$. Let $\widetilde{\mathcal{T}}$ be the restriction of $\widetilde{\Gamma}_s$ to $\mathcal{T}$.

Corollary 3.5. *The restriction of the representation ρ of $\widetilde{\Gamma}$ on $V(\Sigma)$ to the extended Torelli group $\widetilde{\mathcal{T}}$ is block diagonal (with respect to the direct sum decomposition (2)).*

Remark 3.6. This result may also be deduced directly from section 7 of [6]. From the definition of the submodules $V(\Sigma,s)$ it is clear that $\widetilde{\Gamma}_s$ commutes with the orthogonal projector onto $V(\Sigma,s)$. It follows that $\widetilde{\Gamma}_s$ leaves $V(\Sigma,s)$ fixed, whence corollary 3.5.

4. The Birman-Craggs-Homomorphisms

In what follows we shall deduce G. Wright's theorem from corollary 3.4. We specialize to the case $k = 1$, and assume Σ is a *connected* surface of genus g. Then the module $V_8(\Sigma, s)$ is free of rank one, if s has Arf invariant zero, and $V_8(\Sigma, s)$ is zero otherwise [6]. Corollary 3.5 implies the following

Corollary 4.1. *In the 'Rochlin case' (i.e. $k = 1$), the restriction of the representation ρ of $\widetilde{\Gamma}$ on $V_8(\Sigma)$ to the extended Torelli group $\widetilde{\mathcal{T}}$ is diagonal.*

For a spin structure s with Arf invariant zero, let us denote by $\tilde{r}_s$ the homomorphism $\widetilde{\mathcal{T}} \to Aut(V_8(\Sigma, s)) = \Lambda^\star$. Let r_s be the Birman-Craggs-homomorphism associated to s (we shall recall its definition below).

It is well known that the extension $\widetilde{\mathcal{T}}$ is trivial, i.e. $\widetilde{\mathcal{T}} \approx \mathcal{T} \times \mathbf{Z}$. (This is because the extension $\widetilde{\Gamma}$ is a pullback of an extension of the symplectic group $\mathrm{Sp}(2g, \mathbf{Z})$.[6])

Proposition 4.2. *There is a 'natural' section $j : \mathcal{T} \to \widetilde{\mathcal{T}}$ such that $r_s = \tilde{r}_s \circ j$.*

Corollary 4.3 (G. Wright[23]). *The representation $\rho \circ j$ of the Torelli group on $V_8(\Sigma)$ is isomorphic to the direct sum of the Birman-Craggs homomorphisms.*

Remark 4.4. G. Wright proves this by explicit computation of the representation on generators of the Torelli group.

By work of Johnson [10], the Birman-Craggs homomorphism r_s can be defined as follows. Since s has Arf invariant zero, there is a Heegaard embedding h of Σ into the 3-sphere S^3 such that s is induced from the unique spin structure on S^3. Let $f \in \mathcal{T}$. Inserting the mapping cylinder C_f between the two complementary handlebodies produces a homology sphere $M(h, f)$. Then

$$r_s(f) \;=\; \tilde{\mu}(M(h, f))$$

(Note that $\tilde{\mu}(M(h, f))$ is well defined and lies in $\{\pm 1\}$, since the homology sphere $M(h, f)$ has a unique spin structure (see remark 1.1).)

The homomorphisms $\tilde{r}_s$ can be expressed in a similar way, as follows. Consider an element $\tilde{f}$ of $\widetilde{\mathcal{T}}$ represented by a mapping cylinder C_f (as before, C_f is equipped with a relative p_1-structure which is suppressed from the notation). Pick a w_2-structure σ on Σ representing s. Since $V_8^s(\Sigma, \sigma)$ has rank one, $Z_8^s(C_f)$ acts on $V_8^s(\Sigma, \sigma)$ as a multiple of the identity; by corollary 3.4, this multiple is precisely $\tilde{r}_s(\tilde{f})$. By a standard TQFT

[6] The class of the extension $\widetilde{\Gamma}$ is three times the class of the signature extension. See e.g. [19].

argument, we can compute this from any two spin bordisms M_1, M_2 with boundary Σ such that $< M_1 \underset{\Sigma}{\cup} (-M_2) >_8^{spin}$ is non-zero, in the following way:

$$\tilde{r}_s(\tilde{f}) = \frac{< M_1 \underset{\Sigma}{\cup} C_f \underset{\Sigma}{\cup} (-M_2) >_8^{spin}}{< M_1 \underset{\Sigma}{\cup} (-M_2) >_8^{spin}}$$

We take for M_1, M_2 the two complementary handlebodies of the Heegaard embedding h considered above. Since $< S^3 >_8^{spin} = 1$, it follows that

$$\tilde{r}_s(\tilde{f}) = < M(h,f) >_8^{spin}$$

Now the invariant $< >_8^{spin}$ is almost the Rochlin invariant $\tilde{\mu}$. The precise relationship is[7]

$$< M >_8^{spin} = (-A)^{\sigma(M)} \tilde{\mu}(M) \tag{4}$$

Here, $\sigma(M) \in \mathbf{Z}$ is the invariant of closed 3-manifolds with p_1-structure defined by

$$\sigma(M) = 3 \text{ signature } W - p_1(W, M)$$

where $\partial W = M$ and $p_1(W, M) \in \mathbf{Z}$ is the obstruction to extending the p_1-structure on M to W, evaluated on the fundamental class of (W, M). (See [6]. A similar invariant was considered by Atiyah [3] in the language of 2-framings.[8])

Given $f \in \mathcal{T}$, we define $j(f) \in \widetilde{\mathcal{T}}$ to be the element represented by the mapping cylinder C_f equipped with a relative p_1-structure such that

$$\sigma(M(h,f)) = 0$$

Then formula (4) yields $r_s = \tilde{r}_s \circ j$, as asserted.

Remark 4.5. One can show that the section $j : \mathcal{T} \to \widetilde{\mathcal{T}}$ is a group homomorphism. This provides a pleasant explanation, from the TQFT viewpoint, for the fact that the assignment $f \mapsto r_s(f) = \tilde{\mu}(M(h,f))$ is indeed a homomorphism when restricted to the Torelli group.

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[7]The formula in [7] contains numbers η_s and κ. They are determined by the choice of κ which is subject to the condition $\kappa^6 = A^{-6-4k}$. Here, we have made the special choice $\kappa = -A$, which implies $\eta_s = 1$.

[8]The relationship is described in section 4 of [19].

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