Relative ranks of some partial transformation semigroups

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Abstract: Let $P_n$, $T_n$, $I_n$, and $S_n$ be the partial transformation semigroup, the (full) transformation semigroup, the symmetric inverse semigroup, and the symmetric group on $X_n = \{1, \ldots, n\}$, respectively. For $1 \leq r \leq n - 1$, let $PK_{n,r}$ be the subsemigroup consisting $\alpha \in P_n$ such that $|\text{im} \alpha| \leq r$ and let $SPK_{n,r} = PK_{n,r} \setminus T_n$. In this paper, we first examine the subsemigroup $I_{n,r} = I_n \cup PK_{n,r}$ and we find the necessary and sufficient conditions for any nonempty subset of $PK_{n,r}$ to be a (minimal) relative generating set of the subsemigroup $I_{n,r}$ modulo $I_n$. Then we examine the subsemigroups $PI_{n,r} = SI_n \cup PK_{n,r}$ and $SI_{n,r} = SI_n \cup SPK_{n,r}$ for $1 \leq r \leq n - 1$, where $SI_n = I_n \setminus S_n$ and compute their relative rank.

Key words: (Partial) transformation semigroup, symmetric inverse semigroup, symmetric group, (minimal) generating set, relative rank

1. Introduction

The partial transformation semigroup $P_X$, the (full) transformation semigroup $T_X$ and the symmetric inverse semigroup $I_X$ on a set $X$ have been extensively studied over the last sixty years, both in the finite and in the infinite cases. Among recent contributions are [1–6, 13, 16]. Here we are concerned solely with the case where $X = X_n = \{1, \ldots, n\}$, and we denote the semigroups $P_{X_n}$, $T_{X_n}$, and $I_{X_n}$, by $P_n$, $T_n$, and $I_n$, respectively. Moreover, we denote the subsemigroup $I_n \setminus S_n$ by $SI_n$ where $S_n$ is the symmetric group on $X_n$.

It is well known that $I_n$ is an inverse semigroup and every finite inverse semigroup $S$ is embeddable in $I_n$, the analog of Cayley’s theorem for finite groups. Hence, as emphasized in [1], the importance of $I_n$ to inverse semigroup theory is similar to that of the symmetric group $S_n$ to group theory. Moreover, Gomes and Howie remarked in [11] that very little has been written on the symmetric inverse semigroups. Despite the appearance of the books of Lipscomb [18], and Ganyushkin and Mazorchuk [8], as well as a handful of papers (for example, [10]), the study of $I_n$ is still in its infancy compared to that of $T_n$.

An element $\alpha$ of $P_n$ is called an idempotent if $\alpha^2 = \alpha$. We denote the set of all idempotents in any subset $U$ of any semigroup by $E(U)$. Let $S$ be a semigroup and let $A$ be a nonempty subset of $S$. Then the subsemigroup generated by $A$, that is the smallest subsemigroup of $S$ containing $A$, is denoted by $\langle A \rangle$. If a semigroup $S$ has a finite subset $A$ such that $S = \langle A \rangle$, then $S$ is called a finitely generated semigroup. The rank of a finitely generated semigroup $S$ is defined by $\text{rank}(S) = \min \{ |A| : \langle A \rangle = S \}$. For a fixed subset $G$ of

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a semigroup $S$, if there exists a subset $A$ of $S$ such that $\langle A \cup G \rangle = S$, then $A$ is called a \textit{relative generating set of $S$ modulo $G$}. Then the \textit{relative rank} of a finitely generated semigroup $S$ modulo $G$ is defined by

$$\text{rerank}(S : G) = \min\{|A| : \langle A \cup G \rangle = S\}.$$ 

For $1 \leq r \leq n$, let

$$K_{n,r} = \{\alpha \in T_n : |\text{im}(\alpha)| \leq r\}, \quad T_{n,r} = S_n \cup K_{n,r},$$

$$PK_{n,r} = \{\alpha \in P_n : |\text{im}(\alpha)| \leq r\}, \quad PT_{n,r} = S_n \cup PK_{n,r},$$

$$SPK_{n,r} = PK_{n,r} \setminus T_n = PK_{n,r} \setminus K_{n,r}, \quad A_{n,r} = A_n \cup K_{n,r},$$

$$PA_{n,r} = A_n \cup PK_{n,r}, \quad I_{n,r} = I_n \cup PK_{n,r},$$

$$SI_{n,r} = SI_n \cup SPK_{n,r} \quad \text{and} \quad PI_{n,r} = SI_n \cup PK_{n,r},$$

where $A_n$ denotes the alternating group on $X_n$.

Howie and McFadden proved in [15] that the rank of $K_{n,r}$ is $S(n,r)$, the Stirling number of the second kind, for $2 \leq r \leq n - 1$. Recall that the Stirling number $S(n,r)$ of the second kind is defined by

$$S(n,1) = S(n,n) = 1 \quad \text{and} \quad S(n,r) = S(n-1,r-1) + r \cdot S(n-1,r)$$

for $2 \leq r \leq n-1$. Moreover, Garba proved in [9] that the rank of the subsemigroup $PK_{n,r}$ of $P_n$ is $S(n+1,r+1)$ for $2 \leq r \leq n-1$.

For $n, r \in \mathbb{Z}^+$ with $r \leq n$, let $P_r(n)$ be the set of all integer solutions of the equation

$$x_1 + x_2 + \cdots + x_r = n \quad \text{with} \quad x_1 \geq x_2 \geq \cdots \geq x_r \geq 1,$$

and let $p_r(n) = |P_r(n)|$. If an $r$-tuple $(n_1, n_2, \ldots, n_r)$ is a solution of the equation given above, then it is called a \textit{partition of $n$ with $r$ terms} (see [12]). Ayk et al. developed a notation for certain primitive elements of $T_n$, called path-cycle, and described an algorithm to decompose an arbitrary transformation $\alpha$ in $T_n$ into a product of path-cycles in [2]. In addition, they used these techniques to obtain some informations about generators of $T_n$, and proved that, rerank$(T_{n,r} : S_n) = p_r(n)$ for $1 \leq r \leq n - 1$ (see also [17, Theorem 8]).

In [19], we obtained the necessary and sufficient conditions for any nonempty subset $U$ of $K_{n,r}$ (or $PK_{n,r}$) to be a (minimal) relative generating set of $T_{n,r}$ (or $PT_{n,r}$) modulo $S_n$ for $1 \leq r \leq n - 1$. Then we concluded the same result in [2, 17] that rerank$(T_{n,r} : S_n) = p_r(n)$ and, we obtained the new result

$$\text{rerank}(PT_{n,r} : S_n) = \sum_{s=0}^{n-r} p_r(n-s)$$

for $1 \leq r \leq n - 1$. Moreover, we showed that

$$\text{rerank}(A_{n,r} : A_n) = p_r(n) \quad \text{and} \quad \text{rerank}(PA_{n,r} : A_n) = \sum_{s=0}^{n-r} p_r(n-s)$$

for each $1 \leq r \leq n - 1$.

In this paper, we first find the necessary and sufficient conditions for any nonempty subset $U$ of $PK_{n,r}$ to be a (minimal) relative generating set of $I_{n,r}$ (respectively $PI_{n,r}$) modulo $I_n$ (respectively $SI_n$).
for \(1 \leq r \leq n - 1\). Moreover, we find the necessary and sufficient conditions for any subset \(U\) of \(SPK_{n,r}\) to be a (minimal) relative generating set of \(SI_{n,r}\) modulo \(SI_n\). Then we conclude that
\[
\text{rerank}(I_{n,r} : I_n) = p_r(n),
\]
\[
\text{rerank}(SI_{n,r} : SI_n) = p_r(n - 1) \quad \text{and}
\]
\[
\text{rerank}(PI_{n,r} : SI_n) = S(n, r)
\]
for \(1 \leq r \leq n - 1\).

2. Preliminaries

The \textit{height} and the \textit{kernel} of any partial transformation \(\alpha \in P_n\) are defined by
\[
h(\alpha) = |\text{im}(\alpha)| \quad \text{and}
\]
\[
\ker(\alpha) = \{(x, y) \in X_n \times X_n : \text{either } x, y \in \text{dom}(\alpha) \text{ and } x\alpha = y\alpha
\]
\[
\quad \text{or } x, y \notin \text{dom}(\alpha)\},
\]
respectively. For any \(\alpha, \beta \in P_n\) (also \(\alpha, \beta \in T_n\)), recall that \(\text{dom}(\alpha\beta) \subseteq \text{dom}(\alpha), \text{im}(\alpha\beta) \subseteq \text{im}(\beta), \ker(\alpha) \subseteq \ker(\alpha\beta)\), and that
\[
(\alpha, \beta) \in \mathcal{D} \iff h(\alpha) = h(\beta)
\]
\[
(\alpha, \beta) \in \mathcal{H} \iff \text{im}(\alpha) = \text{im}(\beta), \ker(\alpha) = \ker(\beta) \text{ and } \text{dom}(\alpha) = \text{dom}(\beta)
\]
where the equivalences \(\mathcal{D}\) and \(\mathcal{H}\) denote Green’s relations (see, for examples [14] and [8, Theorem 4.5.1]). For \(1 \leq r \leq n\), we denote that Green’s \(\mathcal{D}\)-class, consists of all elements in \(T_n\) (respectively \(P_n\) of height \(r\), by \(D^r\)) (respectively \(D^r_P\)). If the implied semigroup is clear from the context, we will use the simpler notation \(D_r\).

It is shown in [7] that Green’s \(\mathcal{D}\)-class \(D^r\) is generated by its idempotents, and so it follows from [15, Lemma 4] that \(K_{n,r} = \langle E(D^r) \rangle\) for \(2 \leq r \leq n - 1\). It is also shown in [9] that a subset \(A\) of Green’s \(\mathcal{D}\)-class \(D^r_P\) is a generating set of \(PK_{n,r}\) if and only if \(E(D^r_P) \subseteq \langle A \rangle\) for \(2 \leq r \leq n - 1\). Therefore, to show a subset \(A\) of \(D^r_P\) is a generating set of \(PK_{n,r}\), it is enough to prove \(D^r_P \subseteq \langle A \rangle\).

For a given nonempty set \(X\) and a positive integer \(r\) where \(1 \leq r \leq |X|\), let \(A_1, \ldots, A_r\) be a collection of nonempty disjoint subsets of \(X\). Then \(\xi = \{A_1, \ldots, A_r\}\) is called a \textit{partition} of \(X\) (with \(r\) terms) if \(X = \bigcup_{i=1}^{r} A_i\). A partition \(\{A_1, \ldots, A_r\}\) of \(X\) is called an \textit{ordered partition} if \(|A_1| \geq \cdots \geq |A_r|\), and is denoted by \((A_1, \ldots, A_r)\). For \(1 \leq r \leq n\), it is clear that \(\alpha \in D^r_P\) if and only if there exists a unique partition \(\{A_1, \ldots, A_r\}\) of \(\text{dom}\)(\(\alpha\)) such that \(\ker(\alpha) = \bigcup_{i=1}^{r+1} (A_i \times A_i)\) where \(A_{r+1} = X_n \setminus \text{dom}(\alpha) = \text{cdom}(\alpha)\); or equivalently, there exists a unique subset \(\{a_1, \ldots, a_r\}\) of \(X_n\) with cardinality \(r\), such that \(\text{im}(\alpha) = \{a_1, \ldots, a_r\}\). Without loss of generality suppose that \(A_i\alpha = a_i\) for each \(1 \leq i \leq r\), and so \(\alpha\) can be written in the following tabular form:
\[
\alpha = \left( \begin{array}{ccc} A_1 & \cdots & A_r \\ a_1 & \cdots & a_r \end{array} \right) \quad \text{and} \quad \alpha = \left( \begin{array}{ccc} A_1 & \cdots & A_r \\ a_1 & \cdots & a_r \end{array} \right) \quad \text{if } \alpha \in D^r_P.
\]

For \(\alpha \in D^r_P\), written in the above tabular form, there clearly exists a permutation \(\sigma \in S_r\) such that \((|A_1\sigma|, \ldots, |A_r\sigma|)\) is a partition of \(n\) with \(r\) terms. In this case the \textit{partition} of \(\alpha\) is defined by
\[
\text{part}(\alpha) = (|A_1\sigma|, \ldots, |A_r\sigma|).
\]
For \( \alpha \in D_r^P \) if \(|\text{cdom}(\alpha)| = |A_{r+1}| = s \geq 1\), similarly there exists a permutation \( \sigma \in S_r \) such that \((|A_{1\sigma}|, \ldots, |A_{r\sigma}|)\) is a partition of \( n-s \) with \( r \) terms. In this case, the \textit{co-partition} of \( \alpha \) is defined by

\[
\text{copart}(\alpha) = (|A_{1\sigma}|, \ldots, |A_{r\sigma}| : |A_{r+1}|).
\]

If \(|\text{dom}(\alpha)| = |A_{r+1}| = s = 0\), then for convenience the \textit{copartition} of \( \alpha \) is defined by \( \text{copart}(\alpha) = \text{part}(\alpha) \).

From now on, we consider the case \( 1 \leq r \leq n-1 \), since \( D_n^P = D_r^P = S_n \). We also assume that \( \alpha \in D_r^P \) is in the above tabular form unless stated otherwise.

First, recall Proposition 1 and Lemma 2 in [19]:

**Proposition 2.1** For \( 1 \leq r \leq n-1 \), let \( \alpha, \beta \in D_r \). Then \( \alpha \beta \in D_r \) if and only if \( \ker(\alpha \beta) = \ker(\alpha) \). Moreover, \( \alpha \beta \in D_r \) implies \( \text{cdom}(\alpha \beta) = \text{cdom}(\alpha) \).

**Lemma 2.2** For \( 1 \leq r \leq n-1 \), let \( \alpha, \beta \in D_r \). Then \( \text{copart}(\alpha) = \text{copart}(\beta) \) if and only if there exist \( \lambda, \mu \in S_n \) such that \( \alpha = \lambda \beta \mu \). In particular, for \( \alpha, \beta \in T_n \), \( \text{part}(\alpha) = \text{part}(\beta) \) if and only if there exist \( \lambda, \mu \in S_n \) such that \( \alpha = \lambda \beta \mu \).

Next we state and prove the following similar lemma which will be used throughout this paper:

**Lemma 2.3** For \( 1 \leq r \leq n-1 \), let \( \alpha \in D_r^P \) and let \( \text{copart}(\alpha) = (n_1, n_2, \ldots, n_r : s) \) with \( s \geq 1 \).

(i) For any \( \beta \in D_r^P \) with \( \text{part}(\beta) = (n_1 + s, n_2, \ldots, n_r) \), there exist \( \lambda, \mu \in SI_n \) such that \( \alpha = \lambda \beta \mu \).

(ii) For any \( \beta \in D_r^P \) with \( \text{copart}(\beta) = (n_1 + s - k, n_2, \ldots, n_r : k) \) where \( 1 \leq k \leq s \), there exist \( \lambda, \mu \in SI_n \) such that \( \alpha = \lambda \beta \mu \).

(iii) For any \( \beta \in D_r^P \), \( \text{copart}(\alpha) = \text{copart}(\beta) \) if and only if \( |\text{dom}(\alpha)| = |\text{dom}(\beta)| \) and there exist \( \lambda, \mu \in SI_n \) such that \( \alpha = \lambda \beta \mu \).

**Proof** Without loss of generality, let

\[
\alpha = \left( \begin{array}{ccc}
A_1 & \cdots & A_r \\
\vdots & \ddots & \vdots \\
A_{r+1}
\end{array} \right)
\]

where \( |A_i| = n_i \) for \( 1 \leq i \leq r \) and \( |A_{r+1}| = s \geq 1 \).

(i)-(ii) For any \( \beta \in D_r^P \) without loss of generality let

\[
\beta = \left( \begin{array}{ccc}
B_1 & \cdots & B_r \\
\vdots & \ddots & \vdots \\
B_{r+1}
\end{array} \right)
\]

where

\[
|B_i| = \begin{cases}
  n_i + s & \text{if } |B_{r+1}| = 0 \\
  n_i + s - k & \text{if } 1 \leq |B_{r+1}| = k \leq s
\end{cases}
\]

for each \( 2 \leq i \leq r \). Then it is clear that there exists \( \lambda \in SI_n \) such that \( A_1 \lambda \subseteq B_1 \), \( A_i \lambda = B_i \) for each \( 2 \leq i \leq r \) and \( \text{cdom}(\lambda) = \text{cdom}(\alpha) \). Moreover, consider the partial injective transformation \( \mu : \text{im}(\beta) \rightarrow \text{im}(\alpha) \) in \( SI_n \) defined by \( b_i \mu = a_i \) for each \( 1 \leq i \leq r \). Since \( A_i(\lambda \beta \mu) = a_i \) for each \( 1 \leq i \leq r \) and \( \text{cdom}(\lambda \beta \mu) = A_{r+1} \), we have \( \alpha = \lambda \beta \mu \), as required.
Theorem 3.1
Notice that, since \( I \subseteq \text{dom}(\alpha) \) and that there exist \( \lambda, \mu \in \text{SI}_n \) such that \( \alpha = \lambda \beta \mu \). Then \( \text{dom}(\alpha) \subseteq \text{dom}(\lambda) \) and \( \text{dom}(\alpha) \lambda \subseteq \text{dom}(\beta) \), and so \( \text{dom}(\alpha) \lambda = \text{dom}(\beta) \). Since \( \alpha, \beta \in D^P_r \), it follows that \( \text{im}(\beta) \subseteq \text{dom}(\mu) \), and so \( \text{im}(\beta) \mu = \text{im}(\alpha) \). Thus, it is easy to see that \( \text{dom}(\beta) = \bigcup_{i=1}^n A_i \lambda \) and that

\[
\beta = \begin{pmatrix} A_1 \lambda & \cdots & A_r \lambda \\ a_1 \mu^{-1} & \cdots & a_r \mu^{-1} \end{pmatrix} B_{r+1},
\]

where \( B_{r+1} = \text{cdom}(\beta) \). Therefore, since \( |A_i| = |A_i \lambda| \) for each \( 1 \leq i \leq r \), we have \( \text{copart}(\alpha) = \text{copart}(\beta) \), as required.

For \( 1 \leq r \leq n - 1 \), let \( \alpha, \beta \in I_n \) with \( h(\alpha) = h(\beta) = r \). Then it is clear that \( h(\alpha \beta) = r \) if and only if \( \text{im}(\alpha) = \text{dom}(\beta) \).

3. Relative ranks
Notice that, since \( I_{n,r} \setminus I_n = PK_{n,r} \) is an ideal of \( I_{n,r} \), any generating set of \( I_{n,r} \) must contain a generating set of \( I_n \) for \( 1 \leq r \leq n - 1 \). Thus, if \( W \subseteq I_{n,r} \) is a generating set of \( I_{n,r} \), then there exist \( U \subseteq D^P_r \cap W \) and \( V \subseteq I_n \cap W \) such that \( I_n = \langle V \rangle \) and \( I_{n,r} = \langle U \cup V \rangle = \langle U \cup I_n \rangle \). Therefore, any minimal relative generating set of \( I_{n,r} \) modulo \( I_n \) must be a subset of \( D^P_r \) for \( 1 \leq r \leq n - 1 \).

Theorem 3.1 Let \( 1 \leq r \leq n - 1 \) and \( U \subseteq PK_{n,r} \). Then \( I_{n,r} = \langle U \cup I_n \rangle \) if and only if, for each partition \( p = (n_1, \ldots, n_r) \in P_r(n) \), there exists \( \beta \in U \cap D^P_r \) such that \( \text{part}(\beta) = p \).

Proof (\( \Leftarrow \)) Let \( 1 \leq r \leq n - 1 \). For each partition \( p = (n_1, \ldots, n_r) \in P_r(n) \), we fix an arbitrary element \( \beta_p \in U \cap D^P_r \) with \( \text{part}(\beta_p) = p \). Then we denote the set of all these fixed elements by \( V = \{ \beta_p \in U : p \in P_r(n) \} \).

For any element \( \alpha \in D^P_r \) either \( \alpha \in K_{n,r} \) or \( \alpha \in SPK_{n,r} \). If \( \alpha \in K_{n,r} \) with \( \text{copart}(\alpha) = \text{part}(\alpha) = (n_1, \ldots, n_r) = p \), then there exists \( \beta_p \in V \) such that \( \text{part}(\beta_p) = p \), and so it follows from Lemma 2.2 that there exist \( \lambda, \mu \in S_n \subseteq I_n \) such that \( \alpha = \lambda \beta_p \mu \). Now suppose that \( \alpha \in SPK_{n,r} \) with \( \text{copart}(\alpha) = (n_1, n_2, \ldots, n_r : s) \) where \( s \geq 1 \). Then there exists \( \beta_p \in V \) such that \( \text{part}(\beta_p) = (n_1 + s, n_2, \ldots, n_r) = p \) and so, it follows from Lemma 2.3 \( (i) \) that there exist \( \lambda, \mu \in SI_n \subseteq I_n \) such that \( \alpha = \lambda \beta_p \mu \). Thus, the set \( V \cup I_n \) and so, the set \( U \cup I_n \) generates \( D^P_r \). Therefore, it follows from Lemma 2.6 given in [9] that \( U \cup I_n \) is a generating set of \( I_{n,r} \).

(\( \Rightarrow \)) For \( 1 \leq r \leq n - 1 \), let \( U \subseteq PK_{n,r} \) be a relative generating set of \( I_{n,r} \) modulo \( I_n \), that is \( I_{n,r} = \langle U \cup I_n \rangle \). Then, for an arbitrary partition \( p \in P_r(n) \), consider an arbitrary element \( \beta \in D^P_r \) such that \( \text{part}(\beta) = p \). Since \( I_{n,r} = \langle U \cup I_n \rangle \), \( \beta \) can be written as a product of finitely many elements of \( U \cup I_n \). It follows from \( \beta \notin I_n \) that either \( \beta = \beta \delta \) or \( \beta = \alpha \beta \delta \) for some \( \beta_1 \in U \), \( \delta \in \langle U \cup I_n \rangle \subseteq I_{n,r} \) and \( \alpha_1 \in I_n \setminus \{\varepsilon\} \) where \( \varepsilon \) is the identity permutation on \( X_n \). Now let

\[
\gamma = \begin{cases} 
\beta_1 & \text{if } \beta = \beta_1 \delta \\
\alpha_1 \beta_1 & \text{if } \beta = \alpha_1 \beta_1 \delta,
\end{cases}
\]

and so \( \beta = \gamma \delta \). Since \( X_n = \text{dom}(\beta) \subseteq \text{dom}(\gamma) \), it follows that \( \gamma \in T_n \) (and \( \alpha_1 \in S_n \) in the second case), and so \( \beta_1 \in T_n \) in both cases. Since \( h(\gamma) \geq h(\beta) = r \) and \( \beta_1 \in K_{n,r} \), it follows that \( \gamma, \beta_1 \in D^P_r \). Thus, since
ker(γ) ⊆ ker(β), we have ker(β) = ker(γ) and so,

\[ p = \text{part}(β) = \text{part}(γ) = \text{part}(β_1). \]

Therefore, β₁ ∈ U ∩ Dᵣ and part(β₁) = p, as required.

As an immediate consequence, we have the following corollary:

**Corollary 3.2** For each 1 ≤ r ≤ n − 1,

\[ \text{rerank}(I_{n,r} : I_n) = p_r(n) \]

where pᵣ(n) is the number of partitions of n with r terms.

Notice that, since SIₙ,r \ SIₙ = SPKₙ,r is an ideal of SIₙ,r, any generating set of SIₙ,r must contain a generating set of SIₙ. Similarly, any minimal relative generating set of SIₙ,r modulo SIₙ must be a subset of Dₚⁿ ∩ SPKₙ,r = Dₚⁿ \ Dᵣ for each 1 ≤ r ≤ n − 1.

**Theorem 3.3** Let 1 ≤ r ≤ n − 2 and U ⊆ SPKₙ,r. Then SIₙ,r = ⟨U ∪ SIₙ⟩ if and only if, for each partition \((n₁, \ldots, n_r) ∈ P_r(n − 1)\), there exists β ∈ U ∩ Dₚⁿ such that copart(β) = (n₁, \ldots, n_r : 1).

**Proof** (⇐) Let 1 ≤ r ≤ n − 2. For each partition \(p = (n₁, \ldots, n_r) ∈ P_r(n − 1)\), we fix an arbitrary element \(β_p ∈ U ∩ Dₚⁿ\) with copart(β_p) = (n₁, \ldots, n_r : 1). Then we denote the set of all these fixed elements by V = \(\{β_p ∈ U : p ∈ P_r(n − 1)\}\).

For any element \(α ∈ Dₚⁿ ∩ SPKₙ,r,\) let copart(α) = \((n₁, \ldots, n_r : s)\). Since s ≥ 1, we have \((n₁ + s − 1, n₂, \ldots, n_r) = p ∈ P_r(n − 1)\). Then there exists \(β_p ∈ V\) such that part(β_p) = p, so it follows from Lemma 2.3 (iii) (when \(k = 1\)) that there exist λ, μ ∈ SIₙ such that \(α = λβ_pμ\). Thus, the set V ∪ SIₙ, and so the set U ∪ SIₙ generates \(Dₚⁿ \setminus Dᵣ\). It follows from Lemma 2.6 in [9] that U ∪ SIₙ is a generating set of SIₙ,r.

(⇒) For 1 ≤ r ≤ n − 2, let U ⊆ SPKₙ,r be a relative generating set of SIₙ,r modulo SIₙ. Then, for an arbitrary partition \(p = (n₁, \ldots, n_r) ∈ P_r(n − 1)\), consider an arbitrary element \(β ∈ Dₚⁿ\) such that copart(β) = (n₁, \ldots, n_r : 1). Since SIₙ,r = (U ∪ SIₙ), β can be written as a product of finitely many elements of U ∪ SIₙ. It follows from the fact \(β \not∈ SIₙ\) that either \(β = β₁δ\) or \(β = α₁β₁δ\) for some \(β₁ ∈ U, α₁ ∈ SIₙ\), and \(δ ∈ SIₙ,r ∪ \{ε\}\) where \(ε\) is the identity permutation on Xₙ. Now let

\[ γ = \begin{cases} β₁ & \text{if } β = β₁δ \\ α₁β₁ & \text{if } β = α₁β₁δ, \end{cases} \]

and so \(β = γδ\). Since |dom(β)| = 1 and dom(β) ⊆ dom(γ) ≠ Xₙ, we have dom(β) = dom(γ). Moreover, since h(γ) ≥ h(β) = r and β₁ ∈ SPKₙ,r, it follows that \(γ, β₁ ∈ Dₚⁿ\). Thus, since ker(γ) ⊆ ker(β) and dom(β) = dom(γ), we have ker(β) = ker(γ), so copart(β) = copart(γ).

Now let \(γ = β₁\), then clearly copart(β) = copart(β₁). Otherwise, since \(α₁ ∈ SIₙ\), we have dom(β) = dom(γ) ⊆ dom(α₁) and |dom(β)| = 1, we have dom(β) = dom(α₁) and im(α₁) = dom(β₁), so

\[ \text{copart}(β) = \text{copart}(γ) = \text{copart}(β₁), \]

as claimed.

As an immediate consequence, we have the following corollary:
Corollary 3.4  For each $1 \leq r \leq n - 2$, 
\[
\text{rerank}(SI_{n,r} : SI_n) = p_r(n - 1)
\]
where $p_r(n - 1)$ is the number of partitions of $n - 1$ with $r$ terms.

Theorem 3.5  Let $1 \leq r \leq n - 1$ and $U \subseteq PK_{n,r}$. Then $PI_{n,r} = \langle U \cup SI_n \rangle$ if and only if, for each partition $\{A_1, \ldots, A_r\}$ of $X_n$, there exists $\beta \in U \cap D_r$ such that 
\[
\text{ker}(\beta) = \bigcup_{i=1}^{r}(A_i \times A_i).
\]

Proof  ($\Rightarrow$) For $1 \leq r \leq n - 1$, let $U \subseteq PK_{n,r}$ be a relative generating set of $PI_{n,r}$ modulo $SI_n$. Then, for an arbitrary partition $\{A_1, \ldots, A_r\}$ of $X_n$, consider an arbitrary element $\beta \in D_r \subseteq K_{n,r} \subseteq PI_{n,r}$ with 
\[
\text{ker}(\beta) = \bigcup_{i=1}^{r}(A_i \times A_i).
\]
Since $PI_{n,r} = \langle U \cup SI_n \rangle$ and $\text{dom}(\beta) = X_n$, there exist $\beta_1 \in U \cap T_n$ and $\delta \in PI_{n,r} \cup \{\varepsilon\}$, where $\varepsilon$ is the identity permutation on $X_n$, such that $\beta = \beta_1 \delta$. Then since $\beta_1 \in K_{n,r}$, $\text{ker}(\beta_1) \subseteq \text{ker}(\beta)$ and $h(\beta_1) \geq h(\beta) = r$, it follows that $h(\beta_1) = r$, so $\text{ker}(\beta) = \text{ker}(\beta_1)$. Therefore, $\text{ker}(\beta_1) = \bigcup_{i=1}^{r}(A_i \times A_i)$ and $\beta_1 \in U \cap D_r$, as required.

($\Leftarrow$) Let $1 \leq r \leq n - 1$. Recall that $PK_{n,r}$ is the disjoint union of $SPK_{n,r}$ and $K_{n,r}$. Then consider any $\alpha \in SPK_{n,r}$ with $\text{copart}(\alpha) = (n_1, \ldots, n_r : s)$. Since $s \geq 1$, it follows from Lemma 2.3 (i) that for any $\beta \in D_r$ with $\text{part}(\beta) = (n_1 + s, n_2, \ldots, n_r)$ there exist $\lambda, \mu \in SI_n$ such that $\alpha = \lambda \beta \mu$. Therefore, since $K_{n,r} = \langle D_r \rangle$, to show $PI_{n,r} = \langle U \cup SI_n \rangle$ it is enough to show that $D_r \subseteq (U \cup SI_n)$.

For each partition $\mathcal{A} = \{A_1, \ldots, A_r\}$ of $X_n$ into $r$ subsets, we fix an arbitrary element $\beta_{\mathcal{A}} \in U \cap D_r$ such that $\text{ker}(\beta_{\mathcal{A}}) = \bigcup_{i=1}^{r}(A_i \times A_i)$. Then we denote the set of all these fixed elements by 
\[
V = \{\beta_{\mathcal{A}} \in U : \mathcal{A} \text{ is a partition of } X_n \text{ into } r \text{ subsets}\}.
\]

For any $\alpha \in D_r$, let $\text{ker}(\alpha) = \bigcup_{i=1}^{r}(A_i \times A_i)$ where $\mathcal{A} = \{A_1, \ldots, A_r\}$ is a partition of $X_n$. Then there exists $\beta_{\mathcal{A}} \in V \subseteq U \cap D_r$ such that $\text{ker}(\beta_{\mathcal{A}}) = \bigcup_{i=1}^{r}(A_i \times A_i)$. Let $A_i \beta_{\mathcal{A}} = b_i$ for $1 \leq i \leq r$. Now consider the map $\delta : \text{im}(\beta_{\mathcal{A}}) \to \text{im}(\alpha)$ defined by $b_i \delta = A_i \alpha$ for each $1 \leq i \leq r$. Then it is clear that $\delta \in SI_n$ and that $\alpha = \beta_{\mathcal{A}} \delta \in \langle U \cup SI_n \rangle$, as required.

As an immediate consequence, we have the following corollary:

Corollary 3.6  For each $1 \leq r \leq n - 1$, 
\[
\text{rerank}(PI_{n,r} : SI_n) = S(n,r)
\]
where $S(n,r)$ is the Stirling number of the second kind.

Proof  The proof follows from the fact that the number of partitions of $X_n$ into $r$ subsets is $S(n,r)$.

References


[4] Ayik H, Bugay L. Generating sets of finite transformation semigroups \( PK(n, r) \) and \( K(n, r) \). Communications in Algebra 2015; 43: 412-422.


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