A blow-up result for nonlocal thin-film equation with positive initial energy

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Abstract: In this note, we consider a thin-film equation including a diffusion term, a fourth order term and a nonlocal source term under the periodic boundary conditions. In particular, a finite time blow-up result is established for the case of positive initial energy provided that

$$\frac{\pi^2}{a^2} \leq \frac{2}{p-1},$$

where \(a\) is the length of the interval and \(p > 1\) is the power of nonlinear force term. Also upper and lower blow-up times are estimated.

Key words: Nonlinear thin film equation, positive initial energy, blow up, periodic boundary condition, Non-local source term

1. Introduction

In this note we consider the following initial and periodic boundary value problem:

$$u_t - u_{xx} + u_{xxxx} = |u|^{p-1}u - \frac{1}{a} \int_0^a |u|^{p-1}u \, dx, \quad x \in \mathbb{R}, \ t > 0,$$  \hspace{1cm} (1.1)

$$u(x, t) = u(x + a, t), \quad \text{for all } x \in \mathbb{R}, \text{ and } t > 0,$$  \hspace{1cm} (1.2)

$$u(x, 0) = u_0(x) \quad x \in \mathbb{R},$$  \hspace{1cm} (1.3)

where \(p > 1\), \(u_0 \in H^2_{\text{per}}(\Omega), \ \Omega = (0, a)\) and \(\int_0^a u_0(x) \, dx = 0\) with \(u_0 \neq 0\). The novelty in the problem above is the existence of the diffusion term and the periodic boundary conditions which are natural boundary conditions for this type models [9].

The following general fourth-order reaction diffusion equation

$$u_t + A_1 \Delta u + A_2 \Delta^2 u + A_3 \nabla \cdot (|\nabla u|^2 \nabla u) + A_4 \Delta |\nabla u|^2 = g(x, t) + \eta(x, t),$$  \hspace{1cm} (1.4)

arises in theories such as the thin film theory, lubrication theory, phase transitions etc. (see [12]). In (1.4), \(u(x, t)\) and \(A_1 \Delta u\) denote the height of a film in epitaxial growth and the diffusion due to evaporation condensation, respectively. The terms \(A_2 \Delta^2 u\) and \(A_3 \nabla \cdot (|\nabla u|^2 \nabla u)\) are the capilarity-driven surface diffusion and the hopping

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of atoms, respectively. The term \( A_4 \Delta |\nabla u|^2 \) describes motion of an atom to a neighbouring kink. The functions \( g(x, t) \) and \( \eta(x, t) \) represent the mean deposition flux, and some Gaussian noise, respectively. For a detailed description of this model we refer the readers to [9].

In [12], Qu and Zhou considered 1D form of the equation in (1.4) and derived a threshold result of global existence and nonexistence of solutions when \( A_1 = A_3 = A_4 = 0 \). In this work, the flux term is the nonlocal-source term

\[
g(x, t) = |u|^{p-1} v - \frac{1}{a} \int_0^a |u|^{p-1} u
\]

and boundary conditions are

\[
u_x(0, t) = u_x(a, t) = 0, \quad u_{xxx}(0, t) = u_{xxx}(a, t) = 0.
\]

In [7], using potential well theory, Zhou established a blow-up result for the same problem in [12] assuming that the initial energy is positive. Also, he derived an upper bound for the blow-up time. Existence of blow up solutions is a long standing topic in the study of nonlinear models of partial differential equations. Interested readers may refer to some or all of the references [1, 3–6, 8, 9, 13, 14]. In this work, using the potential well method the existence of finite time blow up solutions will be studied under the assumption

\[
0 < J(u_0) < E_m \quad \text{and} \quad 0 < I(u_0)
\]

where

\[
J(u) = \frac{1}{2} \| u_x \|^2 + \frac{1}{2} \| u_{xx} \|^2 - \frac{1}{p+1} \| u \|_{p+1}^{p+1},
\]

\[
I(u) = \| u_x \|^2 + \| u_{xx} \|^2 - \| u \|_{p+1}^{p+1},
\]

and \( E_m \) is the potential well depth given below in (1.8). The existence of blow-up solutions and lower bounds for their blow-up times will be estimated. By the zero average of initial function from (1.1), we obtain that

\[
\frac{d}{dt} \int_0^a u \, dx = 0.
\]

Now we present some notations and mathematical tools which we shall need:

Let us denote the \( L^2(\Omega) \)-inner product and the \( L^2(\Omega) \)-norm by \( \langle u, v \rangle = \int_\Omega u(x)v(x) \, dx \) by \( \| \cdot \| \), respectively. Let \( \dot{H}^2_{per}(\Omega) := \left\{ u \in H^2_{per}(\Omega) : \int_\Omega u \, dx = 0 \right\} \). The pair \( \left( \dot{H}^2_{per}(\Omega), \| \cdot \|_{\dot{H}^2_{per}(\Omega)} \right) \) is a Hilbert space with the inner product and the norm \( \langle u, v \rangle_{\dot{H}^2_{per}(\Omega)} = \int_\Omega u_x v_x \, dx + \int_0^a u_{xx} v_{xx} \, dx, \quad \| u \|_{\dot{H}^2_{per}(\Omega)}^2 := \| u_x \|^2 + \| u_{xx} \|^2 \), respectively.

By the Sobolev embedding theorem, the inclusion \( \dot{H}^2_{per}(\Omega) \hookrightarrow L^{p+1}(\Omega) \) is continuous, so there exists an optimal embedding constant \( B \) such that:

\[
\| u \|_{p+1} \leq B \| u_x \|.
\]

In the rest of this text, we shall use \( B \) as the optimal embedding constant. Define the function

\[
g(\alpha) := \frac{1}{2} \left( \frac{a^2 + \pi^2}{a^2} \right) \alpha^2 - \frac{1}{p+1} (B\alpha)^{p+1}.
\]
It is obvious that \( g(\alpha) \) has a critical point at

\[
\alpha_1 = \left[ \frac{a^2 + \pi^2}{a^2} \right]^\frac{1}{p-1} B^{-\frac{p+1}{p}}
\]

and attains its maximum value at this point as

\[
E_m := \frac{p - 1}{2(p + 1)} \left[ \frac{a^2 + \pi^2}{a^2} \right]^{\frac{p+1}{2}} B^{-\frac{2(p+1)}{p-1}} = \left( \frac{a^2 + \pi^2}{a^2} \right) \frac{p - 1}{2(p + 1)} \alpha_1^2,
\]

(1.8)
because \( g(\alpha) \) is increasing on \((0, \alpha_1)\) and is decreasing on \((\alpha_1, \infty)\) with \( \lim_{\alpha \to \infty} g(\alpha) = -\infty \).

The rest of this note is organized as follows: Section 2 is devoted to a local existence result and a regularity theorem. In Section 3 a blow-up result is established. In Section 4 a lower blow-up time is estimated.

## 2. Local Existence

**Definition 2.1** A function \( u(x, t) \) is called a weak solution of (1.1) if

\[
u \in L^\infty(0, T, \dot{H}^2_{\text{per}}(\Omega)) \text{ and } u_t \in L^2(0, T, L^2(\Omega))
\]

and satisfies

\[
\int_0^t \int_\Omega \left[u_t \phi + u_x \phi_x + u_{xx} \phi_{xx} - \left( |u|^{p-1} u - \int_\Omega |u|^{p-1} u \right) \phi \right] \, dx \, ds = 0,
\]

(2.1)

for all \( \phi \in \dot{H}^2_{\text{per}}(\Omega) \).

Now we give the following existence result for weak solutions and its proof:

**Theorem 2.2** Assume that \( p > 1, u_0 \in \dot{H}^2_{\text{per}}(\Omega), \) and \( I(u_0) > 0 \) then the problems (1.1)-(1.3) has a unique local solution \( u(x, t) \) with \( u \in L^\infty([0, T]; \dot{H}^2_{\text{per}}(\Omega)) \) and \( u' \in L^\infty([0, T]; L^2(\Omega)) \).

**Proof** Let \( \{ \omega_n \}_{n \in \mathbb{N}} \) be the set of eigenfunctions of the problem

\[-u_{xx} = \lambda u, \quad u(x + a) = u(x) .\]

The eigenvalues of this has the property \( \lambda_n \leq \lambda_{n+1}, \) for all \( n \in \mathbb{N}, \) and \( \lim_{n \to \infty} \lambda_n = \infty \). The eigenfunctions are orthogonal in the spaces \( \dot{H}^2_{\text{per}}(\Omega), \dot{H}^1_{\text{per}}(\Omega), \) and \( L^2(\Omega) \). We normalize the eigenfunctions in \( L^2(\Omega) \):

\[ (\omega_i, \omega_j) = \delta_{ij}. \]

We proceed by constructing approximate solutions \( u_m := \sum_{i=1}^{m} g_{im}(t) \omega_i(x) \)

satisfying

\[ (u_m, \omega_j) + (u_{mxx}, \omega_{jxx}) + (u_{mx}, \omega_{jx}) = (f(u_m), \omega_j), \]

(2.2)

and

\[ u_m(x, 0) = u_{0m}(x) = \sum_{i=1}^{m} (u_0, \omega_i) \omega_i, \]

(2.3)
where \( f(u_m) = |u_m|^{p-1}u_m - \frac{1}{a}\int_0^a |u_m|^{p-1}u_m\,dx \). The problem (2.2)–(2.3) is equivalent to the following initial
value problem of a system of first order ordinary differential equations for \( \{g_{jm}(t)\}_{j=1}^m \):
\[
ge'_{jm}(t) = (\lambda_j^2 + \lambda_j)g_{jm}(t) + f_{jm}(t), \quad g_{jm}(0) = (u_0, w_j).
\] (2.4)
For \( p > 1 \), the function \( f_{jm}(t) \) is a continuously differentiable function of \( g_{jm} \). Thus, the problem (2.4) has a
unique local solution \( g_{jm}(t) \) on \([0, T_1]\) for \( j = 1, 2, \ldots, m \).

Now we multiply (2.2) by \( g'_{jm}(t) \) and sum from 1 to \( m \)
\[
\|u_m'(t)\|^2 + \frac{d}{dt}\left[ \frac{1}{2}\|u_{mx}\|^2 + \frac{1}{2}\|u\|^2 - \frac{1}{p+1}|u_m|^{p+1}\right] = 0.
\] (2.5)
Integrating from 0 to \( t \) we obtain
\[
\int_0^t \|u_m'(\tau)\|^2\,d\tau + J(u_m) = J(u_{0m}).
\] (2.6)
By the convergence of \( u_m(x, 0) \to u_0(x) \) in \( \hat{H}^2_{per}(\Omega) \), we get \( J(u_m(x, 0)) \to J(u_0) < d \). Then for sufficiently
large \( m \), we have
\[
\int_0^t \|u_m'(\tau)\|^2\,d\tau + J(u_m) < d
\]
for \( 0 \leq t \leq T_1 \). By the assumption \( I(u_0) > 0, J(u_m(t)) \) is positive on some interval \([0, T_2]\). Let \( T \) be the
minimum of \( T_1 \) and \( T_2 \). By
\[
J(u_m) = \frac{p-1}{2(p+1)} \left( \|u_{mx}\|^2 + \|m\|^2 \right) + \frac{1}{p+1}I(u_m),
\]
for sufficiently large \( m \) and any \( t \in [0, T] \), we obtain
\[
\int_0^t \|u_m'(\tau)\|^2\,d\tau + \frac{p-1}{2(p+1)}\|u_{mx}\|^2 < d.
\]
Hence, we obtain the following a priori estimates
\[
\left\{
\begin{array}{l}
\int_0^t \|u_m'(\tau)\|^2\,d\tau < d, \quad \text{for} \quad t \in [0, T],
\sup_{[0, T]} \|u_m'(t)\|^2 < d,
\|u_{mx}\|^2 < \left( \frac{2(p+1)}{p-1}d \right)^{\frac{1}{2}}, \quad \text{for} \quad t \in [0, T],
\|u_m\|^{p+1}_{p+1} \leq B\|u_{mx}\|^p < B\left( \frac{2(p+1)}{p-1}d \right)^{\frac{1}{2}}, \quad \text{for} \quad t \in [0, T].
\end{array}
\right.
\]
Therefore, the sequence \( \{u_m\} \) has a subsequence, which is denoted by itself has the following convergence
properties:
\[
\begin{align*}
&u'_m \overset{w^*}{\to} u', \quad \text{in} \ L^2([0, T]; L^2(\Omega)),
&u_m \overset{w^*}{\to} u, \quad \text{in} \ L^\infty([0, T]; L^2(\Omega)),
&u_m \overset{w}{\to} u, \quad \text{in} \ L^\infty([0, T]; \hat{H}^2_{per}(\Omega)),
&|u_m|^{p-1}u_m \overset{w}{\to} |u|^{p-1}u, \quad \text{in} \ L^\infty([0, T]; L^2(\Omega)).
\end{align*}
\]
Hence, the problem admits a unique local weak solution on \([0, T]\). 
\[\square\]
Now we adapt the following regularity theorem for the smoothness of weak solutions from [2] (Chapter 6.3, Theorem 4):

**Theorem 2.3** Suppose \( f \in L^2(\Omega) \) and the boundary \( \partial \Omega \) is \( C^2 \) and \( u \in \dot{H}^1_{\text{per}}(\Omega) \) is a weak solution of the elliptic boundary value problem

\[
\begin{aligned}
-u_{xx} &= f \quad \text{in } (0,a) \\
u(x) &= u(x + a)
\end{aligned}
\]

Then \( u \in \dot{H}^2_{\text{per}}(\Omega) \) and \( \|u\|^2_{\dot{H}^2_{\text{per}}(\Omega)} \leq C(\|f\|^2 + \|u\|^2_{\dot{H}^1_{\text{per}}(\Omega)}) \), where \( C \) depends on \( \Omega \).

**Theorem 2.4** Let \( u_0 \in \dot{H}^2_{\text{per}}(\Omega) \), \( f \in L^2([0,T];L^2(\Omega)) \) and \( u \in L^\infty([0,T];\dot{H}^2_{\text{per}}(\Omega)) \) be a weak solution of (1.1)-(1.3). Then \( u \in \dot{H}^4_{\text{per}}(\Omega) \).

**Proof** For a.e \( t \) we have the identity

\[(u', v) + (u_{xxxx}, v) - (u_{xx}, v) = (f, v) \quad \text{for each } v \in \dot{H}^2_{\text{per}}(\Omega).
\]

We rewrite \((u_{xxxx}, v) = (h, v)\) for \( h = f + u_{xx} - u' \) for a.e. \( t \) in \([0,T]\). By \((*)\) \( h \in L^2([0,T];L^2(\Omega)) \) and hence \( u \in \dot{H}^4_{\text{per}}(\Omega) \) follows from the previous theorem.

### 3. main result

For the establishment of blow-up solution and an upper bound for the blow-up time we have the following result:

**Theorem 3.1** Assume that \( 0 < J(u_0) < E_m \) and \( \|u_{0x}\| > \alpha_1 \), then the solution \( u(x,t) \) of (1.1)-(1.3) blows up at a finite time

\[T_* \leq T_{\text{max}} = \frac{2(\|u_0\|^2 + \|u_{0x}\|^2)^{\frac{\frac{p-1}{2}}{2}}}{C(p+1)},\]

where \( C = C_1/C_2 \) with

\[C_1 = \frac{p-1}{p+1} \left(1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{p+1}\right)\quad \text{and} \quad C_2 = (2^{-\frac{p+1}{2}})(\frac{p+1}{2})\]

and \( T_{\text{max}} \) is the an upper bound for the blow up time.

First we introduce the following lemmata which are analogous to the ones in [7] and are necessary for the proof of this theorem.

**Lemma 3.2** The potential energy functional functional \( J(u)(t) \) given in (1.5) is nonincreasing in \( t \) because of \( J'(u(t)) = -\|u_t\|^2 \leq 0 \) and

\[J(u) = J(u_0) - \int_0^t \|u_s\|^2 \, ds.\]

This lemma is a corollary of Theorem 2.2. However, we shall include the proof of the following lemma because its use differs slightly in our case:
Lemma 3.3 Assume that the axioms of Theorem 3.1 hold. Then there exists a positive constant \( \alpha_2 > \alpha_1 \) such that

\[
\|u_x(.,t)\| \geq \alpha_2, \quad \text{for all } t \geq 0,
\]  
and

\[
\|u_x(.,t)\|_{p+1} \geq B\alpha_2, \quad \text{for all } t \geq 0.
\]

Proof Let \( \alpha = \|u_x\| \). Using the Wirtinger’s inequality and Sobolev imbedding theorem we deduce that

\[
J(u) = \frac{1}{2} \|u_x\|^2 + \frac{1}{2} \|u_{xx}\|^2 - \frac{1}{p+1} \|u\|_{p+1}^{p+1}
\]
\[
\geq \frac{1}{2} \left( \frac{a^2 + \pi^2}{a^2} \right) \|u_x\|^2 - \frac{1}{p+1} (B\|u\|_{p+1}^{p+1}) \tag{3.3}
\]
\[
= \frac{1}{2} \left( \frac{a^2 + \pi^2}{a^2} \right) \alpha^2 - \frac{1}{p+1} (B\alpha)^{p+1}
\]

\[
= g(\alpha).
\]

Since \( J(u_0) < E_m \), there exists \( \alpha_2 > \alpha_1 > 0 \) such that \( J(u_0) = g(\alpha_2) \). Let \( \alpha_0 = \|u_{0x}\| > \alpha_1 \). By (3.3), we have \( g(\alpha_0) \leq J(u_0) = g(\alpha_2) \). Since \( \alpha_0, \alpha_2 \geq \alpha_1 \), we obtain \( \alpha_0 \geq \alpha_2 \). Hence, (3.1) is true for \( t = 0 \).

To prove that (3.1) is true for \( t > 0 \) we assume that (3.1) is not true for some \( t_0 \). Using the continuity of \( \|u_x(.,t)\| \), which follows from (*), and \( \alpha_1 < \alpha_2 \) we may choose \( t_0 \) so that \( \alpha_1 < \|u_x(.,t_0)\| < \alpha_2 \). Then from (3.3) it follows that

\[
J(u_0) = g(\alpha_2) < g(\|u_x(.,t_0)\|) \leq J(u(t_0))
\]

which contradicts the fact that \( J(u)(t) \) is nonincreasing.

From Lemma 3.2 it follows that \( J(u_0) \geq J(u) \). When this is combined with (3.3) we find

\[
\frac{1}{p+1} \|u\|_{p+1}^{p+1} \geq \frac{1}{2} \|u_x\|^2 + \frac{1}{2} \|u_{xx}\|^2 - J(u_0)
\]
\[
\geq \frac{1}{2} \left( \frac{a^2 + \pi^2}{a^2} \right) \alpha_2^2 - J(u_0)
\]
\[
= \frac{1}{2} \left( \frac{a^2 + \pi^2}{a^2} \right) \alpha_2^2 - g(\alpha_2) \tag{3.4}
\]
\[
= \frac{1}{p+1} (B\alpha_2)^{p+1}.
\]

Hence, (3.2) follows.

Lemma 3.4 Under the assumptions of Theorem 3.4 we have

\[
\frac{\alpha_2}{\alpha_1} \geq \left[ (p+1) \left( \frac{a^2 + \pi^2}{2a^2} - \frac{J(u_0)}{\alpha_1^2} \right) \right]^{\frac{1}{p+1}} > 1 + \frac{\pi^2}{a^2} \tag{3.5}
\]
Proof  Let $\beta = \frac{\alpha_2}{\alpha_1} > 1$. Now we have

$$J(u_0) = g(\alpha_2) = g(\alpha_1, \beta) = (\alpha_1 \beta)^2 \left[ \frac{a^2 + \pi^2}{a^2} - \frac{1}{p + 1} B^{p+1}(\beta \alpha_1)^{p-1} \right]$$

$$= (\alpha_1 \beta)^2 \left( \frac{a^2 + \pi^2}{2a^2} - \frac{1}{p + 1} \beta^{p-1} \right).$$

(3.6)

Dividing both sides the previous equality by $(\alpha_1 \beta)^2$, we obtain

$$\left( \frac{a^2 + \pi^2}{2a^2} - \frac{1}{p + 1} \beta^{p-1} \right) = \frac{J(u_0)^2}{(\beta \alpha_1)} < \frac{J(u_0)}{\alpha_1^2}.$$ 

By this inequality, we have

$$(p + 1)^{1-p} \left[ \frac{a^2 + \pi^2}{2a^2} - \frac{J(u_0)}{\alpha_1} \right]^{\frac{1}{p-1}} \leq \beta = \frac{\alpha_2}{\alpha_1}.$$ 

Since $J(u_0) < E_m = \left( \frac{a^2 + \pi^2}{a^2} \right)^{\frac{p-1}{2(p+1)}} \alpha_1^2$,

$$\frac{J(u_0)}{\alpha_1^2} \leq \frac{a^2 + \pi^2}{2a^2} \frac{p - 1}{p + 1}.$$ 

So

$$(p + 1) \left[ \frac{a^2 + \pi^2}{2a^2} \right] \left( 1 - \frac{p - 1}{p + 1} \right) = \frac{a^2 + \pi^2}{a^2}.$$ 

\[\square\]

**Lemma 3.5** Let $H(u) = E_m - J(u)$. Under the assumptions of Theorem 3.1 the functions $H(u)$ enjoys the property

$$0 < H(u_0) \leq H(u) \leq \frac{1}{p + 1} \|u\|_{p+1}^{p+1},$$

(3.7)

provided that $\frac{\pi^2}{a^2} \leq \frac{2}{p - 1}$.

**Proof**  Since $J(u)$ is nonincreasing in $t$, $H(u)(t)$ is nondecreasing in $t$. By the assumption $J(u_0) < E_m$, we have

$$0 < E_m - J(u_0) = H(u_0) \leq H(u).$$

(3.8)
Now, for $\alpha_2 > \alpha_1$ and by the help of (3.1), we derive

\[
H(u) = E_m - \frac{1}{2} \|u_x\|^2 - \frac{1}{2} \|u_{xx}\|^2 + \frac{1}{p+1} \|u\|_{p+1}^{p+1} \\
\leq E_m - \frac{1}{2} \left( \frac{a^2 + \pi^2}{a^2} \right) \|u_x\|^2 + \frac{1}{p+1} \|u\|_{p+1}^{p+1} \\
\leq E_m - \frac{1}{2} \alpha_1^2 + \frac{1}{p+1} \|u\|_{p+1}^{p+1} \\
= \left( \frac{a^2 + \pi^2}{a^2} \right) \frac{p-1}{2(p+1)} \alpha_1^2 - \frac{1}{2} \alpha_1^2 + \frac{1}{p+1} \|u\|_{p+1}^{p+1} \\
= \left( \frac{a^2 + \pi^2}{a^2} \right) \frac{p-1}{2(p+1)} \alpha_1^2 + \frac{1}{p+1} \|u\|_{p+1}^{p+1} \\
\leq \frac{1}{p+1} \|u\|_{p+1}^{p+1}.
\]

Since $\frac{\pi^2}{a^2} \leq \frac{2}{p-1}$, the inequality (3.7) follows. \hfill \Box

Now we can prove our main result:

**Proof** Define $\phi(t) = \frac{1}{2} \int_0^t u^2 \, dx$. Then

\[
\phi'(t) = -\|u_x\|^2 - \|u_{xx}\|^2 + \|u\|_{p+1}^{p+1} \\
= -2J(u) - 2J(u) + \frac{2}{p+1} \|u\|_{p+1}^{p+1} + 2H(u) - \frac{p-1}{p+1} \|u\|_{p+1}^{p+1} \\
= 2H(u) - 2E_m + \frac{p-1}{p+1} \|u\|_{p+1}^{p+1}.
\]

Now, using

\[
E_m := \frac{p-1}{2(p+1)} \left[ \frac{a^2 + \pi^2}{a^2} \right]^{\frac{p+1}{p+1}} B^{-2(p-1)}
\]

and (3.2) we have

\[
2E_m = \frac{p-1}{p+1} \left[ \frac{a^2 + \pi^2}{a^2} \right]^{\frac{p+1}{p+1}} B^{-2(p-1)} = \frac{p-1}{p+1} \left[ \frac{a^2 + \pi^2}{a^2} \right]^{\frac{p+1}{p+1}} (BB^{-2(p-1)})^{p+1} \\
= \frac{p-1}{p+1} (B\alpha_1)^{p+1} = \frac{p-1}{p+1} \left( \frac{\alpha_1}{\alpha_2} \right)^{p+1} (B\alpha_2)^{p+1} \\
\leq \frac{p-1}{p+1} \left( \frac{\alpha_1}{\alpha_2} \right)^{p+1} \|u\|_{p+1}^{p+1}.
\]

Hence, we obtain

\[
\phi'(t) \geq C_1 \|u\|_{p+1}^{p+1} + 2H(u),
\]

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where
\[ C_1 = \frac{p - 1}{p + 1} \left[ 1 - \frac{\alpha_1}{\alpha_2} \right]^{p+1}, \]
is a positive number. On the other hand, by Hölder’s inequality, we have
\[ \phi^{\frac{p+1}{p}}(t) \geq C_2 \|u\|_{p+1}^{p+1}, \]  
(3.13)
where \( C_2 = (2^{\frac{p+1}{p}})(a^{\frac{p+1}{p}}) \). Combining (3.12) and (3.13), we obtain
\[ \phi'(t) \geq C \phi^{\frac{p+1}{p}}(t), \]
where \( C = C_1/C_2 \), and
\[ \phi(t) \geq \left( \phi^{\frac{p-1}{p}}(0) - \frac{p - 1}{2} Ct \right)^{-\frac{p^2}{2}}. \]  
(3.14)
with \( \phi(0) = \frac{1}{2} \|u_0\|^2 \). Let
\[ T_{\text{max}} := \frac{2^{\frac{p+1}{p}}}{C(p-1)} \|u_0\|^{-(p-1)}. \]  
(3.15)
Hence, \( \phi(t) \) blows up at some finite time \( T_* \leq T_{\text{max}} \). By (3.15) and (3.5), we easily estimate \( T_* \) as
\[ T_* \leq T_{\text{max}} = \frac{2^{\frac{p+1}{p}}}{C(p-1)} \|u_0\|^{-(p-1)} \left( p + 1 \right) \left( 1 - \frac{\alpha_1}{\alpha_2} \right)^{p+1}. \]  
(3.16)
\[ \square \]

4. A lower blow-up time

In this section by adapting a result of Phillipin[10] we will obtain a lower blow-up time estimate. Our goal is to show the existence of a time interval \((0, T_0)\) in which \( \|u\|_{\mathbb{H}^{p+1}_0}^2 \) remains bounded. Here is our result:

**Theorem 4.1** Let \( u(x, t) \) be a solution of the problem (1.1)–(1.3). Assume that the constant \( p > 1 \). Then
\[ \phi(t) = \int_0^a (u_{xx})^2 \, dx, \]
remains bounded for \( t \in (0, T_{\text{min}}) \) such that
\[ T_{\text{min}} = \frac{1}{\phi^{p-1}(0)(p-1)\gamma}, \]  
(4.1)
where \( \gamma \) is the best optimal constant of the Kondrachov inequality.

In the proof of this theorem we will use \( u_{xxx}, u_{xxxx} \in \mathbb{L}^2(0, a) \) due to Theorem 2.4.
Proof Differentiating $\phi(t)$, we obtain

$$
\phi'(t) = 2 \int_0^a u_{xx} u_{xxt} \, dx = \int_0^a u_t u_{xxx} \, dx.
$$

Plugging $u_t = u_{xx} - u_{xxxx} + |u|^{p-1} u - \frac{1}{a} \int_0^a |u|^{p-1} u \, dx$ into above equality and using integration by parts, we obtain

$$
\phi'(t) = -\|u_{xxx}\|^2 - \|u_{xxxx}\|^2 + \int_0^a u|u|^{p-1} u_{xxxx} \, dx. \tag{4.2}
$$

Applying the arithmetic-geometric mean inequality to the last term above, we obtain

$$
\int_0^a u|u|^{p-1} u_{xxxx} \, dx \leq \frac{1}{4} \int_0^a |u|^{2p} \, dx + \int_0^a (u_{xx})^2 \, dx. \tag{4.3}
$$

Thus, we have

$$
\phi'(t) \leq \frac{1}{4} \int_0^a |u|^{2p} \, dx.
$$

Thanks to Kondrachov inequality $\int_0^a |u|^{2p} \, dx \leq \gamma \|u_{xx}\|^{2p}$, for $p > 1$. Thus,

$$
\phi'(t) \leq \gamma (\phi(t))^p, \quad p > 1.
$$

Solving the previous inequality we obtain:

$$
\phi^{1-p}(t) \geq \phi^{1-p}(0) - (p-1)\gamma t. \tag{4.4}
$$

Hence, (4.1) follows from (4.4). \[\Box\]

Acknowledgements

We would like to thank the reviewer for helpful comments that served to improve the paper.

References


