Recursive Relations for the Cohomology Ring of Moduli Spaces of Stable Bundles

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The moduli spaces under consideration basically arise in two different ways: Algebraically as spaces of representations \( \rho : \pi_1(S_g \setminus \{P\}) \to \text{SU}(2), \rho(\prod_i A_i B_i A_i^{-1} B_i^{-1}) = -I \) modulo conjugation by \( \text{SU}(2) \). Here \( S_g \) is a closed oriented surface of genus \( g \) (\( g \geq 2 \) in the sequel), and \( A_i, B_i \) \((i = 1, \ldots, g)\) are canonical generators for the fundamental group of \( S_g \setminus \{P\} \) (a free group on \( 2g \) generators: \( \prod_i A_i B_i A_i^{-1} B_i^{-1} \) represents a simple loop around the removed point \( P \in S_g \)). Or geometrically as spaces of (isomorphism classes of) stable bundles \( E \) over a Riemann surface \( \Sigma \) of genus \( g \) with fixed determinant bundle \( L = \det E \) of odd degree. It was an observation of Mumford [Mu] that in the latter instance the moduli space is in a natural way a non-singular projective variety (of dimension \( 3g - 3 \)). Let us denote this variety by \( \mathcal{M}(\Sigma, L) \) to indicate its dependence on the particular choices of \( \Sigma \) and \( L \). The space of representations on the other hand depends only on \( g \) and has the structure of a differentiable manifold which we denote by \( \mathcal{N}_g \) [NaSe]. Then the celebrated theorem of Narasimhan and Seshadri says that \( \mathcal{N}_g \) is the manifold underlying \( \mathcal{M}(\Sigma, L) \) for any \( \Sigma, L \) [NaSe].

We are dealing here with the simplest non-trivial moduli spaces of stable bundles (they are non-singular, complete, Fano, with Picard group infinite cyclic of rank 1) and quite a bit is known about their cohomology. For example there are several ways of computing the Betti-numbers [Ne1], [HaNa], [AtBo], [Ki1], the cohomology is known to be torsion-free [AtBo] and explicit generators for the integral ring have been found [Ne1], [AtBo]. More recently Thaddeus [Th] found a beautiful formula for the highest intersection products relating to the Verlinde formula. Around the same time Kirwan succeeded in proving Mumford's conjecture that a certain (huge) canonical set of relations for the cohomology ring is complete [Ki2].

What is unsatisfactory about the picture is that still a big computational effort is required to find generators for the relations explicitly. And already low genus, say \( g = 5 \), demands the use of computers which in turn will not reach beyond \( g = 15 \) or so.

The content of this paper remedies this situation and should be good news for everybody interested in \( \mathcal{N}_g \): The basic part of \( H^*(\mathcal{N}_g) \) (the subring generated by Newstead's classes \( \alpha, \beta, \gamma \)) is a complete intersection ring and there is a very simple inductive formula for generators of the relation ideal! Namely,
Theorem 0.1. \( \langle \alpha, \beta, \gamma \rangle = Q[\alpha, \beta, \gamma]/(f_1^g, f_2^g, f_3^g) \) with
\[
\begin{align*}
  f_1^{g+1} &= \alpha f_1^g + g f_2^g, \\
  f_2^{g+1} &= \beta f_1^g + \frac{2g}{g+1} f_3^g, \\
  f_3^{g+1} &= \gamma f_1^g
\end{align*}
\]
for \( g \geq 1 \) and formally setting \((f_1^1, f_2^1, f_3^1) = (\alpha, \beta, \gamma)\). The \( f_i^g \) are uniquely determined by their respective initial terms \( \alpha^0, \alpha^{g-1}\beta \) and \( \alpha^{g-1}\gamma \).

As a consequence the monomials \( \alpha^a \beta^b \gamma^c \) with \( a + b + c < g \) form a \( Q \)-basis for this subring. Moreover, there is a generating function \( \Phi(t) \) for generators of all the relation ideals, i.e. simultaneously for all \( g \), s.t. \((f_1^g, f_2^g, f_3^g) = (\Phi^{(g)}, \Phi^{(g+1)}, \Phi^{(g+2)})\) as ideals (\( \Phi^{(r)} \) the \( r \)-th derivative at \( t = 0 \)). \( \Phi \) is characterized by the simple (formal) functional equation
\[
\Phi'(t) = \frac{\alpha + \beta t + 2\gamma t^2}{1 - \beta t^2} \cdot \Phi(t).
\]
For an explicit expression of \( \Phi \), cf. Definition 2.4.

Besides its aesthetical appeal, why is this exciting? First of all our approach is technically simpler and yields even more insight into the cohomology ring than any other approach: The only input we have to use is the embedding \( \varphi : M(\Sigma, L) \to G(g + 3, 2g + 2) \) of Desale and Ramanan (\( \Sigma \) hyperelliptic) together with an intrinsic characterization of the pull-back of the tautological bundle [DeRa]; and the total dimension of \( H^{2*}(\mathcal{N}_g) \) for which various methods of computation are available. From the recursion relations one actually should be able to derive without much additional effort more or less everything that is known on the cohomology of \( \mathcal{N}_g \), e.g. the vanishing of the Pontryagin ring and the Chern classes of \( \mathcal{N}_g \) above degree \( 4g - 4 \) [Ki2], [Th], [Gi].

Our method should also be applicable to moduli spaces of bundles of even degree, once the technical problem of non-existence of a universal bundle in the literal sense is overcome [Ra]. Furthermore, there are chances that the method of proving completeness of certain relations by an explicit computation of the Leitideal generalizes to moduli spaces of higher rank bundles (where none of the other methods apply at the moment).

Recently the cohomology of \( \mathcal{N}_g \) occurred as instanton Floer homology on the three-manifold \( S^1 \times S_g \) [DoSa]. It might well be that there are applications of our formulas to this circle of ideas or even to Donaldson invariants via gluing formulas [BrDo].

Finally, and this was the main motivation for the authors to search for a minimal set of relations, finding a good presentation for the cohomology ring is the first essential step in the computation of the quantum cohomology of \( \mathcal{N}_g \), cf. [SiTi]. The latter should compute the so-called Fukaya-Floer homology of a trivial product \( S^1 \times S_g \), cf. [Do]. Classically there follows at least a new integral formula for the highest intersection products on \( \mathcal{N}_g \), hence for Bernoulli numbers by the formula of [Th], cf. [SiTi].

After this paper had been finished and put on the electronic bulletin board alg-geom the authors learnt of work of several other mathematicians proving similar results [Za], [Ba], [BaKiNe], [KiNe]. There preprints are all not available at the moment, but we can say the following: V. Baranovsky also uses the Desale-Ramanan model to get relations,
while D. Zagier starts with the original Mumford relations of minimal degree. V. Balaji, A. King and P. E. Newstead in turn set up the recursion relations in their form from a geometric argument and compute the coefficients by studying explicit families of bundles. Previously, P. E. Newstead computed the relations modulo $\beta$ [Ne3].

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1. Method and Notation

Let us now fix a Riemann surface $\Sigma$ of genus $g$ and a line bundle $L$ on $\Sigma$ of odd degree. We write $\mathcal{M}_g = \mathcal{M}_g(\Sigma, L)$. Generators for $H^*(\mathcal{M}_g, Q)$ occur as coefficients in the Künneth decomposition of a characteristic class associated to the universal bundle $\mathcal{U}$ over $\mathcal{M}_g \times \Sigma$

$$c_2(\text{End} \mathcal{U}) = -\beta + 4\psi + 2\alpha \otimes \omega.$$ 

Here $\omega \in H^2(\Sigma, Z)$ is the normalized volume form and $\psi$ is the part of type $(3, 1)$. Choosing a standard basis $e_i$ for $H^1(\Sigma, Z)$, $i = 1, \ldots , 2g$ (s.th. $e_i e_j = 0$ unless $i \equiv j \mod g$ and $e_i e_{i+g}[\Sigma] = 1$), $\psi$ decomposes further $\psi = \sum_{i=1}^{2g} \psi_i \otimes e_i$. The classes $\alpha \in H^2(\mathcal{M}_g, Q)$, $\psi_i \in H^3(\mathcal{M}_g, Q)$ and $\beta \in H^4(\mathcal{M}_g, Q)$ are actually integral and generate the rational cohomology ring [Ne1], [AtBo].

There is an interesting subring of $H^*(\mathcal{N}_g, Q)$ to which the intersection pairing may easily be reduced by geometric arguments as noted by Thaddeus [Th]. Namely, any orientation preserving diffeomorphism of $S_g$ (wlog. leaving $P \in S_g$ fixed) induces a diffeomorphism of $\mathcal{N}_g$ by acting on $\pi_1$. The corresponding action on $H^*(\mathcal{M}_g, Q)$ leaves $c_2(\text{End} \mathcal{U})$ fixed. Thus $\alpha$ and $\beta$ are invariant and the $\psi_i$ transform dually to the $e_i$. It is not hard to show that the ring of such transformations is precisely the subring generated by $\alpha$, $\beta$ and a newly defined class $\gamma := 2 \sum_{i=1}^{2g} \psi_i \psi_{i+g}$, or more intrinsically $\psi^2 = \gamma \omega$ (in view of the functional equation for the generating function $\Phi$ it might be more natural to take twice this class, but to be consistent with the work of Newstead and Thaddeus we keep this definition).

On the other hand, there is a method introduced by Mumford to construct relations among the generators, cf. [AtBo]: Letting $L$ vary among the line bundles of fixed (odd) degree $D$ (the space of which we denote by $\text{Pic}^D(\Sigma)$), one gets a moduli space $\mathcal{M}(\Sigma)$ and a fibration $\mathcal{M}(\Sigma) \to \text{Pic}^D(\Sigma)$ with fibre $\mathcal{N}_g$. In rational cohomology this fibration is trivial, i.e. $H^*(\mathcal{M}(\Sigma), Q) \simeq H^*(\mathcal{N}_g, Q) \otimes Q H^*(\text{Pic}^D(\Sigma), Q)$. But $\text{Pic}^D(\Sigma) \simeq \text{Pic}^0(\Sigma)$ and $H^*(\text{Pic}^0(\Sigma), Q)$ is a free alternating algebra in $2g$ generators $\delta_i$ of degree 1 (the push-forwards of $e_i$ under the Jacobi map $\Sigma \to \text{Pic}^0(\Sigma)$). Now let $\mathcal{U}$ be the universal bundle over $\mathcal{M}_g \times \Sigma$ and let $\pi : \mathcal{M}_g \times \Sigma \to \mathcal{M}_g$ be the projection. If $D = 4g - 3$, $R^1\pi_*\mathcal{U} = 0$
and \( \tilde{\pi}_* \mathcal{U} \) is locally free of rank \( 2g - 1 \). By Grothendieck-Riemann-Roch then the Chern classes of \( \tilde{\pi}_* \mathcal{U} = \tilde{\pi}_1 \mathcal{U} \) are expressed as polynomials in \( \alpha, \beta, \psi_i \) and \( \tilde{e}_1 \). Equating to zero the coefficients of \( \tilde{e}_1, \ldots, \tilde{e}_r \) in \( c_r(\tilde{\pi}_* \mathcal{U}) \), \( r \geq 2g \) (which vanish for rank reasons) gives a number of relations among the generators \( \alpha, \beta \) and \( \psi_i \). The point of letting \( L \) vary is of course to lower the degrees of the relations by up to \( 2g \). The smallest degree of a relation we thus obtain is \( 4g - 2g = 2g \).

The conjecture of Mumford which Kirwan recently succeeded to prove as remarked in the introduction, is that this set of relations is complete. But in view of the explicit formula for the intersection pairing of Thaddeus the authors could not believe in this end of the story. In fact, computer evidence (up to \( g = 18 \) using “Macaulay” [BaSt]) showed that for the subring generated by \( \alpha, \beta, \gamma \) there should be only three independent relations of degrees \( g, g + 1, g + 2 \) respectively, coming from the lowest degree equation \( c_{2g} = 0 \). Unfortunately, explicit computations are rather arduous, e.g. due to the presence of the odd degree classes \( \tilde{e}_i \).

To find relations of low degrees without bringing \( \text{Pic}(\Sigma) \) into the game we first remark that because the \( \mathcal{M}(\Sigma, L) \) are all diffeomorphic as long as we fix the genus of \( \Sigma \) and the degree of \( L \) modulo 2, we may restrict ourselves to a hyperelliptic curve \( \Sigma \). In this case, there is a closed embedding \( \varphi : \mathcal{M}_g \hookrightarrow \mathcal{G}(g + 3, 2g + 2) \) into a Grassmannian as follows [DeRa]: Let \( p : \Sigma \to IP^1 \) represent \( \Sigma \) as two-fold covering of \( IP^1 \), \( \iota : \Sigma \to \Sigma \) the corresponding hyperelliptic involution (s.th. \( p \circ \iota = p \)) and \( B \subset IP^1 \) the branch locus of \( p (B) = 2g + 1 \). Now let \( L \) be a line bundle of degree \( 2g + 1 \) (as opposed to \( d = 4g - 3 \) in Mumford’s method) and let \( E \) be a stable 2-bundle over \( \Sigma \) with determinant \( L \). Applying \( \iota^* \) to \( E \otimes \iota^* E \) and switching factors induces an involution \( J : E \otimes \iota^* E \to E \otimes \iota^* E \). Denote by \( p_*(E \otimes \iota^* E)^\mathbb{Z} \) the \( J \)-anti-invariant subsheaf of \( p_*(E \otimes \iota^* E) \). One shows \( h^0(IP^1, p_*(E \otimes \iota^* E)^\mathbb{Z}) = g + 3 \) (loc. cit., Prop. 2.2). In a branch point \( t \in B \) we have the identification

\[
p_*(E \otimes \iota^* E)^\mathbb{Z} \cong p_*(E \wedge E)_t = (p_* L)_t.
\]

The map \( \varphi : \mathcal{M}_g \to \mathcal{G}(g + 3, 2g + 2) \) is then defined as

\[
E \mapsto \left( H^0(IP^1, p_*(E \otimes \iota^* E)^\mathbb{Z}) \right)_{|B} \subset H^0(IP^1, p_* L_B) \cong C^{2g+2}.
\]

Now let \( S \) and \( Q \) be the universal bundle and the universal quotient bundle on \( \mathcal{G}(g + 3, 2g + 2) \) respectively. The key observation is that \( Q \) has rank \( g - 1 \)! Similarly to Mumford’s method we “just” have to express the Chern classes of \( \varphi^* Q \) in terms of \( \alpha, \beta, \gamma \) (essentially by Grothendieck-Riemann-Roch of course) to get relations \( c_r(\varphi^* Q) = 0, r = g, g + 1, g + 2 \).

The Chern class computations are based on the following exact sequence.

**Lemma 1.1.** On \( \mathcal{M}_g \times \Sigma \), there is an exact sequence

\[
0 \longrightarrow \hat{p}^* \hat{p}_*(\mathcal{U} \otimes \iota^* \mathcal{U})^\mathbb{Z} \longrightarrow \mathcal{U} \otimes \iota^* \mathcal{U} \longrightarrow S^2 \mathcal{U}|_{\mathcal{M}_g \times \mathcal{P}^{-1}(B)} \longrightarrow 0,
\]

where \( \hat{p} = Id \times p : \mathcal{M}_g \times \Sigma \to \mathcal{M}_g \times IP^1 \).
Proof. The restriction map \( \hat{p}^* \hat{p}_* E \to E \) is injective for any locally free sheaf \( E \) by flatness of \( p \). Composing this map with the inclusion \( \hat{p}^* \hat{p}_* (U \otimes \epsilon^* U)^3 \hookrightarrow \hat{p}^* \hat{p}_* (U \otimes \epsilon^* U) \) yields exactness at the first place. Outside of the branch points this map is obviously an isomorphism (by anti-invariance elements of \( \hat{p}^* \hat{p}_* (U \otimes \epsilon^* U)^3 \) are uniquely determined by their behaviour on one branch). It is then a matter of linear algebra to check that at a branch point \( y \in \Sigma \) the cokernel is given by mapping a germ of sections of \( U \otimes \epsilon^* U \) to \( (s + J(s))(y) \in U \otimes \epsilon^* U \mid_{M_y \times \mathbb{P}^{-1}(B)} \cong S^2 U \mid_{M_y \times \mathbb{P}^{-1}(B)}. \)

2. Computations of Chern Classes

This section contains the computational heart of the paper, summarized in the following proposition. We adopt the convention that in writing Chern classes or Chern characters as analytic functions in certain cohomology classes we understand to evaluate on these classes the corresponding power series expansion.

**Proposition 2.1.** Denote by \( c(\varphi^* Q) = \sum_{i \geq 0} c_i(\varphi^* Q) \) the total Chern class of \( \varphi^* Q \). Then

\[
c(\varphi^* Q) = (1 - \beta)^{-1/2} \exp \left[ \alpha + \frac{2\gamma}{\beta} \sum_{m \geq 1} \frac{\beta^m}{2m + 1} \right].
\]

Note that the \( c_i(\varphi^*(Q)) \) are really **polynomials** in \( \alpha, \beta, \gamma \), the denominator \( \beta \) cancels. Also, if one prefers, one could write \( \arctanh(\sqrt{\beta})/\sqrt{\beta} - 1 \) (with \( \sqrt{\beta} \) formally adjoint to \( H^*(M_g) \)) instead of the infinite sum.

Before turning to the proof we will need some preparations. Letting \( \pi : M_g \times \mathbb{P}^1 \to M_g \) and \( \tilde{\pi} = \pi \circ \hat{p} : M_g \times \Sigma \to M_g \) be the projections, we know \( \varphi^* S = \tilde{\pi}_* (U \otimes \epsilon^* U)^3 \otimes H^{-1} \) with \( H \in \text{Pic}(M_g) \cong \mathbb{Z} \) the ample generator and \( \tilde{\pi}_*(U \otimes \epsilon^* U)^3 := \pi_* (\hat{p}_* (U \otimes \epsilon^* U)^3) [\text{DeRa}]. \)

To apply Grothendieck-Riemann-Roch to this sheaf we first need a closed formula for the Chern character of \( U \otimes \epsilon^* U \). We will use a computational trick which the authors learned from [Ki2, p.862f] to represent most of the terms in exponential form. This will be convenient when we transform back to Chern classes. For that fix a large number \( N \) such that \( \beta^N = 0 \) (e.g. for dimension reasons) and let \( \mu_1, \ldots, \mu_N \in C \) be such that

\[
N_k(\mu_1, \ldots, \mu_N) := \sum_{\nu=1}^N \mu^k \nu = \frac{1}{k + 1} \quad \text{for } 1 \leq k \leq N.
\]

The existence of \( \mu_1, \ldots, \mu_N \) is clear either by an elementary argument or from finiteness of the map \( (N_1, \ldots, N_N) : C^N \to C^N. \)

**Lemma 2.2.** Formally adjoining \( \sqrt{\beta} \) and \( \alpha' := \alpha + 2\gamma/\beta \) (both of real degree 2) to...
$H^{2\ast}(\mathcal{M}_g, Q)$ the following holds:

$$
ch(U \otimes i^*U) = e^\alpha \left[ (2 + e^{\sqrt{\beta}} + e^{-\sqrt{\beta}})(1 + D\omega) - 2\alpha\omega \\
- \alpha'\omega \sum_{\nu} (e^{\mu\nu\sqrt{\beta}} + e^{-\mu\nu\sqrt{\beta}} - 2) \right].
$$

Proof. We have $c(U \otimes \iota^*U) = 1 + c_2(\text{End } U) = 1 - \beta + 4\psi + 2\alpha\omega$ with $\psi = \sum_i \psi_i \otimes e_i$. Note that $\iota^*: H^1(\Sigma) \rightarrow H^1(\Sigma)$ is just multiplication by $-1$. In fact, if $\delta$ is a closed 1-form on $\Sigma$, then $\delta + i^\ast\delta$ is closed and $i^\ast$-invariant, hence $\delta + i^\ast\delta = p^\ast df = dp^\ast f$ for $f \in \mathcal{C}^\infty(\mathbb{P}^1)$ since $H^1(\mathbb{P}^1) = 0$. But any orientation preserving diffeomorphism leaves $c(\text{End } U)$ unchanged, hence $\iota^\ast\psi_i = -\psi_i$. Now it is almost clear and can be easily verified by a standard Chern class computation that $c_2(U^* \otimes \iota^*U) = -\beta + 2\alpha\omega$, i.e. the factor $4\psi$ drops out. Instead, there is a non-trivial $c_4$, namely $c_4(U^* \otimes \iota^*U) = 4\psi^2 = 4\gamma\omega$. Summarizing, we have

$$
c(U^* \otimes \iota^*U) = 1 + (-\beta + 2\alpha\omega) + 4\gamma\omega.
$$

Next, for any bundle $E$ with only $c_2$ and $c_4$ non-vanishing

$$
ch_{2k}(E) = 2(-1)^k c_2(E)^k - 2c_4(E), \quad ch_{2k+1}(E) = 0
$$

(induction on $k$). Thus for $k \geq 2$ ($k = 1$: $ch_2(U^* \otimes \iota^*U) = -2c_2 = 2\beta - 4\alpha\omega$)

$$
ch_{2k}(U^* \otimes \iota^*U) = 2(-1)^k \left( (-\beta)^{k-2} + (k - 2)(-\beta)^{k-3}2\alpha\omega \right) \left( \beta^2 - 4\alpha\beta\omega - 4k\gamma\omega \right)
= 2\beta^k - 4k(\alpha\beta + 2\gamma)\beta^{k-2}\omega.
$$

Formally adjoining $\sqrt{\beta}$ and $\alpha' = \alpha + 2\gamma/\beta$ (in the denominator always cancels in the following) we get

$$
ch(U^* \otimes \iota^*U) = 2 + 2 \sum_{k \geq 0} \frac{\beta^k}{(2k)!} - 2(\alpha\beta + 2\gamma)\omega \sum_{k \geq 2} \frac{\beta^{k-2}}{(2k)!} - \frac{4\alpha\omega}{2}
= 2 + e^{\sqrt{\beta}} + e^{-\sqrt{\beta}} - 2\alpha\omega - 2\alpha'\omega \sum_{k \geq 1} \frac{1}{2k + 1} \frac{\beta^k}{(2k)!}.
$$

The computational trick consists in writing

$$
2 \sum_{k \geq 1} \frac{1}{2k + 1} \frac{\beta^k}{(2k)!} = \sum_{\nu=1}^N \left( e^{\mu\nu\sqrt{\beta}} + e^{-\mu\nu\sqrt{\beta}} - 2 \right).
$$

Finally using $c_1(U) = \alpha + D\omega$ ($D = 2g + 1$) together with the isomorphism $U \cong U^* \otimes \det U$ and the multiplicativity of the Chern character we deduce $ch(U \otimes \iota^*U) = e^\alpha ch(U^* \otimes \iota^*U)$ which upon inserting the previous computations gives the stated formula. $\square$

Pushing-forward the exact sequence from Lemma 1.1 we get

$$
0 \rightarrow \hat{p}_*(U \otimes \iota^*U)^3 \otimes \hat{p}_*\mathcal{O} \rightarrow \hat{p}_*(U \otimes \iota^*U) \rightarrow \hat{p}_*(\mathcal{S}^2 U_{\mathbb{P}^1}(B)) \rightarrow 0
$$

136
(for the first term apply the projection formula).

**Lemma 2.3.** Denoting $\tilde{\omega}$ the normalized volume form on $IP^1$ the following holds

$$
ch(\hat{p}_*(\mathcal{U} \otimes \iota^*\mathcal{U})^g) = e^\alpha \left[ (2 + e^{\sqrt{\beta}} + e^{-\sqrt{\beta}})(1 - \frac{\tilde{\omega}}{2}) + \tilde{\omega} \left( -\alpha - \frac{\alpha'}{2} \sum_{\nu} (e^{\nu \sqrt{\beta}} + e^{-\nu \sqrt{\beta}} - 2) + (g + 1) \right) \right].
$$

**Proof.** From the above exact sequence, we see

$$
ch(\hat{p}_*(\mathcal{U} \otimes \iota^*\mathcal{U})^g) = (ch\hat{p}_*(\mathcal{U} \otimes \iota^*\mathcal{U}) - ch\hat{p}_*(S^2\mathcal{U}|_{M_g \times p^{-1}(B)}) \div ch(\hat{p}_*\mathcal{O})
$$

which by Grothendieck-Riemann-Roch applied to $\hat{p}$ and writing $\tilde{g} = g - 1$ and $ch(\mathcal{U} \otimes \iota^*\mathcal{U}) - ch(S^2\mathcal{U}|_{M_g \times p^{-1}(B)}) = A + B \omega$, equals

$$
\hat{p}_*[\tilde{g} = g - 1; A + B \omega](1 - \tilde{g} \omega)/(1 - \tilde{g} \omega) = \hat{p}_* \left[ A + (B - \tilde{g}A)\omega \right]/(2 - \tilde{g} \omega) = \left( 2A + (B - \tilde{g}A)\omega \right)/2 \left( 1 + \frac{\tilde{g} \omega}{2} \right) = A + B \omega.
$$

To compute the Chern character of $S^2\mathcal{U}|_{M_g \times p^{-1}(B)}$ we restrict the exact sequence of Lemma 1.1 to $M_g \times p^{-1}(B)$. Then since $\hat{p}^*\hat{p}_*(\mathcal{U} \otimes \iota^*\mathcal{U})^g|_{M_g \times p^{-1}(B)} \simeq det \mathcal{U}|_{M_g \times p^{-1}(B)}$ we get

$$
ch(S^2\mathcal{U}|_{M_g \times p^{-1}(B)}) = [ch(\mathcal{U} \otimes \iota^*\mathcal{U}) - ch(det \mathcal{U})] (2g + 2) \omega.
$$

Thus

$$
A + B \omega = ch(\mathcal{U} \otimes \iota^*\mathcal{U}) \left( 1 - (2g + 2) \omega \right) + ch(det \mathcal{U})(2g + 2) \omega = e^{\alpha} \left[ (2 + e^{\sqrt{\beta}} + e^{-\sqrt{\beta}})(1 - \omega) + 2\omega \left( -\alpha - \frac{\alpha'}{2} \sum_{\nu} (e^{\nu \sqrt{\beta}} + e^{-\nu \sqrt{\beta}} - 2) + (g + 1) \right) \right],
$$

hence the claim. □

**Proof of Proposition 2.1.** It follows from Proposition 2.2 of [DeRa] applied to $\mathcal{U} \otimes \iota^*\mathcal{U}$ that $R^1p_*(\hat{p}_*(\mathcal{U} \otimes \iota^*\mathcal{U})^g) = 0$. The Grothendieck-Riemann-Roch theorem for pushing-forward $\hat{p}_*(\mathcal{U} \otimes \iota^*\mathcal{U})^g$ by $\pi : M_g \times IP^1 \to M_g$ thus reads

$$
ch(\pi_*(\mathcal{U} \otimes \iota^*\mathcal{U})^g) = \pi_* \left( ch\hat{p}_*(\mathcal{U} \otimes \iota^*\mathcal{U})^g \cdot (1 + \tilde{\omega}) \right) = e^{\alpha} \left[ \frac{1}{2} (2 + e^{\sqrt{\beta}} + e^{-\sqrt{\beta}}) - \alpha - \frac{\alpha'}{2} \sum_{\nu} (e^{\nu \sqrt{\beta}} + e^{-\nu \sqrt{\beta}} - 2) + (g + 1) \right].
$$

137
Plugging in the relation expressing the pull-back of $S$, i.e. $\varphi^*S = \pi_*(U \otimes \iota^*U)^\# \otimes H^{-1}$, we get

$$
\text{ch} \varphi^*Q = (2g + 2) - \text{ch} \varphi^*S = (2g + 2) - \text{ch} \pi_* (U \otimes \iota^*U)^\# / \text{ch}(H)
= (g - 1) - \frac{1}{2} \left( e^{\sqrt{\beta}} - e^{-\sqrt{\beta}} \right) + \alpha + \frac{\alpha'}{2} \sum_{\nu} \left( e^{\mu_{\nu} \sqrt{\beta}} + e^{-\mu_{\nu} \sqrt{\beta}} - 2 \right).
$$

Next we need to make use of the computational trick of Kirwan again: Assume $M \gg 0$ s.th. $\alpha^M = \alpha'^M = 0$ and find $\lambda_k \in C$ with $N_1(\lambda_1, \ldots, \lambda_M) = 1$, $N_k(\lambda_1, \ldots, \lambda_M) = 0$ for $2 \leq k \leq M$. Then $\alpha = \sum_{\kappa} (\lambda^{\kappa}_{\kappa} \alpha - 1)$, $\alpha' = \sum_{\kappa} (\lambda^{\kappa}_{\kappa} \alpha' - 1)$. Inserting we get

$$
\text{ch} \varphi^*Q = (g - 1) - \frac{1}{2} \left( e^{\sqrt{\beta}} - e^{-\sqrt{\beta}} \right) + \sum_{\kappa} \left( e^{\lambda^{\kappa}_{\kappa} \alpha + \mu_{\nu} \sqrt{\beta}} + e^{\lambda^{\kappa}_{\kappa} \alpha' - \mu_{\nu} \sqrt{\beta}} - e^{\mu_{\nu} \sqrt{\beta}} - e^{-\mu_{\nu} \sqrt{\beta}} - 2 \right) - 1.
$$

This is a sum of exponentials and as such easily transformed into the corresponding total Chern class:

$$
c(\varphi^*Q) = \left[ \left( 1 + \sqrt{\beta} \right) \left( 1 - \sqrt{\beta} \right) \right]^{-1/2} \prod_{\kappa} \left( 1 + \lambda^{\kappa}_{\kappa} \alpha \right)
= \prod_{\kappa, \nu} \frac{1 + \mu_{\nu} \sqrt{\beta} + \lambda^{\kappa}_{\kappa} \alpha'}{1 + \mu_{\nu} \sqrt{\beta}} \cdot \frac{1 - \mu_{\nu} \sqrt{\beta} + \lambda^{\kappa}_{\kappa} \alpha'}{1 - \mu_{\nu} \sqrt{\beta}} \cdot \frac{1}{(1 + \lambda^{\kappa}_{\kappa} \alpha')^2}^{1/2}.
$$

To get rid of the $\lambda_k$ we observe that $\sigma_k(\lambda_1, \ldots, \lambda_M) = 1/k!$ [Ki2, p.862]. The product over $\kappa$ can thus be carried out, e.g.

$$
\prod_{\kappa, \nu} \frac{1 + \mu_{\nu} \sqrt{\beta} + \lambda^{\kappa}_{\kappa} \alpha'}{1 + \mu_{\nu} \sqrt{\beta}} = \prod_{\kappa, \nu} \left( 1 + \lambda^{\kappa}_{\kappa} \frac{\alpha'}{1 + \mu_{\nu} \sqrt{\beta}} \right) = \prod_{\nu} \exp \frac{\alpha'}{1 + \mu_{\nu} \sqrt{\beta}}
= \exp \alpha' \sum_{\nu} \sum_{l \geq 0} \left( - \mu_{\nu} \sqrt{\beta} \right)^l = \exp \alpha' \sum_{l \geq 0} \frac{(-\sqrt{\beta})^l}{l + 1}.
$$

Inserting into our last formula we thus find (the term $(1 + \lambda^{\kappa}_{\kappa} \alpha')^{-2}$ cancels the summand for $l = 0$)

$$
c(\varphi^*Q) = (1 - \beta)^{-1/2} \exp \left( \alpha + \frac{\alpha'}{2} \sum_{l \geq 1} \frac{(-\sqrt{\beta})^l + (\sqrt{\beta})^l}{l + 1} \right)
= (1 - \beta)^{-1/2} \exp \left( \alpha + \alpha' \sum_{m \geq 1} \frac{1}{2m + 1} \beta^m \right)
$$

as claimed. □

It is convenient to gather the relations in a generating function.
Definition 2.4. We define $\Phi \in Q[\alpha, \beta, \gamma][t]$ by

$$
\Phi(t) := (1 - \beta t^2)^{-1/2} \exp \left[ \alpha t + \left( \frac{2\gamma}{\beta} \right) t \sum_{m \geq 1} \frac{\beta^m t^{2m}}{2m + 1} \right].
$$

For later use let us also state a functional equation that $\Phi$ obeys. This equation is actually equivalent to the recursion formula to be proved below (Proposition 3.2).

Proposition 2.5. $\Phi$ obeys the following differential equation:

$$
\Phi'(t) = \frac{\alpha + \beta t + 2\gamma t^2}{1 - \beta t^2} \cdot \Phi(t).
$$

Proof. Direct computation. \qed

Let us add that with the same methods, it is not hard to deduce also a closed formula for the Chern classes of $N_g$. The result is:

$$
c(N_g) = (1 - \beta)^g \exp \left( \frac{-4\gamma}{1 - \beta} \right) \cdot c(\varphi^* Q)^2.
$$

Note the simple dependence on the genus!

3. A Minimal Set of Relations

We have remarked in the introduction that the three generating relations $f_1^g$, $f_2^g$, $f_3^g$ of degrees $g$, $g+1$ and $g+2$ are uniquely determined by their initial terms $\alpha^g$, $\alpha^{g-1}/\beta$ and $\alpha^{g-2}/\gamma$ respectively (w.r.t. the reverse lexicographic order; we will prove this as an easy consequence of the recursion relations, see Proposition 4.1). It is then an exercise in calculus to find the following

Definition 3.1. Writing $\Phi^{(r)} = \frac{d^r}{dt^r} \Phi(0)$ we define for $g \geq 1$

$$
\begin{align*}
  f_1^g & := \Phi^{(g)} \\
  f_2^g & := \frac{1}{g^2} \left( \Phi^{(g+1)} - \alpha \Phi^{(g)} \right) \\
  f_3^g & := \frac{1}{2g(g+1)} \left( \Phi^{(g+2)} - \alpha \Phi^{(g+1)} - (g + 1)^2 \beta \Phi^{(g)} \right).
\end{align*}
$$

We are now in a position to prove the recursion relations.
Proposition 3.2. \((f_1^1, f_2^1, f_3^1) = (\alpha, \beta, \gamma)\) and inductively for \(g \geq 1\)

\[
\begin{align*}
    f_1^{g+1} &= \alpha f_1^g + g^2 f_2^g \\
    f_2^{g+1} &= \beta f_1^g + \frac{2g}{g+1} f_3^g \\
    f_3^{g+1} &= \gamma f_1^g.
\end{align*}
\]

Proof. The first claim is by direct check. Next, the recursion relations for \(f_1^{g+1}\) and \(f_2^{g+1}\) are immediate consequences of their definition. All the work is thus shifted to the innocuous looking formula for \(f_3^{g+1}\). What we have to show is the vanishing of

\[
2(g+1)(g+2) (f_3^{g+1} - \gamma f_1^g) = \Phi^{(g+3)} - \alpha \Phi^{(g+2)} - (g+2)^2 \beta \Phi^{(g+1)} - 2(g+1)(g+2) \gamma \Phi^{(g)} = (g+2)! [(g+3) \varphi_{g+3} - \alpha \varphi_{g+2} - (g+2) \beta \varphi_{g+1} - 2 \gamma \varphi_g]
\]

with \(\varphi_k\) the \(k\)-th Taylor coefficient of \(\Phi\) at \(t = 0\). Multiplying with \(t^{g+3}\) and taking the sum this will follow from

\[
\sum_{g \geq 1} (g+3) \varphi_{g+3} t^{g+3} = \alpha t \sum_{g \geq 1} \varphi_{g+2} t^{g+2} + \beta t^2 \sum_{g \geq 1} (g+2) \varphi_{g+1} t^{g+1} + 2 \gamma t^3 \sum_{g \geq 1} \varphi_g t^g
\]

which is the part of order larger 3 of

\[
t \cdot \Phi' = \alpha t \Phi + \beta t^2 (\Phi \cdot t)' + 2 \gamma t^3 \Phi = (\alpha t + \beta t^2 + 2 \gamma t^3) \Phi + \beta t^3 \Phi',
\]

the functional equation for \(\Phi\) (Proposition 2.5). \(\square\)

4. The Leitideal

The decisive step in the proof of completeness of our relations is that the Leitideal (initial ideal) can be computed completely and has a particularly simple form. In all that follows we use the (graded) reverse lexicographic order in \(Q[\alpha, \beta, \gamma]\) and write \(\text{In}(f)\) (\(\text{In}(\mathcal{I})\)) for the initial term of \(f \in Q[\alpha, \beta, \gamma]\) (resp. the initial ideal of an ideal \(\mathcal{I} \subset Q[\alpha, \beta, \gamma]\)). As warm-up let us check that the initial terms of the \(f_i^g\) are as promised in the last chapter:

Proposition 4.1. In the reverse lexicographic order \(\text{In}(f_1^g) = \alpha^g, \text{In}(f_2^g) = \alpha^{g-1} \beta, \text{In}(f_3^g) = \alpha^{g-1} \gamma\).

Proof. By induction on \(g\). \(g = 1\) is clear by the first line of Proposition 3.2. Applying our recursion relations and the induction hypothesis, we get

\[
\begin{align*}
    f_1^{g+1} &= \alpha^{g+1} + \alpha^g \beta + \ldots, \\
    f_2^{g+1} &= \alpha^g \beta + \frac{2g}{g+1} \alpha^{g-1} \gamma + \ldots, \\
    f_3^{g+1} &= \alpha^g \gamma + \ldots,
\end{align*}
\]

where "..." mean terms of lower order. \(\square\)

Now setting \(\mathcal{I}_g := (f_1^g, f_2^g, f_3^g) \subset Q[\alpha, \beta, \gamma]\), the ideal spanned by \(f_i^g, i = 1, 2, 3\), then

Proposition 4.2. \(\text{In}(\mathcal{I}_g) = (\alpha^a \beta^b \gamma^c, a + b + c \geq g)\).
\textbf{Proof}. By induction on \( g \), \( g = 1 \) being trivially true. From
\[
\begin{align*}
\gamma f_1^g &= f_3^{g+1} \\
g^2 \gamma f_2^g &= \gamma f_1^{g+1} - \alpha \gamma f_1^g = \gamma f_2^{g+1} - \alpha f_3^{g+1} \\
\frac{4g}{g+1} \gamma f_3^g &= \gamma f_2^{g+1} - \beta \gamma f_1^g = \gamma f_3^{g+1} - \beta f_3^{g+1}
\end{align*}
\]
we see that \( \gamma I_g \subset I_{g+1} \) (this is also clear from the observation that \( \gamma \in H^*(\mathcal{M}_{g+1}) \) is Poincaré dual to \( 2g \) copies of \( \mathcal{M}_g \), cf. below). By induction hypothesis the claim is thus true for \( c > 0 \). But in any homogeneous expression (with \( \alpha, \beta, \gamma \) having weights 1, 2, 3 respectively) the monomials containing \( \gamma \) have lower order than those without. Therefore, we can reduce modulo \( \gamma \) (indicated by a bar) and have only to show \( \text{In}(\overline{I}_g) = (\overline{\alpha}^\alpha \overline{\beta}^\beta, a + b \geq g) \). Modulo \( \gamma \) the recursion relations read
\[
\overline{f}_1^{g+1} = \overline{\alpha} \overline{f}_1^g + g^2 \overline{f}_2^g, \quad \overline{f}_2^{g+1} = \overline{\beta} \overline{f}_1^g.
\]
Now we are able to repeat the argument from above with \( \overline{\beta} \) instead of \( \gamma \), because
\[
\begin{align*}
\overline{\beta} f_1^g &= \overline{f}_2^{g+1} \\
g^2 \overline{\beta} f_2^g &= \overline{\beta} f_1^{g+1} - \overline{\alpha} \overline{f}_1^g = \overline{\beta} f_2^{g+1} - \overline{\alpha} f_3^{g+1}.
\end{align*}
\]
This leaves us with the case \( b = 0, c = 0 \), which is clearly true since \( \alpha \beta = \text{In}(f_1^g) \) is the smallest power of \( \alpha \) contained in \( I_g \) (for \( \deg f_i^g \leq g, i = 1, 2, 3 \)). \( \square \)

5. Completeness

The strategy of showing that \( I_g = (f_1^g, f_2^g, f_3^g) \subset C[\alpha, \beta, \gamma] \) is really the ideal of relations between \( \alpha, \beta, \gamma \) is a simple dimension count. But although the subring \( \langle \alpha, \beta, \gamma \rangle \subset H^*(\mathcal{M}_g, Q) \) generated by \( \alpha, \beta, \gamma \) is the invariant ring under the action of the orientation preserving diffeomorphisms, the authors do not know a direct way to compute \( \dim Q(\alpha, \beta, \gamma) \). Instead we are viewing the even cohomology \( H^{2*}(\mathcal{M}_g, Q) \) as module over \( C[\alpha, \beta, \gamma]/I_g \) and check injectivity of the structure map \( C[\alpha, \beta, \gamma]/I_g \to H^{2*}(\mathcal{M}_g, Q) \) by refining the basis \( \{ \alpha^a \beta^b \gamma^c \mid a + b + c < g \} \) of \( C[\alpha, \beta, \gamma]/I_g \) to a basis of \( H^{2*}(\mathcal{M}_g, Q) \). As a by-product we will actually find an explicit basis of the latter, which in a sense explains the inductive formulas for the even Betti numbers found by Newstead [Ne2].

\textbf{Proposition 5.1}. \( H^{2*}(\mathcal{M}_g, Q) \) is generated by elements of the form
\[
\alpha^a \beta^b \psi_i \ldots \psi_{i_{2l}} \quad \text{with} \quad a + b + 2l < g - 1,
\]
\[
\text{and} \quad \alpha^a \beta^b \gamma^k \psi_i \ldots \psi_{i_{2l}} \quad \text{with} \quad a + b + k + 2l = g - 1, \quad k \geq 0,
\]
where \( 1 \leq i_1 < \ldots < i_{2l} \leq 2g \).

As an intermediate notion between the \( \psi_i \) and \( \gamma \) let us introduce the classes \( \gamma_j := \psi_j \psi_{j+g}, j = 1, \ldots, g \) (then \( \gamma = 2 \sum_j \gamma_j \)). Each of the \( \gamma_j \) is Poincaré dual to a diffeomorphic image \( N_j \) of \( \mathcal{M}_{g-1} \) (by "contracting a handle", cf. no.26 in [Th]). Moreover, \( \mathcal{U}/N_j \) is topologically a universal bundle on \( \mathcal{M}_{g-1} \), so \( \alpha, \beta, \gamma \) restrict to generators \( \tilde{\alpha}, \tilde{\beta}, \tilde{\psi}_i \) (\( i \neq j \)).
of $H^*(M_{g-1}, Q) \psi_j|N_j = 0 = \psi_{j+g}|N_j$ since $\psi_j \gamma_j = 0 = \psi_{j+g} \gamma_j$ trivially). We will also use the fact that intersection products $\alpha^a \beta^b \psi_{i_1} \ldots \psi_{i_2t}[M_g] (a + 2b + 3l = 3g - 3)$ are zero unless $\{i_1, \ldots, i_{2t}\} = \{j_1, j_1 + g, \ldots, j_l, j_l + g\}$ in which case

$$\alpha^a \beta^b \psi_{i_1} \ldots \psi_{i_{2t}}[M_g] = \frac{1}{g!} \alpha^a \beta^b \gamma^l[M_g],$$

i.e. only depending on the length of the sequence $(j_1, \ldots, j_l)$, cf. [Th].

Proof of proposition. We want to refine the result of Proposition 4.2 that a monomial $\alpha^a \beta^b \gamma^c$ is equivalent (= may be reduced modulo $I_g$) to a polynomial of lower order if $a + b + c \geq g$. For this we will use the reverse lexicographic order $\alpha > \beta > \psi_1 > \ldots > \psi_{2g} > \gamma_1 > \ldots > \gamma_{2g} > \gamma$.

Let $1 \leq i_1 < \ldots < i_k \leq 2g$ and $1 \leq j_1 < \ldots < j_k \leq g$ with $\{i_1, \ldots, i_k\} \cap \{j_1, j_1 + g, \ldots, j_l, j_l + g\} = \emptyset$.

Claim: If $a + b + k + l \geq g$ then $\alpha^a \beta^b \psi_{i_1} \ldots \psi_{i_k} \gamma_{j_1} \ldots \gamma_{j_l}$ is equivalent to a polynomial of lower order modulo $I_g$, which can be taken of the form $F(\alpha, \beta) \psi_{i_1} \ldots \psi_{i_k} \gamma_{j_1} \ldots \gamma_{j_l}$.

The claim certainly holds if $k + l = 0$ by Proposition 4.2. For $l > 0$ let $\iota : M_{g-1} \hookrightarrow M_g$ have image $N_{j_k}$ (Poincaré dual to $\gamma_{j_k}$) and use a hat to denote pull-back by $\iota$. By descending induction on $g$ then ($l < g$ because $\gamma_1 \ldots \gamma_g = 0$ for dimension reasons),

$$\iota^* \left( \alpha^a \beta^b \psi_{i_1} \ldots \psi_{i_k} \gamma_{j_1} \ldots \gamma_{j_{l-1}} \right) = \hat{\alpha}^a \hat{\beta}^b \hat{\psi}_{i_1} \ldots \hat{\psi}_{i_k} \hat{\gamma}_{j_1} \ldots \hat{\gamma}_{j_{l-1}} = F(\hat{\alpha}, \hat{\beta}) \hat{\psi}_{i_1} \ldots \hat{\psi}_{i_k} \hat{\gamma}_{j_1} \ldots \hat{\gamma}_{j_{l-1}} = \iota^* \left( F(\alpha, \beta) \psi_{i_1} \ldots \psi_{i_k} \gamma_{j_1} \ldots \gamma_{j_{l-1}} \right)$$

with order($F$) $< a + b$. This means $\alpha^a \beta^b \psi_{i_1} \ldots \psi_{i_k} \gamma_{j_1} \ldots \gamma_{j_l} = F(\alpha, \beta) \psi_{i_1} \ldots \psi_{i_k} \gamma_{j_1} \ldots \gamma_{j_l}$ as wanted. Finally the case $l = 0$, $k > 0$: Set $i_k = i_k + g$ if $i_k \leq g$ and $i_k = i_k - g$ if $i_k > g$.

By the previous case, we get $\alpha^a \beta^b \psi_{i_1} \ldots \psi_{i_k} \psi_k = F(\alpha, \beta) \psi_{i_1} \ldots \psi_{i_k} \psi_k$. Then the above remarks on the intersection product show

$$(\alpha^a \beta^b \psi_{i_1} \ldots \psi_{i_k} - F(\alpha, \beta) \psi_{i_1} \ldots \psi_{i_k}) \cdot A[M_g] = 0$$

for all $A \in H^*(M_g)$, i.e. $\alpha^a \beta^b \psi_{i_1} \ldots \psi_{i_k} = F(\alpha, \beta) \psi_{i_1} \ldots \psi_{i_k}$. This proves the claim.

The proposition is now clearly reduced to a second

Claim: Let $M = \alpha^a \beta^b \psi_{i_1} \ldots \psi_{i_k} \gamma_{j_1} \ldots \gamma_{j_l}$ with $a + b + k + l = g - 2$. Then for all $1 \leq i, j \leq g$, $M \gamma_i - M \gamma_j$ may be reduced to lower order modulo $I_g$.

In fact, from the above we know already $M \gamma_i \gamma_j = F \gamma_i \gamma_j$ in $H^*(M_g)$ with order($F$) $< g - 2$. $F \gamma_i - F \gamma_j$ is our candidate for the lower order term. If $\{i, i + g, j, j + g\} \cap \{i_1, \ldots, i_k\} = \emptyset$ and $A = \alpha^a' \beta^b' \psi_{i_1} \ldots \psi_{i_k}$ then

$$(M \gamma_i - M \gamma_j) \cdot A[M_g] = 0 = (F \gamma_i - F \gamma_j) \cdot A[M_g]$$

142
again by the symmetry in the $\gamma_i$ of the intersection pairing. On the other hand
\[
(\mathbb{G} \gamma_i - \mathbb{M} \gamma_j) \gamma_j [\mathcal{M}_g] = \mathbb{M} \gamma_i \gamma_j A[\mathcal{M}_g] = F \gamma_i \gamma_j A[\mathcal{M}_g]
\]
and analogously with $\gamma_i$. Thus $\mathbb{G} \gamma_i = \mathbb{M} \gamma_j + F(\gamma_i - \gamma_j)$ modulo $\mathcal{I}_g$ as claimed.
(Note: This argument fails for the $\psi_i$ because of the presence of $\psi_{i+g}$ respectively $\psi_{i-g}$.)

The only thing we finally need to do is to count the number of generators and compare with the inductive formula for the Betti numbers found by Newstead.

**Proposition 5.2.** The generators for $H^{2*}(\mathcal{M}_g, \mathbb{Q})$ in Proposition 5.1 are linearly independent up to the middle dimension.

**Proof.** Let $G_r$ be the set of generators of (real) degree $r$ from Proposition 5.1, $g_r := \# G_r$. We will show that for $s \leq \left[ \frac{3g - 1}{2} \right]
\[
g_{2s+4} = g_{2s} + \sum_{l=s-g+1}^{[s/3]} \binom{2g}{2l}
\]
which together with $g_0 = 1$ and $g_2 = 1$ is exactly Newstead’s formula for the even Betti numbers [Ne1]. Define a map $\varphi : G_{2s} \to G_{2s+4}$ by
\[
\alpha^a \beta^b \psi_{i_1} \ldots \psi_{i_{2l}} \mapsto \alpha^a \beta^{b+1} \psi_{i_1} \ldots \psi_{i_{2l}} \quad \text{for } a + b + 2l < g - 1
\]
and
\[
\alpha^a \beta^b \gamma^k \psi_{i_1} \ldots \psi_{i_{2l}} \mapsto \alpha^{a-1} \beta^b \gamma^{k+1} \psi_{i_1} \ldots \psi_{i_{2l}} \quad \text{for } a + b + k + 2l = g - 1
\]
with $k \geq 0$ and $a > 0$. Note that the case $a = k = 0$ does not occur (then $b + 2l = g - 1$ and $2b + 3l = s$ imply $s - 2g + 2 = -l \leq 0$ which contradicts $s \leq (3g - 1)/2$). Now $G_{2s+4} \setminus \text{im } \varphi = \{ \alpha^a \psi_{i_1} \ldots \psi_{i_{2l}} \mid a + 2l < g - 1, a + 3l = s \}$ s.th. $l$ runs from $s - g + 1$ to $[s/3]$ ($a$ is determined through $a + 3l = s$). The contribution for $l$ fixed is precisely $\binom{2g}{2l}$. □

**References**


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