Coefficient estimates for a new subclasses of $\lambda$-pseudo biunivalent functions with respect to symmetrical points associated with the Horadam Polynomials

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Abstract: In the present article, we introduce two new subclasses of $\lambda$-pseudo biunivalent functions with respect to symmetrical points in the open unit disk $U$ defined by means of the Horadam polynomials. For functions belonging to these subclasses, estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ are obtained. Fekete–Szegö inequalities of functions belonging to these subclasses are also founded. Furthermore, we point out several new special cases of our results.

Key words: Analytic function, univalent and biunivalent functions, Fekete–Szegö problem, $\lambda$-pseudo biunivalent functions with respect to symmetrical points, Horadam polynomials, coefficient bounds, subordination

1. Introduction and preliminaries

Let $A$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1.1)

which are analytic in the open unit open disk $U = \{ z : z \in \mathbb{C}, |z| < 1 \}$, and denoted by $A$. Let $S$ be class of all functions in $A$ which are univalent and normalized by the conditions

$$f(0) = 0 = f'(0) - 1$$

in $U$. For two functions $f$ and $g$, analytic in $U$, we say that the function $f$ is subordinate to $g$ in $U$, written as $f(z) \prec g(z)$, $(z \in U)$, provided that there exists an analytic function (that is, Schwarz function) $w(z)$ defined on $U$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad \text{for all} \quad z \in U,$$

such that $f(z) = g(w(z))$ for all $z \in U$.

Besides, it is known that

$$f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

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It is well known that every univalent function \( f \) has an inverse \( f^{-1} \), defined by
\[
f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),
\]
and
\[
f^{-1}(f(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \frac{1}{4}),
\]
where
\[
f^{-1}(w) = w + a_2w^2 + (2a_2^2 - 3a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.
\]
(1.2)

A function \( f \in \mathcal{A} \) is said to be biunivalent in \( \mathbb{U} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathbb{U} \), and denoted by \( \Sigma \).

In 1967, the class \( \Sigma \) of biunivalent functions was first investigated by Lewin [12] and it was derived that \( |a_2| < 1.51 \). Brannan and Taha [6] also considered certain subclasses of biunivalent functions, and obtained estimates for the initial coefficients. In 2010, Srivastava et al. [17] revived the investigation of various classes of biunivalent functions. Moreover, many other authors (see [1–4, 7]) have introduced and investigated subclasses of biunivalent functions.

By \( S^*(\varphi) \) and \( K(\varphi) \) we denote the following classes of functions
\[
S^*(\varphi) = \left\{ f : f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \prec \varphi(z), \quad z \in \mathbb{U}, \right\}
\]
and
\[
K(\varphi) = \left\{ f : f \in \mathcal{A}, 1 + \frac{zf'(z)}{f(z)} \prec \varphi(z), \quad z \in \mathbb{U}, \right\}
\]
where \( S^*(\varphi) \) and \( K(\varphi) \) are the class of starlike and convex functions, respectively, were defined and studied by Ma and Minda [14]. It is clear that if \( f(z) \in K \), then \( zf'(z) \in S^* \).

Sakaguchi [16] introduced the class \( S_*^* \) of functions starlike with respect to symmetric points, which consists of functions \( f(z) \in S \) satisfying the condition
\[
\Re \left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in \mathbb{U}.
\]
Moreover, Wang et al. [18] introduced the class \( K_* \) of functions convex with respect to symmetric points, which consists of functions \( f(z) \in S \) satisfying the condition
\[
\Re \left\{ \frac{(zf'(z))'}{(f(z) - f(-z))'} \right\} > 0, \quad z \in \mathbb{U}.
\]
It is easily seen that if \( f(z) \in K_* \), then \( zf'(z) \in S_*^* \). For such a function \( \varphi \), Ravichandran [15] introduced the following subclasses: A function \( f \in \mathcal{A} \) is in the class \( S_*^*(\varphi) \) if
\[
\frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi(z), \quad z \in \mathbb{U},
\]
and in the class $K_s(\varphi)$ if
\[
\frac{2zf'(z)'}{f'(z) + f'(-z)} < \varphi(z) \quad z \in \mathbb{U}.
\]

Recently, Babalola [5] defined the class $L_\lambda$ of $\lambda$-pseudo-starlike functions as follows: Let $f \in A$ and $\lambda \geq 1$ is real. Then $f(z)$ belongs to the class $L_\lambda$ of $\lambda$-pseudo-starlike functions in the unit disc $\mathbb{U}$ if and only if
\[
\Re\left\{\frac{z(f'(z))^\lambda}{f(z)}\right\} \geq 0, \quad z \in \mathbb{U}.
\]

The Horadam polynomials $h_n(x)$ are given by the following recurrence relation (see [10])
\[
h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad (n \in \mathbb{N} \geq 2),
\]
with $h_1 = a$, $h_2 = bx$, and $h_3 = pbx^2 + aq$ where $(a, b, p, q)$ are some real constants.

The characteristic equation of recurrence relation (1.3) is
\[
t^2 - px - q = 0.
\]
This equation has two real roots;
\[
\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2},
\]
and
\[
\beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.
\]

Note that, some particular cases of Horadam polynomials sequence are listed as follows:

- If $a = b = p = q = 1$, the Fibonacci polynomials sequence is obtained
  \[
  F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad F_1(x) = 1, \quad F_2(x) = x.
  \]
- If $a = 2, b = p = q = 1$, the Lucas polynomials sequence is obtained
  \[
  L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x), \quad L_0(x) = 2, \quad L_1(x) = x.
  \]
- If $a = q = 1, b = p = 2$, the Pell polynomials sequence is obtained
  \[
  P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \quad p_1(x) = 1, \quad P_2(x) = 2x.
  \]
- If $a = b = p = 2, q = 1$, the Pell-Lucas polynomials sequence is obtained
  \[
  Q_{n-1}(x) = 2xQ_{n-2}(x) + Q_{n-3}(x), \quad Q_0(x) = 2, \quad Q_1(x) = 2x.
  \]
- If $a = b = 1, p = 2, q = -1$, the Chebyshev polynomials of first kind sequence is obtained
  \[
  T_{n-1}(x) = 2xT_{n-2}(x) + T_{n-3}(x), \quad T_0(x) = 1, \quad T_1(x) = x.
  \]
If \( a = 1, b = p = 2, q = -1 \), the Chebyshev polynomials of second kind sequence is obtained
\[
U_{n-1}(x) = 2xU_{n-2}(x) + U_{n-3}(x), \quad U_0(x) = 1, \quad U_1(x) = 2x.
\]

If \( x = 1 \), the Horadam numbers sequence is obtained
\[
h_{n-1}(1) = ph_{n-2}(1) + qh_{n-3}(1), \quad h_0(1) = a, \quad h_1(1) = b.
\]

For more information associated with these polynomials see [8], ([9, 11, 13]).

**Remark 1.1** [9] Let \( \Omega(x, z) \) be the generating function of the Horadam polynomials \( h_n(x) \). Then
\[
\Omega(x, z) = \frac{a + (b - ap)xt}{1 - pxt - qt^2} = \sum_{n=1}^{\infty} h_n(x) z^{n-1}.
\]

In this paper, we introduce two new subclasses of \( \lambda \)-pseudo biunivalent functions with respect to symmetrical points by using the Horadam polynomials \( h_n(x) \) and the generating function \( \Omega(x, z) \) which are given by the recurrence relation (1.3) and (1.5), respectively. Furthermore, we find the initial coefficients and the Fekete–Szegő inequality for functions belonging to the classes \( L\Sigma(\lambda, \alpha, x) \) and \( M\Sigma(\lambda, \alpha, x) \). Also, several special cases to our results were obtained.

2. Coefficient bounds for the function class \( L\Sigma(\lambda, \alpha, x) \)

**Definition 2.1** A function \( f \in \Sigma \) given by (1.1) is said to be in the class \( L\Sigma(\lambda, \alpha, x) \), if the following conditions are satisfied:
\[
(1 - \alpha) \frac{2z|f'(z)|^\lambda}{f(z) - f(-z)} + \alpha \frac{2|g'(w)|^\lambda}{g(w) - g(-w)} \prec \Omega(x, z) + 1 - \alpha
\]
and
\[
(1 - \alpha) \frac{2w|g'(w)|^\lambda}{g(w) - g(-w)} + \alpha \frac{2|g'(w)|^\lambda}{g(w) - g(-w)} \prec \Omega(x, w) + 1 - \alpha
\]
where the real constants \( a, b \), and \( q \) are as in (1.3) and \( g(w) = f^{-1}(z) \) is given by (1.2).

We first state and prove the following result.

**Theorem 2.2** Let the function \( f \in \Sigma \) given by (1.1) be in the class \( L\Sigma(\lambda, \alpha, x) \). Then
\[
|a_2| \leq \frac{|bx| \sqrt{|bx|}}{\sqrt{\left|((2\lambda^2 + \lambda - 1) + 2\alpha(3\lambda^2 - 1))b - 4p\lambda^2(1 + \alpha)^2|bx^2 - 4qa\lambda^2(1 + \alpha)^2|}}
\]
\[
|a_3| \leq \frac{|bx|}{(3\lambda - 1)(1 + 2\alpha)} + \frac{(bx)^2}{4\lambda^2(1 + \alpha)^2},
\]
and for some \( \eta \in \mathbb{R} \),
\[
|a_3 - \eta a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|bx|}{(3\lambda - 1)(1 + 2\alpha)} & , \quad |\eta - 1| \leq \frac{1}{2(3\lambda - 1)(1 + 2\alpha)} |A| \\
\frac{|bx|^2}{4\lambda^2(1 + \alpha)^2} & , \quad |\eta - 1| \geq \frac{1}{2(3\lambda - 1)(1 + 2\alpha)} |A|
\end{array} \right.
\]
where \( A = (2\lambda^2 + \lambda - 1) + 2\alpha(3\lambda^2 - 1) - 4\lambda^2(1 + \alpha)^2|bx^2 + qa| \).
Proof Let \( f \in \Sigma \) be given by the Taylor-Maclaurin expansion (1.1). Then, for some analytic functions \( \Psi \) and \( \Phi \) such that \( \Psi(0) = \Phi(0) = 0 \), \( |\psi(z)| < 1 \) and \( |\Phi(w)| < 1 \), \( z, w \in U \) and using Definition 2.1, we can write

\[
(1 - \alpha) \frac{2z[f'(z)]^\lambda}{f(z) - f(-z)} + \alpha \frac{2[(z f'(z))^\lambda]}{[f(z) - f(-z)]'} = \omega(x, \Phi(z)) + 1 - \alpha
\]

and

\[
(1 - \alpha) \frac{2w[g'(w)]^\lambda}{g(w) - g(-w)} + \alpha \frac{2[(wg'(w))^\lambda]}{[g(w) - g(-w)]'} = \omega(x, \psi(w)) + 1 - \alpha
\]

or, equivalently,

\[
(1 - \alpha) \frac{2z[f'(z)]^\lambda}{f(z) - f(-z)} + \alpha \frac{2[(z f'(z))^\lambda]}{[f(z) - f(-z)]'} = 1 + h_1(x) - a + h_2(x) \Phi(z) + h_3(x)[\Phi(z)]^3 + \cdots \tag{2.6}
\]

and

\[
(1 - \alpha) \frac{2w[g'(w)]^\lambda}{g(w) - g(-w)} + \alpha \frac{2[(wg'(w))^\lambda]}{[g(w) - g(-w)]'} = 1 + h_1(x) - a + h_2(x) \psi(w) + h_3(x)[\psi(w)]^3 + \cdots \tag{2.7}
\]

From (2.6) and (2.7), we obtain

\[
(1 - \alpha) \frac{2z[f'(z)]^\lambda}{f(z) - f(-z)} + \alpha \frac{2[(z f'(z))^\lambda]}{[f(z) - f(-z)]'} = 1 + h_2(x)p_1z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \cdots \tag{2.8}
\]

and

\[
(1 - \alpha) \frac{2w[g'(w)]^\lambda}{g(w) - g(-w)} + \alpha \frac{2[(wg'(w))^\lambda]}{[g(w) - g(-w)]'} = 1 + h_2(x)p_1w + [h_2(x)q_2 + h_3(x)q_1^2]w^2 + \cdots \tag{2.9}
\]

Notice that if

\[
|\Phi(z)| = |p_1z + p_2z^2 + p_3z^3 + \cdots| < 1 \quad (z \in \mathbb{U})
\]

and

\[
|\psi(w)| = |q_1w + q_2w^2 + q_3w^3 + \cdots| < 1 \quad (w \in \mathbb{U}),
\]

then

\[
|p_i| \leq 1 \quad \text{and} \quad |q_i| \leq 1 \quad (i \in \mathbb{N}).
\]

Thus, upon comparing the corresponding coefficients in (2.8) and (2.9), we have

\[
2\lambda(1 + \alpha)a_2 = h_2(x)p_1 \tag{2.10}
\]

\[
2\lambda(\lambda - 1)(1 + 3\alpha)a_2^2 + (3\lambda - 1)(1 + 2\alpha)a_3 = h_2(x)p_2 + h_3(x)p_1^2 \tag{2.11}
\]

\[
-2\lambda(1 + \alpha)a_2 = h_2(x)q_1 \tag{2.12}
\]

\[
[2(\lambda^2 + 2\lambda - 1) + 2\alpha(3\lambda^3 + 3\lambda - 2)]a_2^2 - (3\lambda - 1)(1 + 2\alpha)a_3 = h_2(x)q_2 + h_3(x)q_1^2. \tag{2.13}
\]
From (2.10) and (2.12), we find that

$$p_1 = -q_1$$  \hspace{1cm} (2.14)

and

$$8\lambda^2(1 + \alpha)^2a_2^2 = h_2(x)(p_1^2 + q_1^2).$$  \hspace{1cm} (2.15)

Moreover, by using (2.13) and (2.11), we obtain

$$[2(2\lambda^2 + \lambda - 1) + 4\alpha(3\lambda^2 - 1)]a_2^2 = h_2(x)(p_2 + q_2) + h_3(x)(p_1^2 + q_1^2).$$  \hspace{1cm} (2.16)

By using (2.14) in (2.16), we get

$$[2(2\lambda^2 + \lambda - 1) + 4\alpha(3\lambda^2 - 1) - 8\lambda^2(1 + \alpha)^2h_3(x)]a_2^2 = h_2(x)(p_2 + q_2).$$  \hspace{1cm} (2.17)

From (1.3) and (2.17), we have the desired inequality (2.3).

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.11) and using (2.14) and (2.15) we get

$$a_3 = \frac{h_2(x)(p_2 - q_2)}{2(3\lambda - 1)(1 + 2\alpha)} + \frac{h_2(x)[p_1^2 + q_1^2]}{8\lambda^2(1 + \alpha)^2}.$$  \hspace{1cm} (2.18)

Hence, using (2.14) and applying (1.3), we get desired inequality (2.4).

Now, by using (2.16) and (2.18) for some $\eta \in \mathbb{R}$, we get

$$a_3 - \eta a_2^2 = \frac{[h_2(x)]^2(1 - \eta)(p_2 + q_2)}{[2(2\lambda^2 + \lambda - 1) + 4\alpha(3\lambda^2 - 1)][h_2(x)]^2 - 8\lambda^2(1 + \alpha)^2h_3(x)} + \frac{h_2(x)(p_2 - q_2)}{2(3\lambda - 1)(1 + 2\alpha)}$$

$$= h_2(x) \left[ \left( \Theta(\eta, x) + \frac{1}{2(3\lambda - 1)(1 + 2\alpha)} \right) p_2 + \left( \Theta(\eta, x) - \frac{1}{2(3\lambda - 1)(1 + 2\alpha)} \right) q_2 \right],$$

where

$$\Theta(\eta, x) = \frac{[h_2(x)]^2(1 - \eta)}{[2(\lambda^2 + \lambda - 1) + 4\alpha(3\lambda^2 - 1)][h_2(x)]^2 - 8\lambda^2(1 + \alpha)^2h_3(x)}.$$  

Thus, we conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{h_2(x)}{2(3\lambda - 1)(1 + 2\alpha)} & \text{if } |\Theta(\eta, x)| \leq \frac{1}{2(3\lambda - 1)(1 + 2\alpha)}, \\ \frac{2|h_2(x)||\Theta(\eta, x)|}{2|h_2(x)||\Theta(\eta, x)|} & \text{if } |\Theta(\eta, x)| \geq \frac{1}{2(3\lambda - 1)(1 + 2\alpha)}. \end{cases}$$

This proves Theorem 2.2.

For $\alpha = 0$ the class $L\Sigma(\lambda, \alpha, x)$ reduced to the class of $\lambda$–pseudo bistarlike functions with respect to symmetrical points. For functions belong to this class we have the following corollary:

**Corollary 2.3** Let the function $f \in \Sigma$ given by (1.1) be in the class $L\Sigma(\lambda, x)$. Then

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|(2\lambda^2 + \lambda - 1)b - 4p\lambda^2|b^2 - 4qa\lambda^2|}}.$$  \hspace{1cm} (2.19)
\[ a_3 \leq \frac{|bx|}{3\lambda - 1} + \frac{(bx)^2}{4\lambda^2}, \tag{2.20} \]

and for some \( \eta \in \mathbb{R} \),
\[
|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{3(3\lambda - 1)} & , \quad |\eta| \leq \frac{1}{2(3\lambda - 1)} |A_0| \\ \frac{|bx|^3|1 - \eta|}{(2(3\lambda + 1)|bx|^2 - 4\lambda^2(pbx^2 + qa))} & , \quad |\eta - 1| \geq \frac{1}{2(3\lambda - 1)} |A_0| \end{cases} \tag{2.21} \]

where \( A_0 = (2\lambda^2 + \lambda - 1) - \frac{4\lambda^2(pbx^2 + qa)}{bx^2} \).

For \( \alpha = 1 \) the class \( L\Sigma(\lambda, 1, x) \) reduced to the class of \( \lambda \)-pseudo biconvex functions with respect to symmetrical points. For functions belong to this class we have the following corollary:

**Corollary 2.4** Let the function \( f \in \Sigma \) given by (1.1) be in the class \( L\Sigma(\lambda, 1, x) \). Then
\[
|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|(2(\lambda^2 + \lambda - 1) + 2(3\lambda^2 - 1))b + 16p\lambda^2|bx^2 - 16qa\lambda^2|}} \tag{2.22} \]
\[
|a_3| \leq \frac{|bx|}{3(3\lambda - 1)} + \frac{(bx)^2}{16\lambda^2}, \tag{2.23} \]

and for some \( \eta \in \mathbb{R} \),
\[
|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{3(3\lambda - 1)} & , \quad |\eta - 1| \leq \frac{1}{16(3\lambda - 1)} |A_1| \\ \frac{|bx|^3|1 - \eta|}{(2(3\lambda + 1)|bx|^2 - 16\lambda^2(pbx^2 + qa))} & , \quad |\eta - 1| \geq \frac{1}{16(3\lambda - 1)} |A_1| \end{cases} \tag{2.24} \]

where \( A_1 = (2\lambda^2 + \lambda - 1) + 2(3\lambda^2 - 1) - \frac{16\lambda^2(pbx^2 + qa)}{bx^2} \).

For \( \lambda = 1 \) the class \( L\Sigma(1, \alpha, x) \) reduced to the class of biunivalent functions with respect to symmetrical points. For functions belonging to this class we have the following corollary:

**Corollary 2.5** Let the function \( f \in \Sigma \) given by (1.1) be in the class \( L\Sigma(1, \alpha, x) \). Then
\[
|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|2(1 + 2\alpha)b - 4p(1 + \alpha)^2|bx^2 - 4qa(1 + \alpha)^2|}} \tag{2.25} \]
\[
|a_3| \leq \frac{|bx|}{2(1 + 2\alpha)} + \frac{(bx)^2}{4(1 + \alpha)^2}, \tag{2.26} \]

and for some \( \eta \in \mathbb{R} \),
\[
|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{2(1 + 2\alpha)} & , \quad |\eta - 1| \leq \frac{1}{4(1 + 2\alpha)} |B| \\ \frac{|bx|^3|1 - \eta|}{2(1 + \alpha)|bx|^2 - 4(1 + \alpha)^2(pbx^2 + qa)|} & , \quad |\eta - 1| \geq \frac{1}{4(1 + 2\alpha)} |B| \end{cases} \tag{2.27} \]

where \( B = 2(1 + 2\alpha) - \frac{4(1 + \alpha)^2(pbx^2 + qa)}{bx^2} \).
3. Coefficient bounds for the function class $M\Sigma(\lambda, \alpha, x)$

**Definition 3.1** A function $f \in \Sigma$ given by (1.1) is said to be in the class $M\Sigma(\lambda, \alpha, x)$, if the following conditions are satisfied:

$$\left( \frac{2zf'(z)}{f(z) - f(-z)} \right)^\alpha \left( \frac{2z[f'(z)]^\lambda}{f(z) - f(-z)} \right)^{1-\alpha} \prec \Omega(x, z) + 1 - \alpha$$

(3.1)

and

$$\left( \frac{2wg'(w)}{g(w) - g(-w)} \right)^\alpha \left( \frac{2[wg'(w)]^\lambda}{g(w) - g(-w)} \right)^{1-\alpha} \prec \Omega(x, w) + 1 - \alpha$$

(3.2)

where the real constants $a$, $b$, and $q$ are as in (1.3) and $g(w) = f^{-1}(z)$ is given by (1.2).

**Theorem 3.2** Let the function $f \in \Sigma$ given by (1.1) be in the class $M\Sigma(\lambda, \alpha, x)$. Then

$$|a_2| \leq \frac{|bx|}{\sqrt{||2\lambda^2(\alpha - 2)^2 + (\lambda + 2\alpha - 3)b - 4\lambda^2(\alpha - 2)]|bx^2 - 4qax(\alpha - 2)^2|}}$$

(3.3)

$$|a_3| \leq \frac{|bx|}{(3\lambda - 1)(3 - 2\alpha)} + \frac{(bx)^2}{4\lambda^2(\alpha - 2)^2},$$

(3.4)

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|bx|}{(3\lambda - 1)(3 - 2\alpha)} & , |\eta - 1| \leq \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} |C| \\
\frac{|bx|^\eta(1-\eta)}{||2\lambda^2(\alpha - 2)^2 + (\lambda + 2\alpha - 3)b|bx^2 - 4\lambda^2(\alpha - 2)^2)|} & , |\eta - 1| \geq \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} |C|
\end{array} \right.$$

(3.5)

where $C = 2\lambda^2(\alpha - 2)^2 + (\lambda + 2\alpha - 3) - \frac{4\lambda^2(\alpha - 2)^2}{bx^2 + qa}.$

**Proof** Let $f \in \Sigma$ be given by the Taylor–Maclaurin expansion (1.1). Then, for some analytic functions $\Psi$ and $\Phi$ such that $\Psi(0) = \Phi(0) = 0$, $|\psi(z)| < 1$ and $|\Phi(w)| < 1$, $z, w \in U$ and using Definition 3.1, we can write

$$\left( \frac{2zf'(z)}{f(z) - f(-z)} \right)^\alpha \left( \frac{2[zf'(z)]^\lambda}{[f(z) - f(-z)]^\lambda} \right)^{1-\alpha} = \omega(x, \psi(z)) + 1 - \alpha$$

and

$$\left( \frac{2wg'(w)}{g(w) - g(-w)} \right)^\alpha \left( \frac{2[wg'(w)]^\lambda}{[g(w) - g(-w)]^\lambda} \right)^{1-\alpha} = \omega(x, \psi(w)) + 1 - \alpha$$

or, equivalently,

$$\left( \frac{2zf'(z)}{f(z) - f(-z)} \right)^\alpha \left( \frac{2[zf'(z)]^\lambda}{[f(z) - f(-z)]^\lambda} \right)^{1-\alpha} = 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^2 + \ldots$$

(3.6)
and
\[
\left( \frac{2w|g'(w)|^\lambda}{g(w) - g(-w)} \right)^\alpha \left( \frac{2[(wg' (w))']^\lambda}{[g(w) - g(-w)]^\nu} \right)^{1-\alpha} = 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^3 + \cdots. \tag{3.7}
\]

From (3.6) and (3.7), we obtain
\[
\left( \frac{2z[f'(z)]^\lambda}{f(z) - f(-z)} \right)^\alpha \left( \frac{2[(zf' (z))']^\lambda}{[f(z) - f(-z)]^\nu} \right)^{1-\alpha} = 1 + h_2(x)p_1z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \cdots \tag{3.8}
\]
and
\[
\left( \frac{2w|g'(w)|^\lambda}{g(w) - g(-w)} \right)^\alpha \left( \frac{2[(wg' (w))']^\lambda}{[g(w) - g(-w)]^\nu} \right)^{1-\alpha} = 1 + h_2(x)p_1w + [h_2(x)q_2 + h_3(x)q_1^2]w^2 + \cdots. \tag{3.9}
\]

Notice that if
\[
|\Phi(z)| = |p_1z + p_2z^2 + p_3z^3 + \cdots| < 1 \quad (z \in \mathbb{U})
\]
and
\[
|\psi(w)| = |q_1w + q_2w^2 + q_3w^3 + \cdots| < 1 \quad (w \in \mathbb{U}),
\]
then
\[
|p_i| \leq 1 \quad \text{and} \quad |q_i| \leq 1 \quad (i \in \mathbb{N}).
\]

Thus, upon comparing the corresponding coefficients in (3.8) and (3.9), we have
\[
-2\lambda(\alpha - 2)a_2 = h_2(x)p_1 \tag{3.10}
\]
\[
[2\lambda^2(\alpha - 2)^2 + 2\lambda(3\alpha - 4)]a_2^2 + (3\lambda - 1)(3\alpha - 2\alpha) = h_2(x)p_2 + h_3(x)p_1^2 \tag{3.11}
\]
\[
2\lambda(\alpha - 2)a_2 = h_2(x)q_1 \tag{3.12}
\]
\[
[2\lambda^2(\alpha - 2)^2 + 2\lambda(5 - 3\alpha) + 2(2\alpha - 3)]a_2^2 + (3\lambda - 1)(2\alpha - 3) = h_2(x)q_2 + h_3(x)q_1^2. \tag{3.13}
\]

From (3.10) and (3.12), we find that
\[
p_1 = -q_1 \tag{3.14}
\]
and
\[
8\lambda^2(\alpha - 2)^2a_2^2 = h_2^2(x)(p_1^2 + q_1^2). \tag{3.15}
\]

Moreover, by using (3.13) and (3.11), we obtain
\[
[4\lambda^2(\alpha - 2)^2 + 2(\lambda + 2\alpha - 3)]a_2^2 = h_2(x)(p_2 + q_2) + h_3(x)(p_1^2 + q_1^2). \tag{3.16}
\]

By using (3.14) in (3.16), we get
\[
\left[ 4\lambda^2(\alpha - 2)^2 + 2(\lambda + 2\alpha - 3) - \frac{8\lambda^2(\alpha - 2)^2h_3(x)}{[h_2(x)]^2} \right]a_2^2 = h_2(x)(p_2 + q_2). \tag{3.17}
\]
From (1.3), and (3.17), we have the desired inequality (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.13) from (3.11) and using (3.14) and (3.15) we get

$$a_3 = \frac{h_2(x)(p_2 - q_2)}{2(3\lambda - 1)(|3 - 2\alpha|)} + \frac{h_2(x)[p_1^2 + q_1^2]}{8\lambda^2(\alpha - 2)^2}. \quad (3.18)$$

Hence, using (3.14) and applying (1.3), we get desired inequality (3.4).

Now, by using (3.16) and (3.18) for some $\eta \in \mathbb{R}$, we get

$$a_3 - \eta a_2 = \frac{[h_2(x)]^3(1 - \eta)(p_2 + q_2)}{[4\lambda^2(\alpha - 2)^2 + 2(\lambda + 2\alpha - 3)]h_2(x)^2} - 8\lambda^2(\alpha - 2)^2h_3(x) + \frac{h_2(x)(p_2 - q_2)}{2(3\lambda - 1)(3 - 2\alpha)}$$

$$= h_2(x) \left[ \left( \Theta(\eta, x) + \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} \right)p_2 + \left( \Theta(\eta, x) - \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} \right)q_2 \right],$$

where

$$\Theta(\eta, x) = \frac{[h_2(x)]^2(1 - \eta)}{[4\lambda^2(\alpha - 2)^2 + 2(\lambda + 2\alpha - 3)]h_2(x)^2 - 8\lambda^2(\alpha - 2)^2h_3(x)}.$$

Thus, we conclude that

$$|a_3 - \eta a_2| \leq \begin{cases} \frac{|h_2(x)|}{2(3\lambda - 1)(3 - 2\alpha)} & , |\Theta(\eta, x)| \leq \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} \\ \frac{|h_2(x)||\Theta(\eta, x)|}{2} & , |\Theta(\eta, x)| \geq \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} \end{cases}.$$

This proves Theorem 3.2. \hfill \Box

For $\lambda = 1$ the class $M\Sigma(\lambda, \alpha, x)$ reduced to the class of biunivalent functions with respect to symmetrical points. For functions belong to this class we have the following corollary:

**Corollary 3.3** Let the function $f \in \Sigma$ given by (1.1) be in the class $M\Sigma(1, \alpha, x)$. Then

$$|a_2| \leq \frac{|bx|}{2\sqrt{|bx|} \sqrt{|(2(\alpha - 2)^2 + 2(\alpha - 1)b - 4p(\alpha - 2)^2)bx^2 - 4qa(\alpha - 2)^2|}} \quad (3.19)$$

$$|a_3| \leq \frac{|bx|}{2(|3 - 2\alpha|)} + \frac{(|bx|)^2}{4(\alpha - 2)^2}, \quad (3.20)$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2| \leq \begin{cases} \frac{|bx|}{2(3 - 2\alpha)} & , |\eta - 1| \leq \frac{1}{2(3 - 2\alpha)} |C_1| \\ \frac{|bx|}{2(3 - 2\alpha)} & , |\eta - 1| \geq \frac{1}{2(3 - 2\alpha)} |C_1| \end{cases} \quad (3.21)$$

where $C_1 = 2(\alpha - 2)^2 + 2(\alpha - 1) - \frac{4(\alpha - 2)^2(bx^2 + qa)}{4(\alpha - 2)^2}$. 2874
References


