Multiplication modules with prime spectrum

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Received: 09.03.2018 • Accepted/Published Online: 19.06.2019 • Final Version: 31.07.2019

Abstract: The subject of this paper is the Zariski topology on a multiplication module $M$ over a commutative ring $R$. We find a characterization for the radical submodule $\text{rad}_M(0)$ and also show that there are proper ideals $I_1, \ldots, I_n$ of $R$ such that $\text{rad}_M(0) = \text{rad}_M((I_1 \cdots I_n)M)$. Finally, we prove that the spectrum $\text{Spec}(M)$ is irreducible if and only if $M$ is the finite sum of its submodules, whose $T$-radicals are prime in $M$.

Key words: Multiplication module, prime submodule, spectrum of module

1. Introduction
Throughout this study, $R$ and $M$ denote a commutative ring with identity and a unitary $R$-module, respectively. We also use $\text{Spec}(M)$ for the spectrum of prime submodules. In [4], the author investigated some properties of Zariski topology of multiplication modules. Motivated by this study, we generalize some important results in [4] and also give a characterization for the intersection of all prime submodules of $M$. Then $M$ is said to be a multiplication $R$-module if for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$. For example, invertible ideals and projective ideals of $R$ are multiplication $R$-modules. Since every cyclic module is a multiplication module and every finitely generated Artinian multiplication module is cyclic, there is a close relationship between multiplication modules and cyclic modules and so there are many studies related to these important concepts in module theory ([1, 4, 6, 8]).

A proper submodule $P$ of an $R$-module $M$ is said to be prime if for $a \in R$ and $m \in M$, $am \in P$ implies that $m \in P$ or $aM \subseteq P$. The radical of a submodule $N$ in $M$ denoted by $\text{rad}_M(N)$ is defined as the intersection of all prime submodules of $M$ containing $N$.

In [11], $V(N)$ was defined as the set $\{P \in \text{Spec}(M) : N \subseteq P\}$ for any submodule $N$ of an $R$-module $M$. Note that $V(M) = \emptyset$, $V(0) = \text{Spec}(M)$ and $\bigcap_{i \in A} V(N_i)$ is equivalent to $V\left(\sum_{i \in A} N_i\right)$ for any family of submodules $N_i$ of $M$.

Let $\Gamma(M) = \{V(N) : N$ is a submodule of $M\}$. If $\Gamma(M)$ is closed under finite union, $\Gamma(M)$ satisfies the axioms of closed subsets of a topological space. Then it is said that $M$ is a module with a Zariski topology.

A topological space $X$ is said to be Noetherian if the closed subsets of $X$ satisfy the descending chain condition. $X$ is said to be irreducible if $X \neq \emptyset$ and for every decomposition $X = X_1 \cup X_2$ with closed subsets $X_1, X_2 \subseteq X$, we have $X = X_1$ or $X = X_2$. $D \subseteq X$ is said to be dense in $X$ if for every nonempty open

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2010 AMS Mathematics Subject Classification: 13C13, 16D99
set \( U \subseteq X \), \( U \cap D \neq \emptyset \) holds. \( X \) is said to be quasi-compact if every open cover of \( X \) has a finite subcover ([7, 10]).

The aim of this paper is to study Zariski topology of multiplication modules over commutative ring with identity.

Section 2 is devoted to the study of a subspace associated with a submodule. We begin by giving a base for complement Zariski topology of a submodule \( N \) in a module \( M \). We show that \( \text{rad}_M(N) = \text{rad}_M(Rm_1 + \ldots + Rm_n) \), where \( m_i \in M \) if \( \mathcal{X}_N \) is quasicompact (Theorem 2.4). We also prove that \( N_N(0) \) is a prime submodule of \( M \) if and only if \( \mathcal{X}_N \) is irreducible (Theorem 2.8). Moreover, we give equivalent conditions for \( \text{Spec}(M) \) (Theorem 2.10).

In Section 3, we are interested in the relationships between the complement Zariski topologies and submodules of a module \( M \) to find some algebraic and topological tools for submodules and find some characterizations for the modules. We show that there are proper ideals \( I_1, \ldots, I_n \) of \( R \) such that \( \text{rad}_M(0) = \text{rad}_M((I_1 \ldots I_n) M) \), where \( M \) is a finitely generated multiplication \( R \)-module satisfying the \( T \)-condition for every submodule (Theorem 3.9). Consequently, we prove that \( \mathcal{X} = \bigcup_{i=1}^{n} \mathcal{X}_{I,M} \), where \( \mathcal{X}_{I,M} \) is irreducible if and only if \( M = \left( \sum_{i=1}^{n} I_i \right) M \) and \( N_{I,M}(0) \) is a prime submodule of \( M \) (Theorem 3.10).

2. The subspace associated with a submodule

Let \( M \) be a multiplication \( R \)-module and let \( N = IM \) be a submodule of \( M \), where \( I \) is an ideal of \( R \). Let \( \mathcal{X}_N = \text{Spec}(M) \setminus V(IM) \) and \( \tilde{V}(JM) = V(JM) \setminus V(IM) \), where \( J \) is an ideal of \( R \). Then

\[
\Gamma_N = \left\{ \tilde{V}(JM) : J \text{ is an ideal of } R \right\}
\]

satisfies the axioms for closed sets of a topological space on \( \mathcal{X}_N \). We name this topology as the complement Zariski topology of \( N \) in \( M \).

**Example 2.1** Let \( R = \mathbb{Z} \), \( M = 6\mathbb{Z} \) and \( N = 30\mathbb{Z} \). Then \( M \) is a multiplication \( \mathbb{Z} \)-module. It is clear that \( \text{Spec}(M) = \{ 6a : a \in \mathbb{P} \} \), \( V(30\mathbb{Z}) = \{ 30\mathbb{Z} \} \), \( V(36\mathbb{Z}) = \{ 12\mathbb{Z}, 18\mathbb{Z} \} \) and \( V(90\mathbb{Z}) = \{ 18\mathbb{Z}, 30\mathbb{Z} \} \), where \( \mathbb{P} \) is the set of prime numbers. Thus we have

\[
\mathcal{X}_N = \text{Spec}(M) \setminus V(30\mathbb{Z}) = \{ 6a : a \in \mathbb{P} \setminus \{ 5 \} \},
\]

\[
\tilde{V}(36\mathbb{Z}) = V(36\mathbb{Z}) \setminus V(30\mathbb{Z}) = \{ 12\mathbb{Z}, 18\mathbb{Z} \} \setminus \{ 30\mathbb{Z} \} = \{ 12\mathbb{Z}, 18\mathbb{Z} \},
\]

\[
\tilde{V}(90\mathbb{Z}) = V(90\mathbb{Z}) \setminus V(30\mathbb{Z}) = \{ 18\mathbb{Z}, 30\mathbb{Z} \} \setminus \{ 30\mathbb{Z} \} = \{ 18\mathbb{Z} \}.
\]

We fix the submodule \( N \) as \( N = IM \), where \( I \) is an ideal of \( R \), and the module \( M \) as a multiplication module in this section.

**Lemma 2.2** Let \( N = IM \) be a submodule of a multiplication \( R \)-module \( M \), where \( I \) is an ideal of \( R \). For any ideal \( J \) of \( R \), the set \( (\mathcal{X}_N)^J = \mathcal{X}_N \setminus \tilde{V}(JM) \) forms a base for the complement Zariski topology of \( N \) in \( M \) on \( \mathcal{X}_N \).

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The following statements hold:

**Proposition 2.3**

Let \( N = IM \) be a submodule of a multiplication \( R \)-module \( M \), where \( I \) is an ideal of \( R \). The following statements hold:

1. \( (X_N)^{JM} = X_N \setminus \hat{V}(JM) = \text{Spec}(M) \setminus V(IJM) \) for \( J \) is an ideal of \( R \).
2. \( (X_N)^{J_1M} \cap (X_N)^{J_2M} = (X_N)^{(J_1J_2)M} \) for every ideal \( J_1, J_2 \) of \( R \).
3. \( (X_N)^{JM} = \emptyset \) if and only if \( \text{rad}_M(IJM) \subseteq \text{rad}_M(0) \) for every ideal \( J \) of \( R \).
4. \( (X_N)^{J_1M} = (X_N)^{J_2M} \) if and only if \( \text{rad}_M(IJ_1M) = \text{rad}_M(IJ_2M) \) for every ideal \( J_1, J_2 \) of \( R \).
5. If \( (X_N)^{JM} = X_N \), then we have \( \text{rad}_M(IJM) = \text{rad}_M(IM) \) for every ideal \( J \) of \( R \).

Let \( M \) be an \( R \)-module and let \( N \) be a proper submodule of \( M \). Then we will say that \( N \) satisfies the condition (*) if there is a finite subset \( \Delta \) of \( \Lambda \) such that \( \text{rad}_M(\{m_i \in M : i \in \Lambda\}) = \text{rad}_M(\{m_j : j \in \Delta\}) \), whenever \( \text{rad}_M(N) \subseteq \text{rad}_M(\{m_i \in M : i \in \Lambda\}) \). It is clear that if \( M/\text{rad}_M(N) \) is a Noetherian module, \( N \) satisfies the condition (*).

In the following theorem, we give an algebraic property belonging to a submodule \( N \) and a topological property belonging to \( X_N \).

**Theorem 2.4**

Let \( N = IM \) be a proper submodule of a multiplication \( R \)-module \( M \), where \( I \) is an ideal of \( R \). Let \( (X_N)^{JM} = X_N \setminus \hat{V}(JM) = \text{Spec}(M) \setminus V(IJM) \) for any ideal \( J \) of \( R \). Then the following statements are true.

1. \( (X_N)^{JM} \) is quasicompact for every ideal \( J \) of \( R \).
2. If \( X_N \) is quasicompact, then \( \text{rad}_M(N) = \text{rad}_M(Rm_1 + \ldots + Rm_n) \), where \( m_i \in M \).
3. If \( N \) satisfies the condition (*), then \( X_N \) is quasicompact.

**Proof**

1. Obvious.

2. Let \( X_N \) be quasicompact.
Let \( N = \langle m_i : i \in \Lambda \rangle \). Then \( V(\{ m_i : i \in \Lambda \}) = V(N) \) and so \( \tilde{V} \left( \sum_{i \in \Lambda} Rm_i \right) = \emptyset \). Thus,

\[
\mathcal{X}_N = \mathcal{X}_N \setminus \emptyset = \mathcal{X}_N \setminus \tilde{V} \left( \sum_{i \in \Lambda} Rm_i \right) = \mathcal{X}_N \setminus \left( \bigcap_{i \in \Lambda} \tilde{V}(Rm_i) \right)
\]

\[
= \bigcup_{i \in \Lambda} \left( \mathcal{X}_N \setminus \tilde{V}(Rm_i) \right) = \bigcup_{i \in \Lambda} (\mathcal{X}_N)^{Rm_i}.
\]

Since \( \mathcal{X}_N \) is quasicompact, there is a finite set \( \Delta = \{1, 2, ..., n\} \subseteq \Lambda \) such that \( \mathcal{X}_N = \bigcup_{i \in \Delta} (\mathcal{X}_N)^{Rm_i} = \mathcal{X}_N \setminus \tilde{V}((m_1, m_2, ..., m_n)) \). Then \( V((m_1, m_2, ..., m_n)) \subseteq V(N) \) and so \( rad_M(N) \subseteq rad_M((m_1, m_2, ..., m_n)) \). On the other hand, we have \( rad_M((m_1, m_2, ..., m_n)) \subseteq rad_M(N) \), which means \( rad_M(N) = rad_M(Rm_1 + ... + Rm_n) \).

iii) Let \( N \) satisfy the condition (\( * \)).

Let \( \{ A_i : i \in \Lambda \} \) be an open cover of \( \mathcal{X}_N \). Since \( A_i \) can be expressed as a union of the sets of \( (\mathcal{X}_N)^{Rm_i} \), we may assume that \( A_i = (\mathcal{X}_N)^{Rm_i} \) for every \( i \in \Lambda \). Then

\[
\mathcal{X}_N = \bigcup_{i \in \Delta} (\mathcal{X}_N)^{Rm_i} = \bigcup_{i \in \Delta} \left( \mathcal{X}_N \setminus \tilde{V}(Rm_i) \right)
\]

\[
= \mathcal{X}_N \setminus \bigcap_{i \in \Delta} \tilde{V}(Rm_i)
\]

\[
= \mathcal{X}_N \setminus \tilde{V} \left( \sum_{i \in \Delta} Rm_i \right).
\]

Thus, \( \tilde{V} \left( \sum_{i \in \Lambda} Rm_i \right) = \emptyset \) and so \( \tilde{V} \left( \sum_{i \in \Lambda} Rm_i \right) \subseteq V(N) \).

In this case, \( rad_M(N) \subseteq rad_M \left( \sum_{i \in \Lambda} Rm_i \right) \). By the condition (\( * \)), there is a finite subset \( \Delta \subseteq \Lambda \) such that \( rad_M \left( \sum_{i \in \Delta} Rm_i \right) = rad_M \left( \sum_{i \in \Delta} Rm_i \right) \). Then \( V \left( \sum_{i \in \Delta} Rm_i \right) \subseteq V(N) \) and so \( \tilde{V} \left( \sum_{i \in \Delta} Rm_i \right) = \emptyset \). Then

\[
\mathcal{X}_N = \mathcal{X}_N \setminus \tilde{V} \left( \sum_{i \in \Delta} Rm_i \right) = \mathcal{X}_N \setminus \bigcap_{i \in \Delta} \tilde{V}(Rm_i)
\]

\[
= \bigcup_{i \in \Delta} \left( \mathcal{X}_N \setminus \tilde{V}(Rm_i) \right) = \bigcup_{i \in \Delta} (\mathcal{X}_N)^{Rm_i}.
\]

Since \( \mathcal{X}_N \) is covered by a finite number \( (\mathcal{X}_N)^{Rm_i} \), \( \mathcal{X}_N \) is quasicompact. \( \square \)

Theorem 2.4 also generalizes Theorem 3.7 in [4].

We now introduce the new submodule class which is a generalization of radical submodule of a module.

**Definition 2.5** Let \( N \) be a submodule of an \( R \)-module \( M \). The set \( \mathcal{N}_N(T) \) is defined as the intersection of all prime submodules containing submodule \( T \) which does not contain \( N \).
It is clear that $N_N(T)$ is equivalent to the radical of a submodule $T$ when $M = N$. Then $N_N(T)$ is a generalization of radical submodule.

**Example 2.6** Let $M = \mathbb{Z}$ be an $\mathbb{Z}$-module. Let $N = 12\mathbb{Z}$ and $T = 20\mathbb{Z}$ be submodules of $M$. Then $N_{12\mathbb{Z}}(20\mathbb{Z}) = 5\mathbb{Z}$ but $rad_{\mathbb{Z}}(20\mathbb{Z}) = 10\mathbb{Z}$. Thus $N_{12\mathbb{Z}}(20\mathbb{Z})$ is different from $rad_{\mathbb{Z}}(20\mathbb{Z})$.

The following lemma deals with algebraic properties of submodule $N_N(T)$.

**Lemma 2.7** Let $N = IM$ be a proper submodule of a multiplication $R$-module $M$, where $I$ is an ideal of $R$. The following statements are true:

i) $N_N(T)$ is a submodule of $M$.

ii) $N_N/K(T/K) = N_N(T)/K$, where $K \subseteq T$ is a submodule of $M$.

iii) $N_N(0) = N_{rad_M(N)}(0)$.

**Proof** The proof is straightforward.

The following theorem gives a connection between topological property of the complement Zariski topology $\mathcal{X}_N$ and algebraic property of submodule $N_N(0)$.

**Theorem 2.8** Let $N = IM$ be a proper submodule of a multiplication $R$-module $M$ and $rad_M(IM) \neq rad_M(0)$. Then $N_N(0)$ is a prime submodule of $M$ if and only if $\mathcal{X}_N$ is irreducible.

**Proof** Let $N_N(0)$ be a prime submodule of $M$ and $K$ be a nonempty open subset of $\mathcal{X}_M$. Then $K = \mathcal{X}_N \setminus \tilde{V}(JM) = \text{Spec}(M) \setminus (V(IM) \cup V(JM))$, where $JM$ is a submodule of $M$. Take $P \in K$. Then we have $P \notin V(IM) \cup V(JM)$, which means that $IM \not\subseteq P$ and $JM \not\subseteq P$. Thus, $N_N(0) \not\subseteq P$, so $JM \not\subseteq N_N(0) \subseteq P$. This implies that $N_N(0) \notin V(JM)$ and by the definition of $N_N(0)$, we get $N_N(0) \notin V(IM)$. Thus, $N_N(0) \subseteq K$.

Therefore, any nonempty open subset of $\mathcal{X}_N$ contains $N_N(0)$. This means that $\mathcal{X}_N$ is irreducible.

Let $\mathcal{X}_N$ be irreducible. Suppose that $N_N(0)$ is not a prime submodule of $M$. Then there exists elements $a \in R$ and $m \in M$ such that $am \in N_N(0)$, $m \notin N_N(0)$ and $aM \subseteq N_N(0)$.

Since $rad_M(N) = rad_M(IM) \neq rad_M(0)$ and $m \in M \setminus N_N(0)$, it follows that $\tilde{V}(Rm) \neq \emptyset$ and $\tilde{V}(Rm) \neq \mathcal{X}_N$, which implies $(\mathcal{X}_N)^{Rm} \neq \emptyset$. This can also be used to prove that $(\mathcal{X}_N)^{aM}$ is a nonempty open subset. Therefore, we get

$$(\mathcal{X}_N)^{aM} \cap (\mathcal{X}_N)^{Rm} = (\mathcal{X}_N)^{Rm \cap aM} \subseteq \mathcal{X}_N \setminus \tilde{V}(am) \subseteq \mathcal{X}_N \setminus \tilde{V}(N_N(0)) \subseteq \text{Spec}(M) \setminus (V(N_N(0)) \cup V(N)) = \emptyset.$$

This contradicts the hypothesis. Thus, $N_N(0)$ is a prime submodule of $M$.

We now need a condition on the submodules $N_N$, which helps us out with going further in finding more connections between topological space and module.

A module $M$ is said to satisfy $\mathcal{T}$-condition for a submodule $N$, if for any chain $N_N(U_1M) \subseteq N_N(U_2M) \subseteq N_N(U_3M) \subseteq \ldots$, where $U_i$ is an ideal of $R$, there is an integer $m$ such that $N_N(U_mM) = N_N(U_{m+i}M)$ for all positive integers $i$. 

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Theorem 2.9 Let $N = IM$ be a proper submodule of a multiplication $R$-module $M$, where $I$ is an ideal of $R$. Then the following statements are equivalent:

i) $M$ satisfies the $T$-condition.

ii) $\mathcal{X}_N$ is a Noetherian topological space.

Proof (i) $\Rightarrow$ (ii) Assume that $M$ satisfies the $T$-condition. Take the sequence $\tilde{V}(U_1M) \supseteq \tilde{V}(U_2M) \supseteq \tilde{V}(U_3M) \supseteq \ldots$, where $U_iM$ is a submodule of $M$. Then we have the sequence $\mathcal{N}_N(U_1M) \subseteq \mathcal{N}_N(U_2M) \subseteq \mathcal{N}_N(U_3M) \subseteq \ldots$ and there exists an integer $m$ such that $\mathcal{N}_N(U_mM) = \mathcal{N}_N(U_{m+i}M)$ for all positive integers $i$ since $M$ satisfies the $T$-condition. Therefore, we have $\tilde{V}(U_mM) = \tilde{V}(U_{m+i}M)$ for all positive integers $i$. Thus, $\mathcal{X}_N$ is Noetherian.

(ii) $\Rightarrow$ (i) Let $\mathcal{X}_N$ be a Noetherian topological space. Take the sequence $\mathcal{N}_N(U_1M) \subseteq \mathcal{N}_N(U_2M) \subseteq \mathcal{N}_N(U_3M)\ldots$, where $U_iM$ is a submodule of $M$. Then this yields the sequence $\tilde{V}(U_1M) \supseteq \tilde{V}(U_2M) \supseteq \tilde{V}(U_3M) \supseteq \ldots$. Since $\mathcal{X}_N$ is Noetherian, there exists an integer $m$ such that $\tilde{V}(U_mM) = \tilde{V}(U_{m+i}M)$ for all positive integers $i$. This implies $\mathcal{N}_N(U_mM) = \mathcal{N}_N(U_{m+i}M)$ for all positive integers $i$. Therefore $M$ satisfies the $T$-condition. \hfill \blacksquare

We close this section with the following theorem, which reveals the connections between algebraic and topological properties.

Theorem 2.10 Let $N = IM$ be a submodule of a multiplication $R$-module $M$, where $I$ is an ideal of $R$. Then the following are equivalent:

i) $\mathcal{X}$ is a Noetherian topological space.

ii) $\mathcal{X}_N$ is a Noetherian topological space for every submodule $N$ of $M$.

iii) $M$ satisfies the $T$-condition.

iv) $M$ satisfies ascending chain condition on the radical submodules of $M$.

Proof (i) $\Rightarrow$ (ii), (ii) $\Leftrightarrow$ (iii) and (iv) $\Leftrightarrow$ (i) are clear.

(ii) $\Rightarrow$ (i) Take the sequence $V(U_1M) \supseteq V(U_2M) \supseteq V(U_3M) \supseteq \ldots$, where $U_i$ is an ideal of $R$. Let $I = \cap U_i$ be an ideal of $R$. Consider the complement Zariski topology $\mathcal{X}_{IM}$. Then we have the sequence $\tilde{V}(U_1M) \supseteq \tilde{V}(U_2M) \supseteq \tilde{V}(U_3M) \supseteq \ldots$. Since $\mathcal{X}_N$ is Noetherian, there exists an integer $m$ such that $\tilde{V}(U_mM) = \tilde{V}(U_{m+i}M)$ for all positive integers $i$. Thus, we have $V(U_mM) = V(U_{m+i}M)$ for all positive integers $i$. Thus, $\mathcal{X}$ is Noetherian. \hfill \blacksquare

3. The connections between subspaces and submodules

This section deals with the relationships between the complement Zariski topologies and submodules of a module to find some algebraic and topological tools for submodules and find some characterizations for modules.

Theorem 3.1 Let $M$ be a multiplication $R$-module and let $I$, $J$, and $K$ be proper ideals of $R$. Then we have the following.

i) Any open set of $\mathcal{X}$ is of the form $\mathcal{X}_{IM}$.

ii) $\mathcal{X}_{IM} = \mathcal{X}_{JM}$ if and only if $\text{rad}_M(IM) = \text{rad}_M(JM)$.

iii) $\mathcal{X}_{IM} \cap \mathcal{X}_{JM} = \mathcal{X}_{KM}$ if and only if $\text{rad}_M(IJM) = \text{rad}_M(KM)$.
iv) \( \mathcal{X}_{iM} \subseteq \mathcal{X}_{JM} \) if and only if \( \text{rad}_M(IM) \subseteq \text{rad}_M(JM) \).

**Proof** It is straightforward. \( \Box \)

**Corollary 3.2** Let \( M \) be a multiplication \( R \)-module and let \( I, J \) be proper ideals of \( R \). Then \( \mathcal{X}_{iM} \cap \mathcal{X}_{JM} = \emptyset \) if and only if \( \text{rad}_M(IJM) = \text{rad}_M(0) \).

**Theorem 3.3** Let \( M \) be a multiplication \( R \)-module and let \( I \) be a proper ideal of \( R \). Then \( \mathcal{X}_{iM} \) is dense in \( \mathcal{X} \) if and only if \( \text{rad}_M(IJM) \neq \text{rad}_M(0) \) for every proper ideal \( J \) such that \( JM \) is not contained in \( \text{rad}_M(0) \).

**Proof** Let \( \mathcal{X}_{iM} \) be dense in \( \mathcal{X} \) and let \( J \) be any proper ideal of \( R \), where \( JM \) is not in \( \text{rad}_M(0) \). Then \( \mathcal{X}_{iM} = \text{Spec}(M) \setminus V(JM) \) is a nonempty open set in the Zariski topology and by the hypothesis, the intersection of \( \mathcal{X}_{iM} \) and \( \mathcal{X}_{JM} \) is nonempty. Thus, \( \text{rad}_M(IJM) \neq \text{rad}_M(0) \) by Corollary 3.2.

Let \( \text{rad}_M(IJM) \neq \text{rad}_M(0) \) for every proper ideal \( J \) of \( R \), where \( JM \) is not in \( \text{rad}_M(0) \). By Corollary 3.2, since \( \mathcal{X}_{iM} \cap \mathcal{X}_{JM} \neq \emptyset \), it follows that \( \mathcal{X}_{iM} \) is dense in \( \mathcal{X} \). \( \Box \)

The following theorem gives a characterization for the module \( M/\text{rad}_M(0) \) by using topological properties.

**Theorem 3.4** Let \( M \) be a faithful multiplication \( R \)-module. The following statements are equivalent:

i) \( \text{rad}_M(0) \) is a prime submodule of \( M \).

ii) \( \text{Spec}(M) \) is irreducible.

iii) Every submodule of \( M/\text{rad}_M(0) \) is essential.

iv) Every open subset of \( \text{Spec}(M) \) is dense.

**Proof** (i) \( \Leftrightarrow \) (ii) By [5], it can be easily proved.

(iii) \( \Rightarrow \) (iv) Let \( \mathcal{X}_{iM} \) and \( \mathcal{X}_{JM} \) be open subsets for any ideals \( I, J \) of \( R \). Then \( (JM + \text{rad}_M(0))/\text{rad}_M(0) \) and \( (IM + \text{rad}_M(0))/\text{rad}_M(0) \) are submodules of \( M/\text{rad}_M(0) \). Since \( \bigcap_{i \in A} (I_iM) = \left( \bigcap_{i \in A} I_i \right)M \), we observe that

\[
\text{rad}_M\left( \bigcap_{i \in A} (I_iM) \right) = \text{rad}_M\left( \left( \bigcap_{i \in A} I_i \right)M \right) = \left( \text{rad}_R\left( \bigcap_{i \in A} I_i \right) \right)M,
\]

and

\[
\text{rad}_M(0) \neq \text{rad}_M\left( (JM + \text{rad}_M(0)) \cap (IM + \text{rad}_M(0)) \right)
\]

\[
= \text{rad}_M\left( (J + \text{rad}_R(0))M \cap (I + \text{rad}_R(0))M \right)
\]

\[
= \text{rad}_M\left( (J + \text{rad}_R(0)) \cap (I + \text{rad}_R(0)) \right)M
\]

\[
= \text{rad}_R\left( J + \text{rad}_R(0) \right)M
\]

\[
= \text{rad}_M\left( IJM + \text{rad}_M(0) \right)
\]

and so \( \text{rad}_M(IJM) \neq \text{rad}_M(0) \), which means that \( \mathcal{X}_{iM} \) is dense.

(iv) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) One can prove it by the above method. \( \Box \)

**Theorem 3.5** Let \( M \) be a finitely generated multiplication \( R \)-module and let \( I_i \) be a proper ideal of \( R \) for all \( i \in A \). Then \( \bigcup_{i \in A} \mathcal{X}_{iM} = \mathcal{X}_{DM} \) for any ideal \( D \) of \( R \) if and only if \( \text{rad}_M(\text{DM}) = \text{rad}_M\left( \left( \sum_{i \in A} I_i \right)M \right) \).

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Corollary 3.7 Let $M$ be a finitely generated multiplication $R$-module and let $I_i$ be a proper ideal of $R$ for all $i \in \Lambda$. Then the following statements are equivalent:

(i) $\bigcup_{i \in \Lambda} X_{I_i} = X_{DM}$.

(ii) There is a finite subset $\Delta$ of $\Lambda$ such that $\bigcup_{i \in \Delta} X_{I_i} = X_{DM}$.

(iii) There is a finite subset $\Delta$ of $\Lambda$ such that $\text{rad}_M\left( \left( \sum_{i \in \Delta} I_i \right) M \right) = \text{rad}_M(DM)$.

Proof (i) $\Rightarrow$ (iii) Let $\text{rad}_M(DM) = \text{rad}_M\left( \left( \sum_{i \in \Delta} I_i \right) M \right)$ and let $D$ be an ideal finitely generated by the set $\{d_1, \ldots, d_t\}$. For each $d_iM$, there is a positive number $n_i$ such that $d_i^{n_i}M \subseteq \left( \sum_{i \in \Delta} I_i \right) M$ and so there is a finite subset $\Delta_i$ of $\Lambda$ such that $d_i^{n_i}M \subseteq \left( \sum_{i \in \Delta_i} I_i \right) M$. If $n = \max\{n_1, \ldots, n_t\}$ and $\Delta = \bigcup_{i=1}^t \Delta_i$ then $\text{rad}_M\left( \left( \sum_{i \in \Delta_i} I_i \right) M \right) = \text{rad}_M(DM)$.

(iii) $\Rightarrow$ (ii) By Theorem 3.5.

(ii) $\Rightarrow$ (i) It is clear. $\Box$

The following corollary is a special case of Theorem 3.6.

Theorem 3.6 Let $M$ be a finitely generated multiplication $R$-module, $I_i$ proper ideals of $R$ for all $i \in \Lambda$ and $D$ a finitely generated ideal of $R$. Then the following statements are equivalent:

(i) $\bigcup_{i \in \Lambda} X_{I_i} = X_{DM}$.

(ii) There is a finite subset $\Delta$ of $\Lambda$ such that $\bigcup_{i \in \Delta} X_{I_i} = X_{DM}$.

(iii) There is a finite subset $\Delta$ of $\Lambda$ such that $\text{rad}_M\left( \left( \sum_{i \in \Delta} I_i \right) M \right) = \text{rad}_M(DM)$.

Proof (i) $\Rightarrow$ (iii) Let $\text{rad}_M(DM) = \text{rad}_M\left( \left( \sum_{i \in \Delta} I_i \right) M \right)$ and let $D$ be an ideal finitely generated by the set $\{d_1, \ldots, d_t\}$. For each $d_iM$, there is a positive number $n_i$ such that $d_i^{n_i}M \subseteq \left( \sum_{i \in \Delta} I_i \right) M$ and so there is a finite subset $\Delta_i$ of $\Lambda$ such that $d_i^{n_i}M \subseteq \left( \sum_{i \in \Delta_i} I_i \right) M$. If $n = \max\{n_1, \ldots, n_t\}$ and $\Delta = \bigcup_{i=1}^t \Delta_i$ then $\text{rad}_M\left( \left( \sum_{i \in \Delta_i} I_i \right) M \right) = \text{rad}_M(DM)$.

(iii) $\Rightarrow$ (ii) By Theorem 3.5.

(ii) $\Rightarrow$ (i) It is clear. $\Box$
Corollary 3.8 Let $M$ be a finitely generated multiplication $R$-module and let $D$ be a finitely generated ideal of $R$. Then $X_{DM}$ is quasicompact.

Using topological properties, we are now ready to prove the following characterization for $\operatorname{rad}_M(0)$.

Theorem 3.9 Let $M$ be a finitely generated multiplication $R$-module satisfying the $T$-condition for every submodule. Then there are proper ideals $I_1, \ldots, I_n$ of $R$ such that $\operatorname{rad}_M(0) = \operatorname{rad}_M((I_1 \ldots I_n)M)$.

Proof Let $X = \operatorname{Spec}(M)$ be Noetherian topological space. By [9], $X$ has only a finite number of distinct irreducible components $U_i$ such that $\bigcup_{i=1}^n U_i = X$. It is well known that any irreducible component in a topological space is closed and so for each $i$, there is an ideal $I_i$ such that $U_i = V(I_iM)$. Then

$$\emptyset = X \setminus \bigcup_{i=1}^n V(I_iM) = \bigcap_{i=1}^n (X \setminus V(I_iM)) = \bigcap_{i=1}^n X_{I_iM}.$$  

Thus, by Theorem 3.1, $\operatorname{rad}_M(0) = \bigcap_{i=1}^n \operatorname{rad}_M(I_iM) = \operatorname{rad}_M((I_1 \ldots I_n)M)$. \hfill \Box

By using Theorems 2.8 and 3.5, we close the paper with the following result.

Theorem 3.10 Let $M$ be a finitely generated multiplication $R$-module and let $I_i$ be an ideal of $R$. Then $X = \bigcup_{i=1}^n X_{I_iM}$, where $X_{I_iM}$ is irreducible, if and only if $M = \left(\sum_{i=1}^n I_i\right)M$ and $X_{I_iM}(0)$ is a prime submodule of $M$.

Acknowledgments

The authors would like to thank the referee for the valuable comments and suggestions on this paper.

References


2008


