On the product of dilation of truncated Toeplitz operators

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Received: 13.08.2019 • Accepted/Published Online: 11.11.2019 • Final Version: 20.01.2020

Abstract: In this paper we study when the product of two dilations of truncated Toeplitz operators gives a dilation of a truncated Toeplitz operator. We will use an approach established in a recent paper written by Ko and Lee. This approach allows us to represent the dilation of the truncated Toeplitz operator via a $2 \times 2$ block operator.

Key words: Model space, truncated Toeplitz operator, dilation of truncated Toeplitz operator

1. Introduction

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. We start by recalling that the Hilbert space $L^2 = L^2(\mathbb{T})$ is the space of all square-integrable functions on the unit circle $\mathbb{T}$ equipped with the normalized Lebesgue measure $dm(e^{i\theta}) = \frac{d\theta}{2\pi}$. This space is endowed with the scalar product $\langle f, g \rangle = \int_{\mathbb{T}} fg dm$.

An orthonormal basis of $L^2$ is given by the set $\{e_n(\theta) : n \in \mathbb{Z}\}$, where $e_n(\theta) = e^{i n \theta}$ for $\theta \in \mathbb{R}$. The following orthonormal expansions are the classical Fourier series:

$$ f = \sum_{n=-\infty}^{+\infty} f_n e_n = \sum_{n=-\infty}^{+\infty} f_n e^{i n \theta}, $$

$$ f_n = \langle f, e_n \rangle = \int_{0}^{2\pi} f(e^{i\theta}) e^{-i n \theta} \frac{d\theta}{2\pi}, n \in \mathbb{Z}. $$

For all $f, g \in L^2$, the tensor product $f \otimes g$ is the rank one operator in $L^2$ and is defined by

$$ (f \otimes g) h = \langle h, g \rangle f $$

for $h \in L^2$. Let $L^\infty$ be the Banach space of essentially bounded functions on $\mathbb{T}$. For any $\varphi \in L^\infty$, the bounded multiplication operator $M_\varphi$ is defined by the formula

$$ M_\varphi f = \varphi f, f \in L^2. $$

An operator $A$ is a multiplication operator if and only if $AM_z = M_z A$. It is well known that, for all $\varphi \in L^\infty$, the multiplication operator $M_\varphi$ is invertible if and only if $\varphi$ is invertible in $L^\infty$. Moreover, $(M_\varphi)^{-1} = M_{\varphi^{-1}}$. The Hardy space of the circle $H^2$ is the set of functions $f \in L^2$ such that $f_n = 0$ for all $n < 0$, and let

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2010 AMS Mathematics Subject Classification: Primary: 47B35; Secondary: 47A05

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Let $H^\infty$ be the set of functions $f \in L^\infty$ such that $f_n = 0$ for all $n < 0$. We introduce now an important class of operators on spaces of analytic functions, which is the class of Toeplitz operators. Let $P$ and $Q = I - P$ indicate the orthogonal projections that map $L^2$ onto $H^2$ and $(H^2)^\perp = \overline{zH^2}$, respectively. Given that $\varphi \in L^\infty$, the Toeplitz operator $T_\varphi : H^2 \to H^2$ is defined by

$$T_\varphi f = P(\varphi f), f \in H^2$$

and the Hankel operator $H_\varphi : H^2 \to (H^2)^\perp$ is defined by

$$H_\varphi f = Q(\varphi f), f \in H^2.$$

Hankel operators play an important role in the study of Toeplitz operators, and vice versa. Note that the Toeplitz operator becomes bounded if and only if $\varphi \in L^\infty$. In this case, we have $\|T_\varphi\| = \|\varphi\|_\infty$ (see [1]). For any $\varphi, \psi \in L^\infty$, the singular integral operator $S_{\varphi, \psi} : L^2 \to L^2$ is defined by

$$S_{\varphi, \psi}(f) = \varphi P(f) + \psi Q(f), f \in L^2.$$

With respect to the decomposition $L^2 = H^2 \oplus (H^2)^\perp$, the operator $S_{\varphi, \psi}$ can be represented as follows:

$$S_{\varphi, \psi} = \begin{pmatrix} T_\varphi & \overline{H_\psi} \\ H_\varphi & T_\psi \end{pmatrix},$$

where $T_\varphi$ and $H_\varphi$ are the Toeplitz operator and Hankel operator, respectively. For more information about the operators $T_\varphi$ and $H_\varphi$, see [6]. Ko and Lee concluded that the operator $S_{\varphi, \psi}$ is the dilation of a Toeplitz operator on $L^2$ [5].

An inner function is an $H^\infty$ function that has unit modulus almost everywhere on $T$. For a nonconstant inner function $u$, the model space $K_u^2$ is defined by

$$K_u^2 = H^2 \ominus uH^2 = \{ f \in H^2 : (f, ug) = 0, \forall g \in H^2 \}.$$

The space $K_u^\infty$ is defined by $K_u^\infty = K_u^2 \cap L^\infty$, which is dense in $K_u^2$. For any $\varphi \in L^\infty$ and an inner function $u$, the truncated Toeplitz operator $A_\varphi^u$ on $K_u^2$ is defined by

$$A_\varphi^u f = P_u(\varphi f), f \in K_u^2, \quad (1.1)$$

where $P_u = P - M_u P M_u$ denotes the orthogonal projection that maps $L^2$ onto $K_u^2$.

For any $\varphi \in L^\infty$ and an inner function $u$, the dual of truncated Toeplitz operator $\widetilde{A_\varphi^u}$ is the operator on $(K_u^2)^\perp$ defined as follows:

$$\widetilde{A_\varphi^u} = Q_u(\varphi f), f \in (K_u^2)^\perp, \quad (1.2)$$

where $Q_u = I - P_u$ refers to the orthogonal projection that maps $L^2$ onto $(K_u^2)^\perp = L^2 \ominus K_u^2 = \overline{zH^2} \ominus uH^2$. The truncated Hankel operator $\Gamma_\varphi^u : K_u^2 \to (K_u^2)^\perp$ is defined by

$$\Gamma_\varphi^u f = Q_u(\varphi f), f \in K_u^2, \quad (1.3)$$

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Let $\tilde{\Gamma}_u^\varphi$ be the operator of $(K_u^2)^\perp$ to $K_u^2$ such that
\[\tilde{\Gamma}_u^\varphi f = P_u(\varphi f), f \in (K_u^2)^\perp.\] (1.4)

From [5], we will use what can be helpful to us in our following work, notably the following identity:
\[\tilde{\Gamma}_u^\varphi = (\Gamma_u^\varphi)^*.\] (1.5)

In 1963, in a famous paper on algebraic properties of Toeplitz operators [1], Brown and Halmos studied when the product of two Toeplitz operators itself becomes a Toeplitz operator. The same issue about truncated Toeplitz operators was solved by Sedlock in 2010 [8]. In 2015, Gu in [3] proved that the product $S_{\varphi_1, \psi_1} S_{\varphi_2, \psi_2}$ on $L^2$ is a singular integral operator if and only if $\varphi_2 \in H^\infty, \psi_2 \in \overline{H^\infty}$.

**Definition 1.1** [5] For $\varphi, \psi \in L^\infty$ and an inner function $u$, the dilation of truncated Toeplitz operator $S_{\varphi, \psi}^u : L^2 \to L^2$ is defined by the formula
\[S_{\varphi, \psi}^u(f) = \varphi P_u(f) + \psi Q_u(f), f \in L^2.\]

Obviously, the operator $S_{\varphi, \psi}^u$ is a bounded operator if and only if $\varphi, \psi \in L^\infty$, such that
\[\|S_{\varphi, \psi}^u(f)\| \leq \|\varphi P_u(f)\| + \|\psi Q_u(f)\| \leq (\|\varphi\|_\infty + \|\psi\|_\infty)\|f\|.\]

Note that for $f \in L^2$, we have
\[S_{\varphi, \psi}^u f = \varphi P_u f + \psi Q_u f = \varphi P_u f + \psi[f - P_u f] = (\varphi - \psi) P_u f + \psi f.\]

Hence, it is easy to see that $S_{\varphi, \psi}^u = M_\psi + S_{\varphi - \psi, 0}^u$ and $S_{\varphi, \psi}^u = M_\varphi$.

The class of dilation of truncated Toeplitz operators was introduced in 2015 by Ko and Lee. For further details of the introduction of this class of operators, see [5]. Moreover, relying on the decomposition $L^2 = K_u^2 \oplus (K_u^2)^\perp$, they proved that the operator $S_{\varphi, \psi}^u$ has the following matrix representation:
\[S_{\varphi, \psi}^u = \begin{pmatrix} A_{\varphi}^u & \tilde{\Gamma}_u^\varphi \\ \Gamma_u^\varphi & A_{\psi}^u \end{pmatrix},\] (1.6)

where $A_{\varphi}^u, \tilde{A}_{\varphi}^u, \Gamma_u^\varphi$, and $\tilde{\Gamma}_u^\varphi$ are defined by equations (1.1), (1.2), (1.3), and (1.4), respectively. We refer to [5, Lemma 3.2] for more details about this representation.

Recently, Gu and Kang gave in [4] a complete characterization when $S_{\varphi, \psi}^u$ is a self-adjoint, isometric, coisometric, and normal operator using their important key observation where $S_{\varphi, \psi}^u$ and $M_z$ are almost commuting. As shown in [4, lemma 3.1], Gu and Kang proved that the operator $S_{\varphi, \psi}^u$ satisfies the following equation:
\[S_{\varphi, \psi}^u - M_z S_{\varphi, \psi}^u M_z^* = (\varphi - \psi) \otimes e_0 - (\varphi - \psi) u \otimes u e_0.\] (1.7)

In this work, we study the product of two dilations of truncated Toeplitz operators $S_{\varphi_1, \psi_1}^u$ and $S_{\varphi_2, \psi_2}^u$. 

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2. Characterization

Let $B(L^2)$ be the algebra of all bounded linear operators on $L^2$. For an operator $A \in B(L^2)$, the operator $A^*$ is called the adjoint of $A$. For an inner function $u \in H^2$, $D_u$ denotes the set of all dilations of truncated Toeplitz operators on $L^2$:

$$D_u = \{ S_{\varphi,\psi}^u \in B(L^2), \varphi, \psi, \in L^\infty \}.$$  

In [4] Gu and Kang gave a full characterization of the class of operators $D_u$ as described in the following lemma.

Lemma 2.1 [4] Let $A \in B(L^2)$. Then $A \in D_u$ if and only if there exists a $\chi \in L^\infty$ such that

$$A - M_zAM_z^* = \chi \otimes e_0 - \chi u \otimes u e_0.$$  

(2.1)

In this case, $A = S_{\chi,\theta,\theta}^u$ for some $\theta \in L^\infty$.

Remark 2.2 [4] Let $\varphi, \psi$ be in $L^\infty$. Then for all $f, g \in L^2$ we have

$$\langle S_{\varphi,\psi}^u f, g \rangle = \langle \varphi P_u(f) + \psi Q_u(f), g \rangle = \langle f, P_u(\overline{\varphi}g) \rangle + \langle f, Q_u(\overline{\psi}g) \rangle.$$  

Therefore,

$$(S_{\varphi,\psi}^u)^* f = P_u(\overline{\varphi}f) + Q_u(\overline{\psi}f), f \in L^2.$$  

Proposition 2.3 Let $\varphi \in L^\infty$ and let $S_{1,0}^u, S_{\varphi,0}^u \in D_u$. Then

$$(S_{1,0}^u S_{\varphi,0}^u)^* = S_{1,0}^u S_{\varphi,0}^u.$$  

Proof Since $S_{\varphi,0}^u = M_\varphi S_{1,0}^u$ and $(S_{1,0}^u)^* = S_{1,0}^u$, we obtain

$$(S_{1,0}^u S_{\varphi,0}^u)^* = (S_{1,0}^u M_\varphi S_{1,0}^u)^* = (S_{1,0}^u)^* M_\varphi^* (S_{1,0}^u)^* = S_{1,0}^u M_\varphi S_{1,0}^u = S_{1,0}^u S_{\varphi,0}^u.$$  

3. Product of dilation of truncated Toeplitz operators

To arrive at the main result of this work, we need the following lemma and proposition.

Lemma 3.1 Letting $\varphi \in L^\infty$, the following statements hold:

1. $A_{\varphi}^u = 0$ if and only if $\varphi \in uH^\infty + \overline{uH^\infty}$.
2. $\widetilde{A_{\varphi}^u} = 0$ if and only if $\varphi = 0$.
3. $\Gamma_{\varphi}^u = 0$ if and only if $\varphi \in K_u^\infty$.
4. $\widetilde{\Gamma_{\varphi}^u} = 0$ if and only if $\varphi \in \overline{K_u^\infty}$.
Proof

1. This statement is an important result in Sarason’s paper; see [7, Theorem 3.1].

2. Since \( \varphi \in L^\infty \), it follows from Property 2.1 in [2] that \( \widetilde{A}_\varphi^u \) is a bounded operator and \( \|\widetilde{A}_\varphi^u\| = \|\varphi\|_\infty \).

Thus, \( \widetilde{A}_\varphi^u = 0 \) if and only if \( \varphi = 0 \).

According to the proof of Theorem 3.14 in [5, p. 15] and equation (1.5), we deduce statements 3) and 4).

\( \square \)

**Proposition 3.2** Let \( u \) be an inner function, \( \varphi_1, \psi_1, \varphi_2, \psi_2 \in L^\infty \). Let \( S_{\varphi_1,\psi_1}^u, S_{\varphi_2,\psi_2}^u \in D_u \), and then the following statements hold:

1. \( S_{\varphi_1,\psi_1}^u S_{\varphi_2,\psi_2}^u \in D_u \) if and only if \( M_{\varphi_1-\psi_1} S_{1,0}^{\varphi_1,\psi_1} S_{\varphi_2,\psi_2}^u \in D_u \).

2. If \( \varphi_1 - \psi_1 \) is invertible in \( L^\infty \) then \( S_{\varphi_1,\psi_1}^u S_{\varphi_2,\psi_2}^u \in D_u \) if and only if \( S_{1,0}^{\varphi_1,\psi_1} S_{\varphi_2,\psi_2}^u \in D_u \).

**Proof**

1. It is clear that

\[
S_{\varphi_1,\psi_1}^u = M_{\psi_1} + S_{\varphi_1-\psi_1,1,0} = M_{\psi_1} + M_{\varphi_1-\psi_1} S_{1,0}^u.
\]

Therefore,

\[
S_{\varphi_1,\psi_1}^u S_{\varphi_2,\psi_2}^u = (M_{\psi_1} + M_{\varphi_1-\psi_1} S_{1,0}^u) S_{\varphi_2,\psi_2}^u = S_{\varphi_2,\psi_2}^u + M_{\varphi_1-\psi_1} S_{1,0}^{\varphi_1,\psi_1} S_{\varphi_2,\psi_2}^u.
\]

We deduce that \( S_{\varphi_1,\psi_1}^u S_{\varphi_2,\psi_2}^u \in D_u \) if and only if \( M_{\varphi_1-\psi_1} S_{1,0}^{\varphi_1,\psi_1} S_{\varphi_2,\psi_2}^u \in D_u \).

2. From the above, we obtain that

\[
M_{\varphi_1-\psi_1} S_{1,0}^{\varphi_1,\psi_1} S_{\varphi_2,\psi_2}^u = S_{\varphi_1,\psi_1}^u S_{\varphi_2,\psi_2}^u - S_{\varphi_2,\psi_2}^u S_{\varphi_1,\psi_1}^u.
\]

If \( \varphi_1 - \psi_1 \) is invertible, then

\[
S_{1,0}^{\varphi_1,\psi_1} S_{\varphi_2,\psi_2}^u = M_{(\varphi_1-\psi_1)^{-1}}(S_{\varphi_1,\psi_1}^u S_{\varphi_2,\psi_2}^u - S_{\varphi_2,\psi_2}^u S_{\varphi_1,\psi_1}^u).
\]

Thus, we conclude that \( S_{\varphi_1,\psi_1}^u S_{\varphi_2,\psi_2}^u \in D_u \) if and only if \( S_{1,0}^{\varphi_1,\psi_1} S_{\varphi_2,\psi_2}^u \in D_u \).

\( \square \)

The main result of this paper is the following theorem.

**Theorem 3.3** Let \( \varphi, \psi \in L^\infty \) and let \( u \) be a nonconstant inner function. Then \( S_{1,0}^u S_{\varphi,\psi}^u \in D_u \) if and only if \( \varphi \in K_u^\infty \cup uH_\infty + \overline{uH_\infty}, \psi \in K_u^\infty \). In this case,

\[
S_{1,0}^u S_{\varphi,\psi}^u = S_{1,0}^u\varphi,0.
\]
Proof By the representation (1.6), we have

\[ S_{\varphi, \psi}^u = \begin{pmatrix} A_{\varphi}^u & \Gamma_{\varphi}^u \\ \Gamma_{\psi}^u & A_{\psi}^u \end{pmatrix} \]

and

\[ S_{1,0}^u = \begin{pmatrix} A_{1}^u & \Gamma_{0}^u \\ \Gamma_{1}^u & A_{0}^u \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \]

This means that

\[ S_{1,0}^u S_{\varphi, \psi}^u = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{\varphi}^u & \Gamma_{\varphi}^u \\ \Gamma_{\psi}^u & A_{\psi}^u \end{pmatrix} = \begin{pmatrix} A_{\varphi}^u & \Gamma_{\varphi}^u \\ 0 & 0 \end{pmatrix}. \]

For each \( \Phi, \Psi \in L^\infty \), we put

\[ S_{1,0}^u S_{\varphi, \psi}^u = S_{\Phi, \Psi}^u = \begin{pmatrix} A_{\Phi}^u & \Gamma_{\Phi}^u \\ \Gamma_{\Psi}^u & A_{\Psi}^u \end{pmatrix}. \]

Then

\[ \begin{pmatrix} A_{\Phi - \varphi}^u & \Gamma_{\Phi - \psi}^u \\ \Gamma_{\Phi}^u & A_{\Psi}^u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \]

Hence,

\[ A_{\Phi - \varphi}^u = 0, \Gamma_{\Phi}^u = 0, \Gamma_{\Phi - \psi}^u = 0, A_{\Psi}^u = 0. \]

Since \( A_{\Phi - \varphi}^u = 0 \) and \( \tilde{A}_{\Phi}^u = 0 \), it follows from Lemma 3.1 that \( \Phi - \varphi \in uH^\infty + \overline{uH^\infty} \) and \( \Psi = 0 \). In the same way, since \( \Gamma_{\Phi}^u = 0 \) and \( \Gamma_{\Phi - \psi}^u = 0 \) and seeing that

\[ 0 = \Gamma_{\Psi - \psi}^u = (\Gamma_{\Psi - \psi}^u)^* \]

is equivalent to \( \Gamma_{\Psi - \psi}^u = 0 \), it results from Lemma 3.1 that \( \Phi \in K^\infty_u \) and \( \Psi - \psi \in K^\infty_u \). From the above, we conclude that

\[ \varphi = \Phi + \varphi_1 \]

for \( \Phi \in K^\infty_u \) and \( \varphi_1 \in uH^\infty + \overline{uH^\infty} \), and

\[ \psi \in K^\infty_u. \]

At last, we have

\[ \varphi \in K^\infty_u + uH^\infty + \overline{uH^\infty} \]
and
\[ \psi \in \overline{K_u}. \]

Observe that \( \Phi = P_u \varphi \) and \( \Psi = Q_u(\overline{\psi}) \). In light of this,
\[ S_{1,0}^u S_{\varphi,\psi}^u = S_{\Phi,\Psi}^u = S_{P_u \varphi, Q_u(\overline{\psi})}^u = S_{P_u \varphi, 0}^u. \]

This finishes the proof of the theorem. \( \square \)

**Corollary 3.4** Let \( \varphi_1, \varphi_2, \psi_1, \psi_2 \in L^\infty \) such that \( \varphi_1 - \psi_1 \) is invertible in \( L^\infty \). Let \( S_{\varphi_1, \psi_1}^u, S_{\varphi_2, \psi_2}^u \in D_u \), and then \( S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u \in D_u \) if and only if \( \varphi_2 \in K_u^\infty + uH^\infty + \overline{uH^\infty} \) and \( \psi_2 \in K_u^\infty \). In this case,
\[ S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u = S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u. \]

**Proof** The result easily follows from Proposition 3.2 and Theorem 3.3, and we also have
\[ S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u = S_{\varphi_1, \psi_1}^u + M_{\varphi_1 - \psi_1} S_{1,0}^u S_{\varphi_2, \psi_2}^u \]
\[ = S_{\varphi_2 \psi_1 + (\varphi_1 - \psi_1) P_u(\varphi_2), \psi_2 \psi_1 + (\varphi_1 - \psi_1) Q_u(\overline{\psi_2})}^u \]
\[ = S_{\varphi_1, \psi_1}^u P_u \varphi_2 + \psi_1 Q_u \varphi_2, \psi_2 \psi_1 \]
\[ = S_{\varphi_1, \psi_1}^u P_u \varphi_2 + \psi_1 Q_u \varphi_2, \psi_2 \psi_1. \]

\[ \square \]

**Remark 3.5** 1) If \( S_{\varphi_1, \psi_1}^u \) is a multiplication operator \( S_{\varphi_1, \psi_1}^u = M_{\varphi_1} \), then \( S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u \in D_u \) for all \( S_{\varphi_2, \psi_2}^u \) and \( S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u = S_{\varphi_1, \psi_1}^u S_{\varphi_1, \psi_1}^u \).

2) Let \( \varphi_1, \psi_1 \in L^\infty \) such that \( \varphi_1 - \psi_1 \) is invertible in \( L^\infty \). If \( S_{\varphi_1, \psi_1}^u \) is not a multiplication operator and \( S_{\varphi_2, \psi_2}^u = M_{\varphi_2} \), and if \( S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u \in D_u \), then, by Theorem 3.3, we have the following two cases:

(a) If \( u(0) = 0 \), then \( \lambda \in K_u^\infty \cap \overline{K_u^\infty} \) for some complex number \( \lambda \). Therefore, \( \varphi_2 = \lambda \) and \( S_{\varphi_1, \psi_1}^u M_{\varphi_2} = S_{\lambda \varphi_1, \lambda \psi_1}^u \).

(b) If \( u(0) \neq 0 \), then \( \lambda \notin K_u^\infty \) and \( \lambda \notin \overline{K_u^\infty} \) for some complex number \( \lambda \). Therefore, \( \varphi_2 = 0 \).

To study particular cases of the product of dilation of truncated Toeplitz operators, we need to construct the subsets \( K_1 \) and \( K_2 \) described below:
\[ K_1 = \{ S_{\varphi, \psi}^u \in D_u, \varphi \in K_u^\infty, \psi \in \overline{K_u^\infty} \} \]
\[ K_2 = \{ S_{\varphi, \psi}^u \in D_u, \varphi \in uH^\infty + \overline{uH^\infty}, \psi \in \overline{K_u^\infty} \}. \]

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Proposition 3.6 Let $\varphi_1, \psi_1 \in L^\infty$ such that $\varphi_1 - \psi_1$ is invertible in $L^\infty$. For $S_{\varphi_1, \psi_1}^u \in D_u$, we have the following cases:

(a) If $S_{\varphi_2, \psi_2}^u \in K_1$ then

$$S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u = S_{\varphi_1 \varphi_2, \psi_1 \psi_2}^u.$$ 

(b) If $S_{\varphi_2, \psi_2}^u \in K_2$ then

$$S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u = S_{\psi_1 \varphi_2, \psi_1 \psi_2}^u.$$

Proof

(a) If $\varphi_2 \in K_u^\infty$ and $\psi_2 \in \overline{K_u^\infty}$, then by theorem 3.3 we have

$$S_{0,0}^u S_{\varphi_2, \psi_2}^u = S_{\varphi_2, \psi_2}^u = S_{\varphi_2, \psi_2}^u.$$

Therefore,

$$S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u = S_{\varphi_1 \varphi_2, \psi_1 \psi_2}^u = S_{\varphi_1 \varphi_2, \psi_1 \psi_2}^u.$$

We are now able to give a sufficient condition under which the operator $S_{\varphi, \psi}^u \in D_u$ becomes invertible and whose inverse is also in $D_u$.

In all the following results we will assume that $\varphi_1 - \psi_1$ is invertible in $L^\infty$.

Corollary 3.7 Assume that $S_{\varphi, \psi}^u$ is not a multiplication operator. If $S_{\varphi, \psi}^u \in K_1$ and $\varphi, \overline{\psi}$ are invertible in $K_u^\infty$, then $S_{\varphi, \psi}^u$ is invertible operator. In this case,

$$(S_{\varphi, \psi}^u)^{-1} = S_{\varphi^{-1}, \psi^{-1}}^u.$$

Proof Let $S_{\varphi_1, \psi_1}^u \in D_u$ be the inverse of $S_{\varphi, \psi}^u$. Then $S_{\varphi, \psi}^u S_{\varphi_1, \psi_1}^u \in K_{1,1}$. Supposing that $\varphi, \overline{\psi} \in K_u^\infty$ are invertible functions, then by Proposition 3.6 we have

$$S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u = S_{\varphi_1 \varphi_2, \psi_1 \psi_2}^u = S_{1,1}^u.$$

Therefore, $\varphi_1 = \varphi^{-1}$ and $\psi_1 = \psi^{-1}$. 

According to Proposition 3.6, we get the following results.

Corollary 3.8 Assuming that $S_{\varphi_1, \psi_1}^u \in D_u$ is not a multiplication operator, we have the following two cases:

1) If $S_{\varphi_2, \psi_2}^u \in K_1$ then the operator $S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u$ is a multiplication operator if and only if $\varphi_1 \varphi_2 = \psi_1 \psi_2$. In this case,

$$S_{\varphi_1, \psi_1}^u S_{\varphi_2, \psi_2}^u = M_{\varphi_1 \varphi_2} = M_{\psi_1 \psi_2}.$$
2) If $S^u_{\varphi_2,\psi_2} \in K_2$ then the operator $S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2}$ is a multiplication operator if and only if $\psi_1\varphi_2 = \psi_1 \psi_2$.

In this case,
\[ S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = M_{\psi_1,\varphi_2} = M_{\psi_1,\psi_2}. \]

The next corollary tells us when $S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = 0$.

**Corollary 3.9** Assuming that $S^u_{\varphi_1,\psi_1} \in D_u$ is not a multiplication operator, we have the following:

1) If $S^u_{\varphi_2,\psi_2} \in K_1$ and $S^u_{\varphi_2,\psi_2} \neq 0$ then
\[ S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = 0 \]

if and only if one of the following two assertions holds:

(a) $\varphi_1 \neq 0, \psi_1 = 0, \varphi_2 = 0, \psi_2 \in K^\infty_u$, 
(b) $\psi_1 \neq 0, \varphi_1 = 0, \psi_2 = 0, \varphi_2 \in K^\infty_u$.

2) If $S^u_{\varphi_2,\psi_2} \in K_2$ and $S^u_{\varphi_2,\psi_2} \neq 0$ then
\[ S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = 0 \]

if and only if one of the following two assertions holds

(a) $\psi_1 = 0, \varphi_2 \neq 0, \psi_2 \neq 0$, 
(b) $\psi_1 \neq 0, \varphi_2 = 0, \psi_2 = 0$.

**Proof**

1) Since $S^u_{\varphi_2,\psi_2} \in K_1$, it follows from Proposition 3.6 that $\varphi_2 \in K^\infty_u$ and $\psi_2 \in K^\infty_u$ and the equation $S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = 0$ is equivalent to $\varphi_1\varphi_2 = \psi_1\psi_2 = 0$.

2) Again using Proposition 3.6, we obtain that the equation $S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = 0$ is equivalent to $\psi_1\varphi_2 = \varphi_1\psi_2 = 0$.

The following corollary shows when $S^u_{\varphi_1,\psi_1}$ commutes with $S^u_{\varphi_2,\psi_2}$.

**Corollary 3.10** The following statements hold:

1) Let $S^u_{\varphi_1,\psi_1}, S^u_{\varphi_2,\psi_2} \in K_1$. Then $S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = S^u_{\varphi_2,\psi_2} S^u_{\varphi_1,\psi_1}$.

2) Let $S^u_{\varphi_1,\psi_1}, S^u_{\varphi_2,\psi_2} \in K_2$. Then $S^u_{\varphi_1,\psi_1} S^u_{\varphi_2,\psi_2} = S^u_{\varphi_2,\psi_2} S^u_{\varphi_1,\psi_1}$ if and only if $\psi_1\varphi_2 = \varphi_1\psi_2$.
References


