The eternal solution to the cross curvature flow exists in 3-manifolds of negative sectional curvature

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Abstract: Given a closed 3-manifold $M^3$ endowed with a radial symmetric metric of negative sectional curvature, we define the cross curvature flow on $M^3$; using the maximum principle theorem, we demonstrated that the solution to the cross curvature flow exists for all time and converges pointwise to a hyperbolic metric.

Key words: Cross curvature flow, geometric evolution equation, negative sectional curvature

1. Introduction

It is conjectured that any closed 3-manifold $M^3$ admits a hyperbolic metric. Owing to the well-developed theory of Ricci flow in [3] and the groundbreaking work by Perelman, which uses the Ricci flow to address geometrization, the Ricci flow represents a powerful tool for analyzing geometrization and resolving this conjecture. After topological surgeries, one can expect the existence of hyperbolic structures in $M^3$. It is not anticipated that the Ricci flow provides direct proof of the conjecture; therefore, exploring alternative flows may provide more information and, specifically, yield a more straightforward hyperbolization theorem in $M^3$.

To prove the hyperbolization theorem in $M^3$, Chow and Hamilton [2] proposed the cross curvature flow for $M^3$ and conjectured that one can obtain a family of negatively curved metrics $g(t)$ under normalized cross curvature flow for any initial Riemannian metric of strictly negative sectional curvatures on $M^3$ where the metrics $g(t)$ converge to a hyperbolic metric as $t \to \infty$. They also took into account the functional

$$ J(g) = \int_{M^3} \left\{ \frac{P}{3} - (\det P)^{1/3} \right\} d\mu. $$

Using an orthonormal basis such that $P^{ij}$ is diagonal and $g_{ij} = \delta_{ij}$, the integrant $\frac{P}{3} - (\det P)^{1/3}$ is nonnegative and identical to zero when $P_{ij} = \frac{1}{3} P g_{ij}$; that is, $g$ is a hyperbolic metric in $M^3$. Furthermore, Chow and Hamilton also proved that $J(g(t))$ is nonincreasing when the metrics $g(t)$ for the cross curvature flow exist for all time. This result supports their conjecture. They appealed to Hamilton’s general existence theorem in [3] and stipulated the integrability condition for studying the solution to the cross curvature flow for a short-time existence. Subsequently, John Buckland [1] used the more classical method, DeTurck diffeomorphisms, to first

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prove the solution to the cross curvature flow for a short-time existence. More examples of solutions to a cross curvature flow feature in [5].

In [4], an asymptotic stability of cross curvature flow was established: Given any hyperbolic metric on a closed 3-manifold and supposing that an initial metric closes to this hyperbolic metric, the solution to a normalized cross curvature flow exists for all time and converges to the given hyperbolic metric. This asymptotic stability implies the feasibility of the conjecture in [2].

2. Fully nonlinear equations

Let \((M^3, g)\) be a Riemannian 3-manifold endowed with a negatively curved metric \(g\). The Einstein tensor and the dual Einstein tensor based on the metric \(g\) are defined as:

\[
P_{ij} = R_{ij} - \frac{1}{2} R g_{ij} P^{ij} = g^{ik} g^{jl} R_{kl} - \frac{1}{2} R g^{ij}.
\]

The cross curvature tensor is defined by

\[
h_{ij} = (\det P)V_{ij}
\]

where \(\det P = \frac{\det P^{kl}}{\det g^{kl}}\) and \(V_{ij} = (P^{ij})^{-1}\).

Due to the symmetry of \(P_{ij}\), we choose a local orthonormal basis such that \(g_{ij} = \delta_{ij}\) and both \(R_{ij}\) and \(h_{ij}\) are diagonal matrices. In the following argument, we also use the sign convention \(R_{ijkl} = R_{ijlk} g_{hk}\) and the sectional curvatures \(R_{ijij}, i \neq j\). Suppose that the eigenvalues of \(P^{ij}\) are \(a = -R_{2323}\), \(b = -R_{1313}\), and \(c = -R_{1212}\); the corresponding eigenvalues of \(R_{ij}\) and \(h_{ij}\) are \((-b + c), -(a + c), -(a + b)\) and \(bc, ac, ab\), respectively. Hence, \(P^{ij}\) and \(h_{ij}\) are positive definite matrices when \(M^3\) admits a metric \(g\) of negative sectional curvature.

Lemma 2.1 (Chow and Hamilton [2]) Let \((M^3, g)\) be a three-dimensional Riemannian manifold with a metric \(g\). The identity for the dual of the Einstein tensor and identity for the cross curvature tensor are as follows:

1. \(\nabla_i P^{ij} = 0\),

2. \((h^{-1})^{ij} \nabla_i h_{jk} = \frac{1}{2} (h^{-1})^{ij} \nabla_k h_{ij}\).

Lemma 2.2 (Chow and Hamilton [2]) Let \((M^n, g)\) be a Riemannian n-manifold with a metric \(g\).

1. If the Ricci curvature is positive, then the identity map \(I : (M, g_{ij}) \to (M, R_{ij})\) is harmonic. By contrast, if the Ricci curvature is negative, then the identity map \(I : (M, g_{ij}) \to (M, -R_{ij})\) is harmonic.

2. If the dimension equals three and the sectional curvature is negative (or positive), then the map \(I : (M, h_{ij}) \to (M, g_{ij})\) is harmonic.
Lemmas 2.1 and 2.2 explain why the cross curvature flow is taken into account for the hyperbolization conjecture. Owing to the duality in Lemma 2.2, the cross curvature flow on \((M^3, g_0)\) with a metric \(g_0\) of negative sectional curvature satisfies
\[
\frac{\partial}{\partial t} g_{ij} = 2h_{ij},
\]
\(g(0) = g_0\).

3. Evolution of the Einstein tensor

For the following arguments, we introduce some notation:
\[
P = P^{ij}g_{ij}, \quad H = g^{ij}h_{ij}, \quad \square = P^{kl}\nabla_k \nabla_l, \quad L = \frac{\partial}{\partial t} - \square.
\]
By denoting the volume form of the metric \(g\) by \(\mu_{ijk}\), we obtain
\[
P_{mn} = -\frac{1}{4} \mu^{ijm} \mu^{kln} R_{ijkl}, \tag{3.1}
\]
where \(\mu^{ijk} = g^{ip}g^{jq}g^{kr} \mu_{pqr}\).

**Lemma 3.1** Let \((M^3, g)\) be a three-dimensional Riemannian manifold with a metric \(g\). The evolution equation of the dual Einstein tensor is given by
\[
\frac{\partial}{\partial t} p^{ij} = \nabla_k \nabla_l (P^{kl} P^{ij} - P^{ik} P^{jl}) - \det Pg^{ij} - H P^{ij}.
\]

**Proof** We first calculate the evolution equations of the Riemannian curvature tensor and the (dual) volume form of \(\mu_{ijk}\):
\[
\frac{\partial}{\partial t} R_{ijkl} = \nabla_i \nabla_l h_{jk} + \nabla_j \nabla_k h_{il} - \nabla_i \nabla_k h_{jl} - \nabla_j \nabla_l h_{ik} + g^{pq}(R_{ijkp} h_{ql} + R_{ijpl} h_{qk}) \tag{3.2}
\]
\[
\frac{\partial}{\partial t} \mu_{ijk} = H \mu_{ijk}. \tag{3.3}
\]
Combining the equations (3.1)–(3.3), we can obtain the evolution equation of the dual Einstein tensor:
\[
\frac{\partial}{\partial t} P^{mn} = -\frac{1}{4} \mu^{ijm} \mu^{kln} (\nabla_i \nabla_l h_{jk} + \nabla_j \nabla_k h_{il} - \nabla_i \nabla_k h_{jl} - \nabla_j \nabla_l h_{ik})
\]
\[
- \frac{1}{4} \mu^{ijm} \mu^{kln} g^{pq}(R_{ijkp} h_{ql} + R_{ijpl} h_{qk}) - 2HP^{mn}
\]
\[
= \mu^{ijm} \mu^{kln} \nabla_i \nabla_k h_{jl} - \frac{1}{2} \mu^{ijm} \mu^{kln} g^{pq} R_{ijpl} h_{qk} - 2HP^{mn}.
\]
The proof is complete when we verify the identity
\[
\frac{1}{2} \mu^{ijm} \mu^{kln} g^{pq} R_{ijpl} h_{qk} + 2HP^{mn} = \det Pg_{mn}.
\]
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Because the tensor is coordinate-free, we can choose an orthonormal basis such that $g_{ij}$ is $\delta_{ij}$ and such that $P^{ij}$ and $h_{ij}$ are diagonal matrices; moreover, the volume form $\mu_{ij}^{kl}$ equals 1. Accordingly, $R_{ijkl}$ is nonzero when $(i,j) = (k,l)$ or $(i,j) = (l,k)$. The equality holds when this orthonormal basis is used for both sides of the identity.

4. Long time existence

Using the identity $\mu_{ijkl}^{pqrs} = -2P_{mi}^{\rho}\delta_{\rho}^{p} \delta_{\rho}^{q} - \delta_{\rho}^{p} \delta_{\rho}^{q} \delta_{\mu}^{m} \delta_{\mu}^{n}$ in [2], we can express the cross curvature $h_{ij}$ in (2.1) as

$$h_{ij} = \frac{1}{2} P^{kl} R_{iklj}. $$

**Theorem 4.1** (Maximum principle theorem) Let $M^3$ be a Riemannian 3-manifold endowed with a radial symmetric metric $g$ of negative sectional curvature. If the function $P \in C^\infty(M_t)$ satisfies $LP \leq 0$ in $M_t$, then

$$\sup_{M_t} P(x,t) \leq \sup_{M_0} P(x,t) = P(x,0),$$

where $M_t = M \times [0,t)$ and $M_0 = M$.

**Proof**

\[
\frac{\partial}{\partial t} P \equiv \frac{\partial P^{ij}}{\partial t} g_{ij} + P_{mn} \frac{\partial g_{mn}}{\partial t} = P^{kl} (\nabla_k \nabla_l P^{ij}) g_{ij} - [\nabla_k \nabla_l (P^{ik} P^{jl})] g_{ij} - \det P P^{ij} g_{ij} - H P^{ij} g_{ij} + P_{mn} \frac{\partial g_{mn}}{\partial t} \\
= P^{kl} (\nabla_k \nabla_l P^{ij}) g_{ij} - (\nabla_k \nabla_l P^{ik}) P^{jl} g_{ij} - (\nabla_l P^{ik} \cdot \nabla_k P^{jl}) g_{ij} - 3 \det P - HP + P^{ij} (2h_{ij}) \\
= \Box (P^{ij} g_{ij}) - (\nabla^i \nabla_k P^{ik}) P^{jl} g_{ij} + (P^{ij} R^{i}_{klh} P^{hl} + P^{ij} R^{i}_{kli} P^{li}) g_{ij} - (\nabla_l P^{ik} \cdot \nabla_k P^{jl}) g_{ij} + 3 \det P - HP \\
= \Box P + (P^{ij} R^{i}_{klh} P^{hl} + P^{ij} R^{i}_{kli} P^{li}) g_{ij} - (\nabla_l P^{ik} \cdot \nabla_k P^{jl}) g_{ij} + 3 \det P - HP.
\]

As the tensor is coordinate-free, we choose a basis such that $g_{ij} = \delta_{ij}$ and such that $P^{ij}$ and $h_{ij}$ are diagonal matrices.

\[
(P^{ij}) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad (h_{ij}) = \begin{pmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{pmatrix}.
\]

where $a = -R_{2323}$, $b = -R_{1313}$, and $c = -R_{1212}$.

**Claim 4.2** $(P^{ij} R^{i}_{klh} P^{hl} + P^{ij} R^{i}_{kli} P^{li}) g_{ij} \leq 0$.

\[
(P^{ij} R^{i}_{klh} P^{hl} + P^{ij} R^{i}_{kli} P^{li}) g_{ij} = \sum_{i,l,h,k} g_{ii} P^{il} R^{i}_{kli} P^{hl} + \sum_{i,l,h} g_{ii} (P^{il} R^{i}_{klh} P^{hl}) \\
= 6 \det P - HP + 3 \det P \leq 0,
\]
where the second equality is obtained by applying the aforementioned orthonormal basis.

Claim 4.3 \(- (\nabla_i P^{jk} \cdot \nabla_k P^{jl}) g_{ij} \leq 0 \).

\[ - (\nabla_i P^{jk} \cdot \nabla_k P^{jl}) g_{ij} = - \sum_{(1,2,3)} g_{11} (\nabla_3 P^{12} [\nabla_2 P^{31} + \nabla_1 P^{23}]) + \text{(nondiagonal terms)} \]

\[ = - \sum_{(1,2,3)} g_{11} [R_{2313,3} (R_{1323,3} - g_{13} R_{1313,1} - g_{23} R_{2323,2})], \]

where \((1,2,3)\) is a cyclic permutation of 1, 2, and 3. The second equality is obtained using expression (3.1) and is negative because \( \{R_{ijij,i} | i \neq j \in \{1,2,3\}\} \) all vanish at the critical points of \( P \).

Hence, we derive \( LP \leq 0 \) and, using the theory of parabolic partial differential equations, we finish the proof.

As we are considering negatively curved metrics accompanied with (3.1), the matrix

\[
(P^{ij}) = \begin{pmatrix}
-R_{2323} & R_{1232} & R_{1323} \\
R_{2313} & -R_{1313} & R_{2131} \\
R_{3212} & R_{3121} & -R_{1212}
\end{pmatrix}
\]

is a positive definite matrix. A positive definite matrix is such that all principle minor determinants are positive numbers, so we have \( R_{2323} R_{1313} > (R_{1323})^2 \), \( R_{1313} R_{1212} > (R_{2131})^2 \), and \( R_{2323} R_{1212} > (R_{1232})^2 \). Using the maximum principle theorem and the estimates, we can give the bounds on \( P^{ij} \) using only the initial sectional curvatures. In 3-manifolds, we know that a Riemannian curvature can be expressed in terms of a Ricci curvature:

\[ R_{ijkl} = g_{ik} R_{jl} - g_{il} R_{jk} - g_{jk} R_{il} + g_{jl} R_{ik} - \frac{1}{2} R (g_{ik} g_{jl} - g_{il} g_{jk}). \]

According to the definition of a Ricci curvature and the expression for a Riemannian curvature, all entries in \( P^{ij} \) provide a bound on Ricci curvatures and Riemannian curvatures. Therefore, Riemannian curvatures are well controlled by the initial condition, and long time existence can be guaranteed under the cross curvature flow.

5. Pointwise convergence

In [2], the cross curvature flow is studied to deform a negatively curved metric to a hyperbolic metric in a 3-manifold. Furthermore, it is proved that \( \frac{2}{3} - \text{(det } P)^{1/3} \) converges to zero in the integral sense. According to the settings in [2], we require some conditions to prove the pointwise convergence.

Lemma 5.1 Given a connection \( \nabla \) that is compatible with metrics, we derive the following identities of the Einstein tensor.

1. \((\text{det } P)^{-1} \nabla_k \text{det } P = (P^{ij})^{-1} \nabla_k P^{ij}\)

2. \(\Box (\text{det } P)^{1/3} = \frac{1}{3} (\text{det } P)^{1/3} V_{ij} \Box P^{ij}\)
where \( \det P = \frac{\det P^{ij}}{\det g^{ij}} \) and \( V_{ij} = (P^{ij})^{-1} \)

**Proof**

\[
(det P)^{-1} \nabla_k \det P = \frac{\nabla_k \det P^{ij} \cdot \det P^{ij} - \det P^{ij} \cdot \nabla_k \det g^{ij}}{(\det g^{ij})^2} \\
= \frac{\det P^{ij} \cdot (P^{ij})^{-1} \nabla_k P^{ij} - \det P^{ij} \cdot \nabla_k \det g^{ij}}{\det g^{ij}} \\
= (P^{ij})^{-1} \nabla_k P^{ij}
\]

\( \Box (\det P)^{1/3} = P^{kl} \nabla_k \nabla_l (\det P)^{1/3} \)

\[
= P^{kl} \nabla_k \left( \frac{1}{3} (\det P)^{2/3} \nabla_l \det P \right) \\
= P^{kl} \nabla_k \left( \frac{1}{3} (\det P)^{2/3} (\det P) V_{ij} \nabla_l P^{ij} \right) \\
= \frac{P^{kl}}{3} (\det P)^{1/3} \left( \frac{1}{3} V_{mn} \nabla_k P^{mn} \cdot V_{ij} \nabla_l P^{ij} + \nabla_k V_{ij} \cdot \nabla_l P^{ij} + V_{ij} \nabla_k \nabla_l P^{ij} \right) \\
= \frac{1}{3} (\det P)^{1/3} V_{ij} \Box P^{ij}
\]

The last equality is derived by \( \nabla_k (V_{ij} P^{ij}) = 0 \).

**Theorem 5.2** Let \((M^3, g_0)\) be a closed Riemannian 3-manifold endowed with a radial symmetric metric of negative sectional curvature. Then \(g\) converges pointwise to a hyperbolic metric under the cross curvature flow.

**Proof** To prove the pointwise convergence, it is sufficient to prove that \( \frac{P}{3} - (\det P)^{1/3} \) converges to zero identically. First, we denote

\[
A := \max_{M_0} \{(\det P)^{1/3} V_{ij}, 1\} \\
= \max_{M_0} \left\{ \left(\frac{abc}{a}\right)^{1/3}, \left(\frac{abc}{b}\right)^{1/3}, \left(\frac{abc}{c}\right)^{1/3}, 1 \right\} \quad \text{(on an orthonormal basis, } g_{ij} = \delta_{ij} \text{)}
\]

and

\[
Q(x, t) := \frac{AP}{3} - (\det P)^{1/3}.
\]
Second, we consider the evolution equation for the function $Q(x, t)$

$$
\frac{\partial}{\partial t} Q(x, t)
= \frac{\partial}{\partial t} \left\{ \frac{AP}{3} - (\det P)^{1/3} \right\}
= A \frac{\partial P}{\partial t} - \frac{1}{3}(\det P)^{1/3} V_{ij} \nabla_k \nabla_l (P^{kl} P^{ij} - P^{ik} P^{jl}) + \frac{2}{3}(\det P)^{1/3} H
\leq \Box Q(x, t) + A(\det P - \frac{HP}{9}) - \frac{2H}{3} Q(x, t)
\leq \Box Q(x, t) - \frac{2H}{3} Q(x, t).
$$

To estimate the commuting part of the covariant derivative, we use a similar argument to the one in Theorem 4.1 along with the expression $A g_{ij} - (\det P)^{1/3} V_{ij} \leq 0$. It is evident that $G(x, t) = \sup_M (Q(x, 0)) \exp^{-\frac{2H}{3} t}$ is a solution to the equation

$$
\begin{cases}
Lu(x, t) = -\frac{2H}{3} u(x, t), \\
u(x, 0) = \sup_M (Q(x, 0)).
\end{cases}
$$

Therefore, $Q(x, t)$ is a subsolution to the equation for $G(x, t)$. Because $\frac{P}{3} - (\det P)^{1/3}$ has an upper bound $\sup_M (Q(x, 0)) \exp^{-\frac{2H}{3} t}$, we obtain the pointwise convergence as $t \to \infty$. \hfill \square

6. Conclusion

Although the cross curvature flow is not a flow preserving the conformal structure, we seek a geometrical quantity indirectly related to conformal structure preserved under the flow in order to construct a correspondence between minimal surfaces in the different ambient spaces. However, we have not yet found an appropriate quantity. Nevertheless, the work of Perelman [6] may be of assistance if we intend to generalize the result to a manifold endowed with any negatively curved metric.

References


