On H-curvature of $(\alpha, \beta)$-metrics

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Abstract: The non-Riemannian quantity $H$ was introduced by Akbar-Zadeh to characterization of Finsler metrics of constant flag curvature. In this paper, we study two important subclasses of Finsler metrics in the class of so-called $(\alpha, \beta)$-metrics, which are defined by $F = \alpha \varphi(s)$, $s = \beta/\alpha$, where $\alpha$ is a Riemannian metric and $\beta$ is a closed 1-form on a manifold. We prove that every polynomial metric of degree $k \geq 3$ and exponential metric has almost vanishing $H$-curvature if and only if $H = 0$. In this case, $F$ reduces to a Berwald metric. Then we prove that every Einstein polynomial metric of degree $k \geq 3$ and exponential metric satisfies $H = 0$. In this case, $F$ is a Berwald metric.

Key words: Polynomial metrics, exponential metric, almost vanishing $H$-curvature.

1. Introduction

Let $(\mathcal{M}, F)$ be a Finsler manifold. Then a global vector field $G$ is induced by the Finsler metric $F$ on slit tangent bundle $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where $G^i = G^i(x, y)$ are scalar functions on $TM_0$ [8].

In [1], Akbar-Zadeh considered a non-Riemannian quantity $H$ obtained from the mean Berwald curvature $E$ by the covariant horizontal differentiation along geodesics. He proved that for a Weyl metric, the flag curvature $K$ is a scalar function on the manifold $K = K(x)$ if and only if $H = 0$ [7]. The quantity $H_y = H_{ij} dx^i \otimes dx^j$ is defined as the covariant derivative of mean Berwald curvature along geodesics. In local coordinates,

$$H_{ij} = \frac{1}{2} \left[ y^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial x^m} - 2G^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial y^m} - \frac{\partial G^m}{\partial y^i} \frac{\partial^5 G^k}{\partial y^j \partial y^k \partial y^m} - \frac{\partial G^m}{\partial y^j} \frac{\partial^5 G^k}{\partial y^i \partial y^k \partial y^m} \right].$$

A Finsler metric $F$ on an $n$-dimensional manifold $\mathcal{M}$ is called of almost vanishing $H$-curvature if

$$H = \frac{n+1}{2} F^{-1} h,$$

where $\theta := \theta_i(x)y^i$ is a 1-form on $\mathcal{M}$ and $h = h_{ij} dx^i \otimes dx^j$ is the angular tensor [7].

Najafi et al. [6] proved that every R-quadratic metric satisfies $H = 0$. Then, Najafi et al. [7] generalized the Akbar-Zadeh theorem and proved that a Finsler metric $F$ has almost isotropic flag curvature $K = 3\theta/F + \sigma$ if and only if it has almost vanishing $H$-curvature, where $\theta = \theta_i(x)y^i$ is a 1-form and $\sigma = \sigma(x)$ is a scalar.

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function on manifold. Mo [4] found a new equation between H-curvature and Riemannian curvature on a Finsler manifold. Tayebi and Najafi [13] showed that every m-th root metric with almost vanishing H-curvature satisfies H = 0. Moreover, Xia [23] proved that a Randers metric has almost isotropic S-curvature if and only if it is of almost vanishing H-curvature. Recently, Zohrehvand and Rezaei [26] have obtained necessary and sufficient conditions for a square metric to be of almost vanishing H-curvature.

Randers metric and square metric belong to the class of (α, β)-metrics. Therefore, in order to find explicit examples of Finsler metrics of almost vanishing H-curvature, we consider (α, β)-metrics. An (α, β)-metric is a Finsler metric of the form $F := αφ(s), \ s = β/α$, where $φ = φ(s)$ is a $C^∞$ function on $(-b_0, b_0)$ with certain regularity, $α = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $β = b_i(x)y^i$ is a 1-form on M (see [9, 14, 16, 18–20, 22]). A polynomial (α, β)-metric of degree k is given by $φ := (1 + s)^k, \ s = β/α$, where $k ∈ \mathbb{N}$. This class of metrics contains Randers metrics ($k = 1$) and square metrics ($k = 2$) as special cases. In this paper, we consider polynomial (α, β)-metrics with almost vanishing H-curvature and prove the following.

**Theorem 1.1** Let $F = αφ(s), \ s = β/α$, be a polynomial (α, β)-metric of degree m ($m \geq 3$) on an n-dimensional manifold M, where $α = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $β = b_i(x)y^i$ is a closed 1-form on M. Then F has almost vanishing H-curvature if and only if $H = 0$. In this case, F is a Berwald metric.

**Example 1.2** Theorem 1.1 does not hold for the polynomial (α, β)-metrics of degree 1, i.e. the Randers-type Finsler metrics. For example, the standard Funk metric on the Euclidean unit ball is defined by

$$F(x, y) := \sqrt{|y|^2 - (|x|^2|y|^2 - <x, y>^2)/1 - |x|^2} + <x, y>/1 - |x|^2, \ y ∈ T_xB^n(1) ∼ \mathbb{R}^n,$$

where $<,>$ and $|.|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. F is a Randers metric and it is easy to see that β is a closed 1-form. By a simple calculation, it follows that $H = 0$ while F is not Berwald metric.

For an (α, β)-metric $F := αφ(s), \ s = β/α$, let us define $b_{ij}$ by

$$b_{ij}θ^j := db_i - b_jθ_i,$$

where $θ^i := dx^i$ and $θ_i^j := Γ^j_{ik}dx^k$ denote the Levi-Civita connection forms of α. Let

$$r_{ij} := \frac{1}{2}(b_{ij} + b_{ji}), \ s_{ij} := \frac{1}{2}(b_{ij} - b_{ji}),$$

$$r_j := b^ibr_{ij}, \ r := b^ib^ir_{ij}, \ s_j := b^is_{ij}, \ r_0 := r_jy^j, \ s_0 := s_jy^j,$$

$$r_{i0} := r_{ij}y^j, \ r_{00} := r_{ij}y^iy^j, \ s_{i0} := s_{ij}y^j, \ s_{00} := s_{ij}y^j, \ r^i_j := a^imr_{mj},$$

$$q_{ij} := r_{im}s^m_j, \ t_{ij} := s_{im}s^m_j, \ q_i := b^iq_{ij}, \ t_i := b^it_{ij}.$$ 

Now, we can give another example.

**Example 1.3** Theorem 1.1 does not hold for the polynomial (α, β)-metrics of degree 2, generally. For example, let $F = (α + β)^2/α$ be the square metric defined by following

$$α := \sqrt{|y|^2(1 - |x|^2^2) + <x, y>^2/(1 - |x|^2)^2}, \ β := <x, y>/(1 - |x|^2)^2.$$  (1.2)
$F$ is an $(\alpha, \beta)$-metric on the unit ball $\mathbb{B}^n(1) \subset \mathbb{R}^n$. We have

$$b_{ij} = 2\tau \{(1 + 2b^2)a_{ij} - 3b_i b_j\},$$

where $\tau = (1 - |x|^2)/2$. Thus, $\beta$ is closed with respect to $\alpha$. $F$ has constant flag curvature then it satisfies $H = 0$. Since $\beta$ is not parallel with respect to $\alpha$, then $F$ is not a Berwald metric.

**Example 1.4** For polynomial $(\alpha, \beta)$-metric $F = (\alpha + \beta)^3/\alpha^2$, we have

$$\Theta = \frac{3(4s - 1)}{2(8s^2 - 6B + s - 1)}, \quad Q = \frac{3}{1 - 2s}, \quad \Psi = \frac{3}{-8s^2 + 6B - s + 1}.$$ 

Suppose that $F$ has almost vanishing $H$-curvature (1.1). Since $\beta$ is a closed 1-form, then

$$2H_{jk} = \left[ \frac{h_3}{A^4 \alpha^2} r_{00} + \frac{h_4}{A^4 \alpha^3} r_{00} r_0 + \frac{h_5}{A^5 \alpha^2} r_{00} r_0 r_0 + \frac{h_6}{A^6 \alpha^2} r_0 r_0 r_0 + \frac{h_7}{A^7 \alpha^2} r_0 r_0 r_0 r_0 + \frac{h_8}{A^8 \alpha^2} r_0 r_0 r_0 r_0 r_0 \right] b_j b_k$$

$$+ \left[ \frac{h_9}{A^4 \alpha^4} r_{00} + \frac{h_{10}}{A^5 \alpha^3} r_{00} r_0 + \frac{h_{11}}{A^6 \alpha^2} r_{00} r_0 r_0 + \frac{h_{12}}{A^7 \alpha^2} r_0 r_0 r_0 r_0 + \frac{h_{13}}{A^8 \alpha^2} r_0 r_0 r_0 r_0 r_0 \right] b_j b_k \left( h_{14} \alpha + \beta \right)$$

where $A := 1 + 6B + 6Bs - 9s^2 - 8s^3$, $B := \|\beta\|_\alpha = \sqrt{\beta b_i}$ and $b_i(i = 1, 2, \ldots, 109)$ are the polynomials of variations $s$ and $B$. By using Lemma 3.1, it follows that $\beta$ satisfies $r_{ij} = 0$ and then it is parallel with respect to $\alpha$. In this case, $F$ reduces to a Berwald metric.

A Finsler metric $F = F(x, y)$ on an $n$-dimensional manifold $M$ is called an Einstein metric if its Ricci curvature satisfies $\text{Ric} = (n - 1)\lambda F^2$, where $\lambda = \lambda(x)$ is a scalar function on $M$. In [2], it is proved that every Einstein polynomial $(\alpha, \beta)$-metric is Ricci-flat. In this paper, we prove the following.

**Theorem 1.5** Let $F = \alpha \varphi(s)$, $s = \beta/\alpha$, be a polynomial $(\alpha, \beta)$-metric of degree $m$ ($m \geq 3$) on an $n$-dimensional manifold $M$, where $\alpha = \sqrt{a_{ij}(x)}y^i y^j$ is a Riemannian metric and $\beta = h_i(x) y^i$ is a closed 1-form on $M$. Suppose that $F$ is an Einstein metric. Then $H = 0$. In this case, $F$ is a Berwald metric.

**Example 1.6** The Funk metric is an Einstein metric with a closed 1-form. It satisfies $H = 0$ while it is not a Berwald metric. Then Theorem 1.5 does not hold for the polynomial $(\alpha, \beta)$-metrics of degree 1.
Example 1.7 The square metric in Example 1.3 is a Ricci-flat Finsler metric. Moreover, $F$ is an Einstein metric. However, $F$ is not a Berwald metric. Thus, Theorem 1.5 does not hold for the polynomial $(\alpha, \beta)$-metrics of degree 2, generally.

Example 1.8 Let $\varphi(s) = (1 + s)^3$ be an Einstein metric. By the Theorem 1.1 in [2], $F$ is Ricci-flat. Suppose that $\beta$ is a closed 1-form. Then $R_m^\alpha = R_m^\beta + T_m^\alpha = 0$, where $\alpha R_m^\beta$ denotes the Riemannian curvature of $\alpha$ and

$$T_m^\alpha = \left( (n - 1) \frac{c_1}{A^3} + \frac{c_2}{A^4} \right) r_{00} + \alpha \left[ \frac{1}{r_0} \left( (n - 1) \frac{c_5}{A^2} + \frac{c_6}{A^3} \right) r_{00} + \left( (n - 1) \frac{c_7}{A} + \frac{c_8}{A^2} \right) r_{00|0} \right]$$

$$+ \frac{c_{11}}{A^2} \left( r r_{00} - r_{0}^2 \right) + \frac{c_{14}}{A^3} \left( r_{00} r_{m}^m - r_{0m} r_{0}^m + r_{00|m} b_{0m} - r_{0m|0} b_{0m} \right),$$

$A := 1 + 6B + 6Bs - 9s^2 - 8s^3$ and $c_i$ $(i = 1, \cdots, 14)$ are polynomials of variations $s$ and $B$ (see [2] for the corrected version of [25]). It follows that $r_{ij} = 0$. Since $\beta$ is a closed 1-form, then it is parallel with respect to $\alpha$. In this case, $F$ reduces to a Berwald metric.

The exponential metric is another important $(\alpha, \beta)$-metric which is given by $\varphi(s) = e^s$, $s = \beta/\alpha$, (see [10, 15, 24]). Here, we consider exponential $(\alpha, \beta)$-metrics with almost vanishing $H$-curvature and prove the following.

Theorem 1.9 Let $F = \alpha \varphi(s)$, $s = \beta/\alpha$, be an exponential metric on an $n$-dimensional manifold $M$, where $\alpha = \sqrt{a_{ij}(x)} y^i y^j$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a closed 1-form on $M$. Then $F$ has almost vanishing $H$-curvature if and only if $H = 0$. In this case, $F$ is a Berwald metric.

Example 1.10 Let $F = \alpha e^{\beta/\alpha}$ be an exponential metric. At a point $x = (x^1, \cdots, x^n) \in \mathbb{R}^n$ and in the direction $y = (y^1, \cdots, y^n) \in T_x \mathbb{R}^n$, consider the following Riemannian metric $\alpha$ and 1-form $\beta$ as follows

$$\alpha(x, y) = \sqrt{(y^1)^2 + e^{2y^1} \left[ (y^2)^2 + \cdots + (y^n)^2 \right]}, \quad \beta(x, y) := y^1. \quad (1.3)$$

Then $s_{ij} = 0$. In this case, $F$ has constant $S$-curvature [5]. Thus, $F$ satisfies $H = 0$ (see [3] and [5]).

Finally, we consider the Einstein exponential metric and prove the following.

Theorem 1.11 Let $F = \alpha \varphi(s)$, $s = \beta/\alpha$, be an exponential metric on a manifold $M$, where $\alpha = \sqrt{a_{ij}(x)} y^i y^j$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a closed 1-form on $M$. Suppose that $F$ is an Einstein metric. Then $H = 0$. In this case, $F$ is a Berwald metric.

Example 1.12 Let $F = \alpha e^{\beta/\alpha}$ be an exponential metric, where $\alpha$ and $\beta$ are defined by (1.3). Suppose that $F$ is an Einstein metric. Thus, $R_m^\alpha = R_m^\beta + T_m^\alpha$, where

$$T_m^\alpha := \left( (n - 1) \frac{c_1}{A^3} + \frac{c_2}{A^4} \right) \alpha^{-2} r_{00} + \alpha^{-1} \left[ \left( (n - 1) \frac{c_5}{A^2} + \frac{c_6}{A^3} \right) r_{00} r_0 + \left( (n - 1) \frac{c_7}{A} + \frac{c_8}{A^2} \right) r_{00|0} \right]$$

$$+ \frac{c_{11}}{A^2} \left( r r_{00} - r_0^2 \right) + \frac{c_{14}}{A^3} \left( r_{00} r_{m}^m - r_{0m} r_{0}^m + r_{00|m} b_{0m} - r_{0m|0} b_{0m} \right), \quad (1.4)$$
2. Preliminary

Let \((M, F)\) be a Finsler manifold. A global vector field \(G\) is induced by \(F\) is given by \(G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}\), where \(G^i = G^i(x, y)\) are given by

\[
G^i = \frac{1}{4} y^{il} \left[ \frac{\partial^2 F^2}{\partial x^i \partial y^l} y^k - \frac{\partial F^2}{\partial x^i} \right].
\]

\(G\) is called the spray of \((M, F)\). The projection of an integral curve of the spray \(G\) is called a geodesic in \(M\) [12, 22].

For \(y \in T_x M_0\), define \(B_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M\) by \(B_y(u, v, w) := B_{ijkl}(y)u^iu^jv^kw^l \frac{\partial}{\partial x^i} \bigg|_x\), where

\[
B_{ijkl} := \frac{\partial^3 G^i}{\partial y^i \partial y^j \partial y^k}.
\]

\(B\) is called Berwald curvature. \(F\) is called a Berwald metric if \(B = 0\).

For \(y \in T_x M_0\), define \(E_y : T_x M \otimes T_x M \rightarrow \mathbb{R}\) by \(E_y(u, v) := E_{ij}(y)u^iv^j\), where

\[
E_{ij} := \frac{1}{2} B_{ijm}^m.
\]

\(E\) is called mean Berwald curvature. \(F\) is called a weakly Berwald metric if \(E = 0\). By definition, every Berwald metric is a weakly Berwald metric.

For \(y \in T_x M_0\), define the linear transformations \(R_y : T_x M \rightarrow T_x M\) with homogeneity \(R_{\lambda y} = \lambda^2 R_y\), \(\forall \lambda > 0\), where \(R_y(u) := R^k_i(y)u^k \frac{\partial}{\partial x^i}\) and

\[
R^k_i(y) = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^i \partial y^k} y^j + 2G^i \frac{\partial^2 G^j}{\partial y^i \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.
\]  

(2.1)

The family \(R := \{R_y\}_{y \in T M_0}\) is called the Riemann curvature (see [11, 17, 21]).

The Ricci curvature \(\text{Ric}(x, y)\) is the trace of the Riemann curvature defined by

\[
\text{Ric}(x, y) := R^m_m(x, y).
\]

A Finsler metric \(F\) on an \(n\)-dimensional manifold \(M\) is called an Einstein metric if the Ricci curvature satisfies

\[
\text{Ric} = (n - 1)\sigma F^2,
\]

(2.2)

where \(\sigma = \sigma(x)\) is a scalar function on \(M\).

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. First, we remark that the spray coefficients \(G^i\) of an \((\alpha, \beta)\)-metric \(F = \alpha \varphi(s)\), \(s = \beta / \alpha\), and the spray coefficients of the Riemannian metric \(\alpha\) are related by

\[\]
following
\[ G^i = G^i_\alpha + Q\alpha s^i_0 + (r_{00} - 2Q\alpha s_0)(\Psi b^i + \Theta^i), \]
where \( l^i := \alpha^{-1}y_i \) and
\[
Q := \frac{\varphi'}{\varphi - s\varphi'} \quad \Theta := \frac{\varphi\varphi' - s(\varphi\varphi'' + \varphi'\varphi')}{2\varphi[(\varphi - s\varphi') + (B^2 - s^2)\varphi'']}, \quad \Psi := \frac{\varphi''}{2[(\varphi - s\varphi') + (B^2 - s^2)\varphi'']}.\]
Moreover, \( B := ||\beta||_\alpha = \sqrt{b_i b_i} \), where \( b^i := a^{ij}b_j \).

**Lemma 3.1** Suppose \( r_{00} \) of an \((\alpha, \beta)\)-metric \( F = \alpha\varphi(s), s = \beta/\alpha \), on a manifold \( M \) satisfies
\[
Ir_{00}^2 \equiv 0 \mod(as^2 + bs + c), \quad \text{and} \quad I \equiv 0 \mod(as^2 + bs + c),
\]
where \( I \) is a polynomial of \( B \), and \( s, a, b, \) and \( c \) are polynomials of \( B \) and \( b \neq 0 \). Suppose that \( r_1 \) and \( r_2 \) are the roots of the equation \( as^2 + bs + c = 0 \) such that \( r_1^2 \neq r_2^2 \). Then \( r_{ij} = 0 \).

**Proof** The following hold
\[
Ir_{00}^2 \equiv 0, \mod(s - r_1) \quad \text{and} \quad Ir_{00}^2 \equiv 0, \mod(s - r_2).
\]
Let us put
\[
I \equiv f_1 \mod(s - r_1) \quad \text{and} \quad I \equiv f_2 \mod(s - r_2),
\]
where \( f_1 \) and \( f_2 \) are polynomials of \( B \). Then we have
\[
f_1r_{00}^2 \equiv 0, \mod(s - r_1) \quad \text{and} \quad f_2r_{00}^2 \equiv 0, \mod(s - r_2)
\]
which imply that
\[
r_{00}^2 \equiv 0, \mod(s - r_1) \quad \text{and} \quad r_{00}^2 \equiv 0, \mod(s - r_2).
\]
It follows that
\[
r_{00} \equiv 0 \mod(s - r_1) \quad \text{and} \quad r_{00} \equiv 0 \mod(s - r_2)
\]
Suppose that \( r_{00} \neq 0 \). Then by the Lemma 4.1 in [25], we get
\[
r_{00} = \sigma_1\alpha^2(s^2 - r_1^2), \quad \text{and} \quad r_{00} = \sigma_2\alpha^2(s^2 - r_2^2),
\]
where \( \sigma_1 = \sigma_1(x) \) and \( \sigma_2 = \sigma_2(x) \) are scalar functions on \( M \). By (3.2), we have
\[
(\sigma_1 - \sigma_2)\beta^2 + (\sigma_1r_1^2 - \sigma_2r_2^2)\alpha^2 = 0.
\]
Then \( \sigma_1 = \sigma_2 \) and \( r_1^2 = r_2^2 \) which contradict with the assumption. Thus, \( r_{00} = 0 \). Taking vertical derivatives of it twice yields \( r_{ij} = 0 \). \( \square \)

**Lemma 3.2** Let \( F = \alpha\varphi(s), s = \beta/\alpha \), be a polynomial \((\alpha, \beta)\)-metric of degree \( m \) on an \( n \)-dimensional manifold \( M \), where \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a 1-form on \( M \). Suppose that \( F \) has almost vanishing \( H \)-curvature. Then the following holds
\[
f_1r_{00}s_0\alpha + f_2r_{00}^2 + f_3s_0^2\alpha^2 \equiv 0, \mod\left( (1 - m^2)s^2 + (2 - m)s + m(m - 1)B + 1 \right),
\]
where \( f_j, (j = 1, 2, 3) \) are polynomials of variations \( s \) and \( B \) and they are homogeneous of degree one with respect to \( s \).
**Proof** For the polynomial metric $\varphi(s) = (1 + s)^m$, we have

$$Q = \frac{m}{1 + s - sm}, \quad \Theta = \frac{m(1 + 2s - 2sm)}{2(-m^2s^2 + s^2 - sm + 2s + 1 - Bm + m^2B)},$$

$$\Psi = \frac{m(m - 1)}{2(-m^2s^2 + s^2 - sm + 2s + 1 - Bm + m^2B)}.$$

By assumption, $F = \alpha \varphi(s)$ has almost vanishing $H$-curvature, i.e. there exists a 1-form $\theta$ on $M$ such that

$$H_{jk} = \frac{n + 1}{2} \theta F_{y'y^k}, \quad (3.4)$$

where

$$F_{y'y^k} = \frac{(1 + s)^{m-2}}{\alpha} \left[ (1 - (m - 2)s - (m - 1)s^2) a_{jk} + (m^2 - m)b_jb_k - (m^2 - m)(b_jl_k + b_kl_j)s 
+ [(m^2 - 1)s^2 + (m - 2)s - 1]l_jl_k \right] \quad (3.5)$$

$l_i := \alpha_{y^i}$, and

$$2H_{jk} = \left[ \frac{h_1}{A^6D^2\alpha} r_{00} s_0 + \frac{h_2}{A^4D^3\alpha^2} s_0 r_{00} + \frac{h_3}{A^6D^3\alpha} r_{000} + \frac{h_4}{A^4D^3\alpha} r_{000} + \frac{h_5}{A^6D^3\alpha} r_{000} + \frac{h_6}{A^5D^3\alpha^2} r_{000} 
+ \frac{h_7}{A^5D^3\alpha^2} r_{000} + \frac{h_8}{A^4D^3\alpha^2} s_0 + \frac{h_9}{A^6D^3\alpha} r_{000} + \frac{h_{10}}{A^5D^3\alpha^2} r_{000} + \frac{h_{11}}{A^5D^3\alpha^2} r_{000} + \frac{h_{12}}{A^5D^3\alpha^2} r_{000} 
+ \frac{h_{13}}{A^4D^3\alpha^2} r_{000} + \frac{h_{14}}{A^5D^3\alpha^2} r_{000} + \frac{h_{15}}{A^6D^3\alpha} r_{000} + \frac{h_{16}}{A^5D^3\alpha} r_{000} + \frac{h_{17}}{A^5D^3\alpha^2} r_{000} + \frac{h_{18}}{A^5D^3\alpha^2} r_{000} 
+ \frac{h_{19}}{A^5D^3\alpha} r_{000} + \frac{h_{20}}{A^4D^3\alpha} r_{000} + \frac{h_{21}}{A^6D^3\alpha} r_{000} + \frac{h_{22}}{A^4D^3\alpha^2} s_0 + \frac{h_{23}}{A^5D^3\alpha^2} r_{000} + \frac{h_{24}}{A^5D^3\alpha^2} r_{000} \right] b_jb_k.$$

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\[ \begin{align*}
 &+ \frac{h_{25}}{A^5D^4\alpha} r^2 s_{00} + \frac{h_{26}}{A^4D^2\alpha} r s_{00} + \frac{h_{27}}{A^3D^4\alpha} r^3 s_{00} + \frac{h_{28}}{A^3D^2\alpha} r^2 s_0 \bigg] \nabla_j l_k
 + \left[ \frac{h_{29}}{A^5D^2\alpha} r_{00}s_{00} + \frac{h_{30}}{A^3D^4\alpha} r_0 s_{00} + \frac{h_{31}}{A^3D^3\alpha} r^2 s_{00} + \frac{h_{32}}{A^3D^3\alpha} r^2 s_{00} + \frac{h_{33}}{A^3D^2\alpha} r^2 s_{00} + \frac{h_{34}}{A^2D^2\alpha} r^2 s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{35}}{A^4D^2\alpha} r^2 s_{00} + \frac{h_{36}}{A^3D^4\alpha} r^2 s_{00} + \frac{h_{37}}{A^3D^3\alpha} r^2 s_{00} + \frac{h_{38}}{A^3D^2\alpha} r^2 s_{00} + \frac{h_{39}}{A^2D\alpha} s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{41}}{A^3D^2\alpha} r s_{00} + \frac{h_{42}}{A^3D^2\alpha} r s_{00} + \frac{h_{43}}{A^2D^2\alpha} r s_{00} + \frac{h_{44}}{A^2D\alpha} r s_{00} + \frac{h_{45}}{A^2D\alpha} r s_{00} + \frac{h_{46}}{A^2D\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{47}}{A^4D^2\alpha} s_{00} + \frac{h_{48}}{A^3D^2\alpha} r s_{00} + \frac{h_{49}}{A^3D\alpha} r s_{00} + \frac{h_{50}}{A^3D\alpha} r s_{00} + \frac{h_{51}}{A^2D\alpha} r s_{00} + \frac{h_{52}}{A^2D\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{53}}{A^3D\alpha} r s_{00} + \frac{h_{54}}{A^3D\alpha} r s_{00} + \frac{h_{55}}{A^3D\alpha} r s_{00} + \frac{h_{56}}{A^2D\alpha} r s_{00} + \frac{h_{57}}{A^2D\alpha} r s_{00} + \frac{h_{58}}{A^2D\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{59}}{A^4D^2\alpha} r s_{00} + \frac{h_{60}}{A^3D\alpha} r s_{00} + \frac{h_{61}}{A^3D\alpha} r s_{00} + \frac{h_{62}}{A^2D\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{63}}{A^4D\alpha} r s_{00} + \frac{h_{64}}{A^4D\alpha} r s_{00} + \frac{h_{65}}{A^4D\alpha} r s_{00} + \frac{h_{66}}{A^3D\alpha} r s_{00} + \frac{h_{67}}{A^3D\alpha} r s_{00} + \frac{h_{68}}{A^3D\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{69}}{A^4D\alpha} r s_{00} + \frac{h_{70}}{A^4D\alpha} r s_{00} + \frac{h_{71}}{A^4D\alpha} r s_{00} + \frac{h_{72}}{A^4D\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{73}}{A^5D\alpha} r s_{00} + \frac{h_{74}}{A^5D\alpha} r s_{00} + \frac{h_{75}}{A^5D\alpha} r s_{00} + \frac{h_{76}}{A^5D\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{77}}{A^6D\alpha} r s_{00} + \frac{h_{78}}{A^6D\alpha} r s_{00} + \frac{h_{79}}{A^6D\alpha} r s_{00} + \frac{h_{80}}{A^6D\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{81}}{A^6D^2\alpha} r s_{00} + \frac{h_{82}}{A^6D^2\alpha} r s_{00} + \frac{h_{83}}{A^6D^2\alpha} r s_{00} + \frac{h_{84}}{A^6D^2\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{85}}{A^6D^2\alpha} r s_{00} + \frac{h_{86}}{A^6D^2\alpha} r s_{00} + \frac{h_{87}}{A^6D^2\alpha} r s_{00} + \frac{h_{88}}{A^6D^2\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{89}}{A^6D^2\alpha} r s_{00} + \frac{h_{90}}{A^6D^2\alpha} r s_{00} + \frac{h_{91}}{A^6D^2\alpha} r s_{00} + \frac{h_{92}}{A^6D^2\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{93}}{A^6D^2\alpha} r s_{00} + \frac{h_{94}}{A^6D^2\alpha} r s_{00} + \frac{h_{95}}{A^6D^2\alpha} r s_{00} + \frac{h_{96}}{A^6D^2\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{97}}{A^6D^2\alpha} r s_{00} + \frac{h_{98}}{A^6D^2\alpha} r s_{00} + \frac{h_{99}}{A^6D^2\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{100}}{A^6D^2\alpha} r s_{00} + \frac{h_{101}}{A^6D^2\alpha} r s_{00} + \frac{h_{102}}{A^6D^2\alpha} r s_{00} + \frac{h_{103}}{A^6D^2\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{104}}{A^6D^2\alpha} r s_{00} + \frac{h_{105}}{A^6D^2\alpha} r s_{00} + \frac{h_{106}}{A^6D^2\alpha} r s_{00} + \frac{h_{107}}{A^6D^2\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{108}}{A^6D^2\alpha} r s_{00} + \frac{h_{109}}{A^6D^2\alpha} r s_{00} + \frac{h_{110}}{A^6D^2\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{111}}{A^6D^2\alpha} r s_{00} + \frac{h_{112}}{A^6D^2\alpha} r s_{00} + \frac{h_{113}}{A^6D^2\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{114}}{A^6D^2\alpha} r s_{00} + \frac{h_{115}}{A^6D^2\alpha} r s_{00} + \frac{h_{116}}{A^6D^2\alpha} r s_{00} \right] \nabla_j l_k
 + \left[ \frac{h_{117}}{A^6D^2\alpha} r s_{00} + \frac{h_{118}}{A^6D^2\alpha} r s_{00} + \frac{h_{119}}{A^6D^2\alpha} r s_{00} \right] \nabla_j l_k
 \end{align*} \]

where

\[ A := 1 + m(m-1)B - (m-2)s - (m^2-1)s^2, \quad D := (m-1)s - 1, \]
and $h_i$ (i = 1, 2, \cdots, 111) are the polynomials of variations $s$ and $B$. Substituting (3.6) in (3.4) and multiplying the result with $A^6D^4\alpha^4$ implies that

$$H_{jk}A^6D^4\alpha^4 - \frac{n+1}{2}\theta F_{y'y''}A^6D^4\alpha^4 = 0. \quad (3.7)$$

The following holds

$$\theta F_{y'y''}A^6D^4\alpha^4 \equiv 0, \pmod{(A)}.$$

Then (3.7) is equivalent to the following

$$\left[ h_{49}r_{00}s_0\alpha + h_{51}D^2r_{00}^2 + h_{56}s_0^2\alpha^2 \right] (l_j b_k + l_k b_j) + \left[ h_{11}r_{00}s_0\alpha + h_3D^2r_{00}^2 + h_8s_0^2\alpha^2 \right] b_j b_k$$

$$+ \left[ h_{15}r_{00}s_0\alpha + h_{17}D^2r_{00}^2 + h_{22}s_0^2\alpha^2 \right] l_j l_k \equiv 0, \pmod{(A)}. \quad (3.8)$$

Multiplying (3.8) with $b^j b^k$ yields

$$I_1r_{00}s_0\alpha + I_2r_{00}^2 + I_3s_0^2\alpha^2 \equiv 0, \pmod{(A)}, \quad (3.9)$$

where $I_i$ (i = 1, 2, 3), are polynomials of $s$ and $B$. Let us put

$$I_1 \equiv f_1 \quad \text{and} \quad I_2 \equiv f_2 \quad \text{and} \quad I_3 \equiv f_3, \pmod{(A)}.$$

Then by (3.9), we get (3.3). \hfill \Box

Now, we can prove Theorem 1.1.

**Proof of Theorem 1.1:** Let $\beta$ be a closed 1-form. Then (3.6) reduces to the following:

$$2H_{jk} = \left[ \frac{h_3}{A^6\alpha^4} + \frac{h_4}{A^6\alpha^4} r_{00} + \frac{h_5}{A^6\alpha^4} r_{00}^2 + \frac{h_6}{A^6\alpha^4} r_{00}^3 + \frac{h_7}{A^6\alpha^4} r_{00}^4 + \frac{h_8}{A^6\alpha^4} r_{00}^5 + \frac{h_9}{A^6\alpha^4} r_{00}^6 \right] b_j b_k$$

$$+ \left[ \frac{h_{10}}{A^6\alpha^4} r_{00}^2 + \frac{h_{11}}{A^6\alpha^4} r_{00}^3 + \frac{h_{12}}{A^6\alpha^4} r_{00}^4 + \frac{h_{13}}{A^6\alpha^4} r_{00}^5 + \frac{h_{14}}{A^6\alpha^4} r_{00}^6 \right] l_j l_k$$

$$+ \left[ \frac{h_{15}}{A^6\alpha^4} r_{00}^2 + \frac{h_{16}}{A^6\alpha^4} r_{00}^3 + \frac{h_{17}}{A^6\alpha^4} r_{00}^4 + \frac{h_{18}}{A^6\alpha^4} r_{00}^5 + \frac{h_{19}}{A^6\alpha^4} r_{00}^6 \right] a_{jk}$$

$$+ \left[ \frac{h_{20}}{A^6\alpha^4} r_{00}^2 + \frac{h_{21}}{A^6\alpha^4} r_{00}^3 + \frac{h_{22}}{A^6\alpha^4} r_{00}^4 + \frac{h_{23}}{A^6\alpha^4} r_{00}^5 + \frac{h_{24}}{A^6\alpha^4} r_{00}^6 \right] r_{00} j k$$

$$+ \left[ \frac{h_{25}}{A^6\alpha^4} r_{00}^2 + \frac{h_{26}}{A^6\alpha^4} r_{00}^3 + \frac{h_{27}}{A^6\alpha^4} r_{00}^4 + \frac{h_{28}}{A^6\alpha^4} r_{00}^5 + \frac{h_{29}}{A^6\alpha^4} r_{00}^6 \right] a_{jk}$$

$$+ \left[ \frac{h_{30}}{A^6\alpha^4} r_{00}^2 + \frac{h_{31}}{A^6\alpha^4} r_{00}^3 + \frac{h_{32}}{A^6\alpha^4} r_{00}^4 + \frac{h_{33}}{A^6\alpha^4} r_{00}^5 + \frac{h_{34}}{A^6\alpha^4} r_{00}^6 \right] r_{00} j k$$

$$+ \left[ \frac{h_{35}}{A^6\alpha^4} r_{00}^2 + \frac{h_{36}}{A^6\alpha^4} r_{00}^3 + \frac{h_{37}}{A^6\alpha^4} r_{00}^4 + \frac{h_{38}}{A^6\alpha^4} r_{00}^5 + \frac{h_{39}}{A^6\alpha^4} r_{00}^6 \right] a_{jk}$$

$$+ \left[ \frac{h_{40}}{A^6\alpha^4} r_{00}^2 + \frac{h_{41}}{A^6\alpha^4} r_{00}^3 + \frac{h_{42}}{A^6\alpha^4} r_{00}^4 + \frac{h_{43}}{A^6\alpha^4} r_{00}^5 + \frac{h_{44}}{A^6\alpha^4} r_{00}^6 \right] r_{00} j k$$

$$+ \left[ \frac{h_{45}}{A^6\alpha^4} r_{00}^2 + \frac{h_{46}}{A^6\alpha^4} r_{00}^3 + \frac{h_{47}}{A^6\alpha^4} r_{00}^4 + \frac{h_{48}}{A^6\alpha^4} r_{00}^5 + \frac{h_{49}}{A^6\alpha^4} r_{00}^6 \right] a_{jk}$$

$$+ \left[ \frac{h_{50}}{A^6\alpha^4} r_{00}^2 + \frac{h_{51}}{A^6\alpha^4} r_{00}^3 + \frac{h_{52}}{A^6\alpha^4} r_{00}^4 + \frac{h_{53}}{A^6\alpha^4} r_{00}^5 + \frac{h_{54}}{A^6\alpha^4} r_{00}^6 \right] r_{00} j k$$

$$+ \left[ \frac{h_{55}}{A^6\alpha^4} r_{00}^2 + \frac{h_{56}}{A^6\alpha^4} r_{00}^3 + \frac{h_{57}}{A^6\alpha^4} r_{00}^4 + \frac{h_{58}}{A^6\alpha^4} r_{00}^5 + \frac{h_{59}}{A^6\alpha^4} r_{00}^6 \right] a_{jk}$$

$$+ \left[ \frac{h_{60}}{A^6\alpha^4} r_{00}^2 + \frac{h_{61}}{A^6\alpha^4} r_{00}^3 + \frac{h_{62}}{A^6\alpha^4} r_{00}^4 + \frac{h_{63}}{A^6\alpha^4} r_{00}^5 + \frac{h_{64}}{A^6\alpha^4} r_{00}^6 \right] r_{00} j k.$$  

(3.10)
By substituting (3.10) in (3.4) and multiplying the result with $A^6\alpha^4$, we get

$$H_{jk}A^6\alpha^4 - \frac{n+1}{2} \theta F_{y'y^k}A^6\alpha^4 = 0.$$  
(3.11)

Since

$$\frac{n+1}{2} \theta F_{y'y^k}A^6\alpha^4 \equiv 0, \text{ mod}(A)$$

then (3.11) is equal to the following

$$\left[ h_{51}(l_jb_k + l_kb_j) + h_{3}b_jb_k + h_{17}l_jl_k \right]r_{00}^2 \equiv 0, \text{ mod}(A).$$  
(3.12)

Multiplying (3.12) with $b^i b^k$ yields

$$I_2r_{00}^2 \equiv 0, \text{ mod}(A),$$

where $I_2$ is a polynomial of s and B. Then we get

$$f_2r_{00}^2 \equiv 0 \text{ mod}(A),$$

where $I_2 \equiv f_2 \text{ mod}(A)$, and $f_2$ is a polynomial of s and B and of degree 1 in s. By Lemma 3.1, it follows that $\beta$ is parallel with respect to $\alpha$. Plugging this in (3.10) yields $H = 0$. The converse is trivial. On the other hand, every regular $(\alpha, \beta)$-metric is a Berwald metric if and only if $\beta$ is parallel with respect to $\alpha$. This completes the proof.

4. Proof of Theorem 1.5

In this section, we are going to prove Theorem 1.5. First, we prove the following.

**Lemma 4.1** Let $F = \alpha \varphi(s)$, $s = \beta/\alpha$, be a polynomial $(\alpha, \beta)$-metric of degree $m$ on an n-dimensional manifold $M$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M$. Suppose that $F$ is an Einstein metric. Then the following holds

$$g_1r_{00}s_0\alpha + g_2r_{00}^2 + g_3s_0^2\alpha^2 \equiv 0, \text{ mod}\left((1 - m^2)s^2 + (2 - m)s + m(m - 1)B + 1\right),$$  
(4.1)

where $g_j$ ($j = 1, 2, 3$), are polynomials of variations $B$ and $s$.

**Proof** Let $\varphi(s) = (1 + s)^m$ ($m \geq 3$) be an Einstein metric. By the Theorem 1.1 in [2], $F$ is Ricci-flat. Then

$$R_m^\alpha = R_m^\alpha + T_m^\alpha = 0,$$  
(4.2)

where $R_m^\alpha$ denotes the Riemannian curvature of $\alpha$ and

$$T_m^\alpha = \left[(n - 1)\frac{c_1}{A^3} + \frac{c_2}{A^4}\right]r_{00}^2 + \frac{1}{\alpha} \left[(n - 1)\frac{c_3}{A^4D} + \frac{c_4}{A^4D^2}\right]r_{00}s_0 + r_0 \left[(n - 1)\frac{c_5}{A^2} + \frac{c_6}{A^3}\right]r_{00}$$

$$+ \left[(n - 1)\frac{c_7}{A} + \frac{c_8}{A^2}\right]r_{000}^2 + \left[(n - 1)\frac{c_9}{A^3D^3} + \frac{c_{10}}{A^4D^3}\right]s_0^2 + \frac{c_{11}}{A^2}(rr_{00} - r_0^2)$$
For the exponential metric variations

Lemma 5.1

In this section, we are going to prove Theorem 5. Proof of Theorem

where \( h \equiv 1 + m(m - 1)B - (m - 2)s - (m^2 - 1)s^2 \), \( D := (m - 1)s - 1 \) and \( c_i (i = 1, \ldots, 26) \), are polynomials of variations \( s \) and \( B \) (see [2] for the corrected version of [25]). Putting (4.3) in (4.2) and multiplying the result with \( A^4D^3\alpha^2 \) implies that

\[ \alpha R_m^m A^4D^3\alpha^2 + T_m^m A^4D^3\alpha^2 = 0. \]

\( \alpha R_m^m \) is a polynomial with respect to \( s \) and \( B \). Since \( \alpha R_m^m A^4D^3\alpha^2 \equiv 0, \text{ mod}(A) \), then we get \( T_m^m A^4D^3\alpha^2 \equiv 0, \text{ mod}(A) \). By (4.3), we obtain

\[ r_{00}s_0^2c_4D^2 + r_{00}^2c_2D^3 + s_0^2\alpha^2c_{10} \equiv 0, \text{ mod}(A). \]

Put

\[ c_4D^2 \equiv g_1 \quad \text{and} \quad c_2D^3 \equiv g_2 \quad \text{and} \quad c_{10} \equiv g_3 \text{ mod}(A). \]

Then we get (4.1).

Proof of the Theorem 1.5: Let \( \beta \) be a closed 1-form on \( M \). By Lemma 4.1, we get

\[ g_2r_{00}^2 \equiv 0, \text{ mod}(A), \]

where \( I_2 \equiv g_2, \text{ mod}(A) \), and \( g_2 \) is polynomials of \( s \) and \( B \) and of degree 1 in \( s \). By Lemma 3.1, it follows that \( \beta \) is Killing. Then \( \beta \) is parallel with respect to \( \alpha \). In this case, \( F \) reduces to a Berwald metric.

5. Proof of Theorem 1.9

In this section, we are going to prove Theorem 1.9. First, we prove the following.

Lemma 5.1 Let \( F = \alpha \varphi(s), \ s = \beta/\alpha, \) be an exponential metric on an \( n \)-dimensional manifold \( M \), where \( \alpha = \sqrt{g_{ij}(x)g^{ij}} \) is a Riemannian metric and \( \beta = b_i(x)g^i \) is a 1-form on \( M \). Suppose that \( F \) has almost vanishing \( H \)-curvature. Then the following holds

\[ h_1r_{00}s_0\alpha + h_2r_{00}^2 + h_3s_0^2\alpha^2 \equiv 0, \quad \text{mod}(-s^2 - s + B + 1), \]

where \( h_j (j = 1, 2, 3) \) are polynomials of variations \( s \) and \( B \) and of degree one in \( s \).

Proof For the exponential metric \( \varphi(s) = e^s \), we have

\[ Q = \frac{1}{1 - s}, \quad \Theta = \frac{2s - 1}{2(s^2 + s - B - 1)}, \quad \Psi = \frac{1}{2(1 + B - s - s^2)}. \]
By assumption, \( F = \alpha \varphi (s) \), \( s = \beta / \alpha \), has almost vanishing \( H \)-curvature, i.e. there exists a 1-form \( \theta \) on \( M \) such that

\[
H_{jk} = \frac{n+1}{2} \theta F_{y^j y^k},
\]

where

\[
F_{y^j y^k} = \frac{e^s}{\alpha} \left[ (1-s)a_{jk} + b_j b_k - s(b_j l_k + b_k l_j) + (s^2 + s - 1) l_j l_k \right],
\]

and

\[
2H_{jk} = \left[ \frac{h_1}{\alpha^3 A^6(s-1)^2} r_{00}s_0 + \frac{h_2}{\alpha^2 A^4(s-1)^3} s_{00} + \frac{h_3}{\alpha^4 A^6} r_{00} + \frac{h_4}{\alpha^3 A^6} r_{00} + \frac{h_5}{\alpha^5 A^6} r_{00} \right]
\]

\[
+ \frac{h_6}{\alpha^4 A^3} r_{00} + \frac{h_7}{\alpha A^4} (s-1)^4 t_0 + \frac{h_8}{\alpha^2 A^6(s-1)^2} s_0 + \frac{h_9}{\alpha^2 A^4} (s-1)^2 q_0 + \frac{h_{10}}{\alpha A^4} (s-1)^2 q_0
\]

\[
+ \frac{h_{11}}{\alpha^2 A^6(s-1)^3} r_{00} + \frac{h_{12}}{\alpha^3 A^4} (s-1)^2 r_{00} + \frac{h_{13}}{\alpha^2 A^4} r_{00} + \frac{h_{14}}{\alpha^2 A^4} r_{00} b_j b_k
\]

\[
+ \left[ \frac{h_{15}}{\alpha^3 A^6(s-1)^3} r_{00} + \frac{h_{16}}{\alpha^2 A^6(s-1)^3} r_{00} + \frac{h_{17}}{\alpha A^4} r_{00} + \frac{h_{18}}{\alpha^2 A^4} r_{00} + \frac{h_{19}}{\alpha^3 A^6} r_{00} \right]
\]

\[
+ \frac{h_{20}}{\alpha^4 A^3} r_{00} + \frac{h_{21}}{\alpha^3 A^4} (s-1)^4 t_0 + \frac{h_{22}}{\alpha^2 A^6(s-1)^2} s_0 + \frac{h_{23}}{\alpha^2 A^4} (s-1)^2 q_0 + \frac{h_{24}}{\alpha A^4} (s-1)^2 q_0
\]

\[
+ \frac{h_{25}}{\alpha^2 A^6(s-1)^3} r_{00} + \frac{h_{26}}{\alpha A^4} (s-1)^2 r_{00} + \frac{h_{27}}{\alpha^2 A^4} r_{00} + \frac{h_{28}}{\alpha^2 A^4} r_{00} l_j l_k
\]

\[
+ a_{jk} \left[ \frac{h_{29}}{\alpha^3 A^6(s-1)^2} r_{00} + \frac{h_{30}}{\alpha^2 A^6(s-1)^3} s_0 + \frac{h_{31}}{\alpha A^4} r_{00} + \frac{h_{32}}{\alpha^2 A^4} r_{00} + \frac{h_{33}}{\alpha^3 A^6} r_{00} \right]
\]

\[
+ \frac{h_{34}}{\alpha^4 A^3} r_{00} + \frac{h_{35}}{\alpha^3 A^4} (s-1)^3 t_0 + \frac{h_{36}}{\alpha^2 A^6(s-1)^3} s_0 + \frac{h_{37}}{\alpha A^4} (s-1)^3 t_0 + \frac{h_{38}}{\alpha^2 A^4} (s-1)^3 t_0
\]

\[
+ \frac{h_{39}}{\alpha^2 A^2(s-1)^3} q_0 + \frac{h_{40}}{\alpha A^3} (s-1)^3 t_0 + \frac{h_{41}}{\alpha^2 A^3} r_{00} + \frac{h_{42}}{\alpha A^4} r_{00} + \frac{h_{43}}{\alpha^2 A^3} r_{00} + \frac{h_{44}}{\alpha^4 A^3(s-1)^3} r_{00} s_0
\]

\[
+ \frac{h_{45}}{\alpha A^4} (s-1)^3 t_0 + \frac{h_{46}}{\alpha A^3} r_{00} + \frac{h_{47}}{\alpha^2 A^4} r_{00} l_j l_k + \frac{h_{48}}{\alpha A^4} r_{00} + \frac{h_{49}}{\alpha^3 A^6(s-1)^3} r_{00} s_0
\]

\[
+ \frac{h_{50}}{\alpha^2 A^4(s-1)^3} s_0 + \frac{h_{51}}{\alpha A^3} r_{00} + \frac{h_{52}}{\alpha^2 A^4} r_{00} + \frac{h_{53}}{\alpha^2 A^4} r_{00} + \frac{h_{54}}{\alpha A^4} r_{00} + \frac{h_{55}}{\alpha^3 A^6} r_{00} s_0
\]

\[
+ \frac{h_{56}}{\alpha^2 A^4(s-1)^2} s_0 + \frac{h_{57}}{\alpha A^4} (s-1)^2 t_0 + \frac{h_{58}}{\alpha^2 A^4(s-1)^2} s_0 + \frac{h_{59}}{\alpha A^4} (s-1)^2 t_0 + \frac{h_{60}}{\alpha A^4} (s-1)^2 t_0
\]

\[
+ \frac{h_{61}}{\alpha^2 A^4} r_{00} + \frac{h_{62}}{\alpha^2 A^4} l_j l_k + \frac{h_{63}}{\alpha A^6(s-1)^3} r_{00} + \frac{h_{64}}{\alpha A^4} r_{00} + \frac{h_{65}}{\alpha A^4} r_{00} s_0
\]

\[
+ \frac{h_{66}}{\alpha^3 A^6(s-1)^2} r_{00}
\]
\[ A = 1 + B - s - s^2 \text{ and } h_i (i = 1, 2, \cdots, 111) \text{ are the polynomials of } s \text{ and } B. \text{ Putting (5.3) in (3.4) and multiplying the result with } A^6(s-1)^4\alpha^4 \text{ implies that} \]

\[ H_{ijk} A^6\alpha^4(s-1)^4 - \frac{n+1}{2} \theta F_{y'y''} A^6\alpha^4(s-1)^4 = 0. \]  

(5.3)

Since \( \theta F_{y'y''} A^6\alpha^4(s-1)^4 \equiv 0, \mod(A) \), then (5.3) is equal to

\[ \left[ h_{49} (s-1)\alpha s_0 r_{00} + h_{51} (s-1)^4r_{00} + h_{56} \alpha^2 s_0^2 \right] (l_j b_k + l_k b_j) + \left[ h_1 (s-1)\alpha s_0 r_{00} + h_8 \alpha^2 s_0^2 \right] l_j l_k \equiv 0, \mod(A). \]

(5.4)

Multiplying (5.4) with \( b^*b^k \) yields \( I_1 r_{00} s_0 \alpha + I_2 r_{00}^2 + I_3 s_0^2 \alpha^2 \equiv 0, \mod(A) \), where \( I_i, (i = 1, 2, 3) \) are polynomials of variations \( s \) and \( B \). Put \( I_1 \equiv h_1, I_2 \equiv h_2 \) and \( I_3 \equiv h_3 \mod(A) \). Then, we get (5.1). \( \square \)

**Proof of Theorem 1.9:** Let \( \beta \) be a closed 1-form on \( M \). By Lemma 5.1, we get \( h_{2r_{00}}^2 \equiv 0, \mod(A) \), where \( I_2 \equiv h_2, \mod(A) \), and \( h_2 \) is a polynomial of \( s \) and \( B \) and of degree 1 in \( s \). By Lemma 3.1, \( \beta \) is Killing. Putting it in (5.3) yields \( H = 0 \). The converse is trivial. In this case, it follows that \( \beta \) is parallel with respect to \( \alpha \). Then, \( F \) reduces to a Berwald metric.

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6. Proof of Theorem 1.11
In this section, we are going to prove Theorem 1.11. For this aim, we need the following.

Lemma 6.1 Let \( F = \alpha \varphi(s) \), \( s = \beta/\alpha \), be an exponential metric on an \( n \)-dimensional manifold \( M \), where \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a \( 1 \)-form on \( M \). Suppose that \( F \) is an Einstein metric. Then the following holds

\[
k_1 r_{00}s_0 \alpha + k_2 r_{00}^2 + k_3 s_0^2 \alpha^2 \equiv 0, \quad \text{mod}(-s^2 - s + B + 1),
\]

(6.1)

where \( k_j \), \( (j = 1, 2, 3) \), are polynomials of variations \( B \) and \( s \).

**Proof** For the exponential metric \( \varphi(s) = e^s \), we have

\[
R_m^a = ^a R_m^a + T_m^a = \text{Ric}(x)F^2,
\]

(6.2)

where

\[
T_m^a := \left( n - 1 \right) \frac{c_1}{A} + \frac{c_2}{A^2} \left[ (n - 1) \frac{c_3}{A^3 D} + \frac{c_4}{A^2 D^2} \right] r_{00} s_0 + \left( n - 1 \right) \frac{c_5}{A^2} + \frac{c_6}{A} \right] r_{00} r_0
\]

\[
+ \left( n - 1 \right) \frac{c_7}{A} + \frac{c_8}{A^2} \left[ r_{00} s_0 \right] + \left[ (n - 1) \frac{c_9}{A^3 D^3} + \frac{c_{10}}{A^2 D^3} \right] s_0^2 + \frac{c_{11}}{A^2} (r_{00} - r_0^2)
\]

\[
+ \left( n - 1 \right) \frac{c_{12}}{A^2 D} + \frac{c_{13}}{A^3 D} \right] r_0 s_0 + \frac{c_{14}}{A} (r_{00} m - r_{00} m + r_{00} m + r_{00} m b_m - r_{00} m b_m)
\]

\[
+ \left( n - 1 \right) \frac{c_{15}}{AD} + \frac{c_{16}}{A D} \right] r_0 s_0 + \left( n - 1 \right) \frac{c_{17}}{A D} + \frac{c_{18}}{A D} \right] s_0 |0| + \frac{c_{19}}{D s_0 m s_0}
\]

\[
+ \left[ \frac{c_{20}}{A D} r s_0 + \left( n - 1 \right) \frac{c_{21}}{A D^2} + \frac{c_{22}}{A D^2} \right] s_0 m + \frac{c_{23}}{AD} \left( 3 s_0 m - 2 s_0 m + 2 r_0 m s_0 m
\]

\[
- 2 s_0 m b_m + s_0 m b_m \right) + \frac{c_{24}}{D s_0 m} \right] \alpha + \left[ \frac{c_{25}}{A D} s_0 m + \frac{c_{26}}{D^2} s_0 m \right] \alpha^2,
\]

(6.3)

\( A = 1 + B - s - s^2 \), \( D = s - 1 \) and \( c_i \), \( (i = 1, \cdots, 26) \), are polynomials of variations \( s \) and \( B \) (see [2]). Putting \( T_m^a \) into (6.2) and multiplying the result with \( A D^3 \alpha^2 \) implies that

\[
^a R_m^a A D^3 \alpha^2 + T_m^a A D^3 \alpha^2 - \text{Ric}(x)F^2 A D^3 \alpha^2 = 0.
\]

\( ^a R_m^a - \text{Ric}(x)F^2 \) is a polynomial of \( s \) and \( B \). Thus,

\[
^a R_m^a A D^3 \alpha^2 - \text{Ric}(x)F^2 A D^3 \alpha^2 \equiv 0, \quad \text{mod}(A).
\]

Then \( T_m^a A D^3 \alpha^2 \equiv 0, \quad \text{mod}(A) \). By (6.3), we get \( r_{00} s_0 c_4 D^2 + r_{00}^2 c_2 D^3 + s_0^2 \alpha^2 c_{10} \equiv 0, \quad \text{mod}(A) \). Put

\[
c_4 D^2 \equiv h_1 \quad \text{and} \quad c_2 D^3 \equiv h_2 \quad \text{and} \quad c_{10} \equiv h_3, \quad \text{mod}(A).
\]

Then, we get (6.1). \( \square \)

**Proof of Theorem 1.11**: By Lemma 6.1, we have (6.1). Let \( \beta = b_i(x)y^i \) be a closed 1-form. Then \( k_2 r_{00}^2 \equiv 0, \quad \text{mod}(A) \), where \( I_2 \equiv k_2 \quad \text{mod}(A) \), and \( k_2 \) is a polynomial of variations \( s \) and \( B \). By Lemma 3.1, \( \beta \) is a Killing 1-form. It follows that \( \beta \) is parallel with respect to \( \alpha \). In this case, \( F \) reduces to a Berwald metric.
References


