Solvability of a system of nonlinear difference equations of higher order

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Abstract: In this paper, we show that the following higher-order system of nonlinear difference equations,

\[ x_{n+1} = \frac{x_{n-k}y_{n-k-l}}{y_{n-l}(a_n + b_n x_{n-k}y_{n-k-l})}, \quad y_{n+1} = \frac{y_{n-k}x_{n-k-l}}{x_{n-l}(a_n + b_n y_{n-k}x_{n-k-l})}, \quad n \in \mathbb{N}_0, \]

where \( k, l \in \mathbb{N}, (a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0}, (\alpha_n)_{n \in \mathbb{N}_0}, (\beta_n)_{n \in \mathbb{N}_0} \) and the initial values \( x_{-i}, y_{-i}, i = 1, k+l, \) are real numbers, can be solved and some results in the literature can be extended further. Also, by using these obtained formulas, we investigate the asymptotic behavior of well-defined solutions of the above difference equations system for the case \( k = 2, l = k. \)

Key words: System of nonlinear difference equations, solution of system of difference equations in closed form, asymptotic behavior

1. Introduction and preliminaries

The study of nonlinear difference equations and systems of difference equations has attracted the attention of many authors in recent years (see, e.g., [1–49]). To find the difference equation or system of difference equations that can be solved in closed form is only one of the challenges. Almost all solvable difference equations or their systems have various generalizations of solvable difference equations and systems. That is, when a solvable equation is found, generalizations such as solvability with parameters, solvability with increasing order, solvability with periodic coefficients, and solvability as two-dimensional or three-dimensional systems can be studied. For example, the following difference equations,

\[ x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_n (\pm 1 \pm x_{n-1}x_{n-2})}, \quad n \in \mathbb{N}_0, \]  
(1.1)

were studied by El-Metwally and Elsayed in [10]. Then, in [35], Eq. (1.1) was generalized to the following difference equation:

\[ x_{n+1} = \frac{x_n x_{n-k}}{x_{n+k+1} (a_n + b_n x_{n-k})}, \quad n \in \mathbb{N}_0, \]  
(1.2)
where \( k \in \mathbb{N} \). Also, in [32], Eq. (1.2) was extended to the following two-dimensional system of difference equation:

\[ x_{n+1} = \frac{x_n y_{n-k}}{y_{n-k+1} (a_n + b_n x_{n-k} y_{n-k})}, \quad y_{n+1} = \frac{y_n x_{n-k}}{x_{n-k+1} (c_n + d_n x_{n-k} y_{n-k})}, \quad n \in \mathbb{N}_0, \]

(1.3)

where \( k \in \mathbb{N} \). Further, system (1.3) is a natural generalization of the difference equation systems given in [12, 15]. Moreover, in [34], the authors studied the next equation:

\[ x_n = \frac{x_{n-2} x_{n-k-2}}{x_{n-k} (a_n + b_n x_{n-2} x_{n-k+2})}, \quad n \in \mathbb{N}_0, \]

(1.4)

where \( k \in \mathbb{N} \), which is a natural generalization of the equations given in [10, 11, 14, 20, 21]. On the other hand, in [1], Eq. (1.1) was extended to the next differences systems:

\[ x_{n+1} = \frac{x_{n-1} y_{n-2}}{y_n (-1 \pm x_{n-1} y_{n-2})}, \quad y_{n+1} = \frac{y_{n-1} x_{n-2}}{x_n (\pm 1 \pm y_{n-1} x_{n-2})}, \quad n \in \mathbb{N}_0. \]

(1.5)

Some of their solution forms were proved by induction. However, the obtained formulas have not been confirmed by some theoretical explanations.

A natural question is to study both the two-dimensional form of equation (1.4) and more general systems of (1.3) and (1.5) solvable in closed form. Here we study such a system. That is, we deal with the following system of difference equations:

\[ x_n = \frac{x_{n-k} y_{n-k-l}}{y_{n-l} (a_n + b_n x_{n-k} y_{n-k})}, \quad y_n = \frac{y_{n-k} x_{n-k-l}}{x_{n-l} (\alpha_n + \beta_n y_{n-k} x_{n-k-l})}, \quad n \in \mathbb{N}_0, \]

(1.6)

where \( k, l \in \mathbb{N} \), \((a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0}, (\alpha_n)_{n \in \mathbb{N}_0}, (\beta_n)_{n \in \mathbb{N}_0}\) and the initial values \( x_{-i}, y_{-i}, \ i = 1, k + l, \) are real numbers. We solve system (1.6) in closed form and determine the asymptotic behavior of solutions for the case \( k = 2, l = k \). Note that system (1.6) is a natural extension of both Eqs. (1.1), (1.2), and (1.4) and systems (1.3) and (1.5).

This paper is organized as follows. In the following section, we obtain the formulas of the solutions of system (1.6) in closed form and give the forbidden set of the initial values of system (1.6). In Section 3, first we write the formulas of the solutions of system (1.6) when the coefficients of system (1.6) are constant and \( k = 2, l = k \). Furthermore, in this case, we investigate the asymptotic behavior of the solutions of system (1.6) in detail according to the case of the coefficients \( a \) and \( \alpha \).

2. The solutions of the system (1.6)

Let \((x_n, y_n)_{n \geq -k-l}\) be a solution of system (1.6). If at least one of the initial values \( x_{-i}, y_{-i}, \ i = 1, k + l, \) is equal to zero, then the solution of system (1.6) is not defined. For example, if \( x_{-k-l} = 0 \), then \( y_0 = 0 \) and so \( x_1 \) is not defined. Similarly, if \( y_{-k-l} = 0 \), then \( x_0 = 0 \) and so \( y_1 \) is not defined. For \( i = 1, k + l - 1 \), the other cases are similar. On the other hand, if \( x_{n_0} = 0 \) (\( n_0 \in \mathbb{N}_0 \)), \( x_n \neq 0 \), for \(-k-l \leq n \leq n_0 - 1 \), and \( x_m \) and \( y_m \) are defined for \(-k-l \leq m \leq n_0 - 1 \), then according to the first equation in (1.6) we get that \( y_{n_0-k-l} = 0 \). If \( n_0 - k-l \leq -1 \), then \( y_{-i_0} = 0 \), for \( i_0 = 1, k+l \). If \( n_0 > k+l-1 \), then according to the second equation in
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(1.6) we have \( y_{n_0 - 2k - l} = 0 \). If \( n_0 - 2k - l \leq -1 \), then \( y_{-i_0} = 0 \) for \( i_0 = \overline{1, k + l} \). Repeating this procedure, we have \( y_{-i_0} = 0 \) for \( i_0 = \overline{1, k + l} \). Similarly, if \( y_{n_1} = 0 \) \( (n_1 \in \mathbb{N}_0) \), \( y_n \neq 0 \), for \( -k - l \leq n \leq n_1 - 1 \), and \( x_m \) and \( y_m \) are defined for \( -k - l \leq m \leq n_1 - 1 \), one can easily show that \( x_{-i_1} = 0 \) for \( i_1 = \overline{1, k + l} \). Thus, for every well-defined solution of system (1.6), we have

\[
x_n y_n \neq 0, \quad n \geq -k - l,
\]  

(2.1)

if and only if \( x_{-i} y_{-i} \neq 0 \), for \( i = \overline{1, k + l} \). Therefore, by employing the substitution

\[
u_n = \frac{1}{x_n y_{n-l}}, \quad v_n = \frac{1}{y_n x_{n-l}}, \quad n \geq -k,
\]  

(2.2)

we transform system (1.6) into the following nonhomogeneous linear \( k \)-order difference equations:

\[
u_n = a_n u_{n-k} + b_n, \quad v_n = \alpha_n v_{n-k} + \beta_n, \quad n \in \mathbb{N}_0.
\]  

(2.3)

If we apply the decomposition of indexes \( n \to km + i \), for some \( m \geq -1 \) and \( i = \overline{0, k-1} \), to (2.3), then they become

\[
u_{km+i} = a_{km+i} u_{k(m-1)+i} + b_{km+i}, \quad v_{km+i} = \alpha_{km+i} v_{k(m-1)+i} + \beta_{km+i}, \quad m \in \mathbb{N}_0,
\]  

(2.4)

which are first-order \( k \)-equations. The solutions of equations in (2.4) are

\[
u_{km+i} = u_{-k} \prod_{j=0}^{m} a_{kj+i} + \sum_{l=0}^{m} b_{kl+i} \prod_{j=l+1}^{m} a_{kj+i}, \quad v_{km+i} = u_{-k} \prod_{j=0}^{m} \alpha_{kj+i} + \sum_{l=0}^{m} \beta_{kl+i} \prod_{j=l+1}^{m} \alpha_{kj+i}, \quad m \geq -1,
\]  

(2.5)

\( i \in \{0,1,\ldots,k-1\} \). If \( a_n = a, b_n = b, \alpha_n = \alpha, \) and \( \beta_n = \beta \), for every \( n \in \mathbb{N}_0 \), then we have

\[
u_{km+i} = \frac{a^{m+1}(u_{-k}(1-a) - b) + b}{1-a}, \quad v_{km+i} = \frac{\alpha^{m+1}(v_{-k}(1-\alpha) - \beta) + \beta}{1-\alpha}, \quad m \geq -1,
\]  

(2.6)

if \( a \neq 1 \) and \( \alpha \neq 1 \), and

\[
u_{km+i} = u_{-k} + b (m+1), \quad v_{km+i} = v_{-k} + \beta (m+1), \quad m \geq -1,
\]  

(2.7)

if \( a = 1 \) and \( \alpha = 1 \). From (2.2) it follows that

\[
x_n = \frac{1}{u_n y_{n-l}} = \frac{v_{n-l}}{u_n} x_{n-2l}, \quad y_n = \frac{1}{v_n x_{n-l}} = \frac{u_{n-l}}{v_n} y_{n-2l},
\]  

(2.8)

for \( n \geq l - k \), and consequently

\[
x_{2lm+i} = x_{i-2l} \prod_{j=0}^{m} \frac{u_{(2j-1)i+l}}{u_{2j+l}}, \quad y_{2lm+i} = y_{i-2l} \prod_{j=0}^{m} \frac{v_{(2j-1)i+l}}{v_{2j+l}},
\]  

(2.9)

for \( m \in \mathbb{N}_0 \) and \( i \in \{l-k, l-k+1, \ldots, 3l-k-1\} \). By the help of the well-known quotient remainder theorem, there exist \( k \in \mathbb{N} \) and \( s \in \mathbb{N}_0 \) such that \( n = ks + j_1 \) and \( j_1 \in \{0,1,\ldots,k-1\} \). From this and system (2.9), we can write

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The change of variables in

Now we give the solution form of system (1536)

By substituting the formulas (1.6)

Note that one can get system (1.6) follows from (2.5) and (2.10) or (2.11).

Now we give the solution form of system (1.6) when all the coefficients in system (1.6) are constant. To do this, we suppose that \( a_n = a, b_n = b, \alpha_n = \alpha, \beta_n = \beta \) for every \( n \in \mathbb{N}_0 \). Then system (1.6) becomes

Then we may assume that \( \gcd(k, l) = 1 \). Indeed, if \( |\gcd(k, l)| = f > 1 \), denoting the greatest common divisor of natural numbers \( k \) and \( l \), then \( k = f k_1 \) and \( l = f l_1 \) for some \( k_1, l_1 \in \mathbb{N} \) such that \( \gcd(k_1, l_1) = 1 \). Since every \( n \in \mathbb{N}_0 \) has the form \( n = mf + i \), for some \( m \in \mathbb{N}_0 \) and \( i \in \{0, 1, \ldots, f - 1\} \), from (2.12) we get

The change of variables

in (2.13) yields \( x_m^{(i)} = x_m f + i, \quad y_m^{(i)} = y_m f + i, \quad m \in \mathbb{N}_0, \quad i \in \{0, 1, \ldots, f - 1\} \) in (2.13) yields \( x_m^{(i)} = x_m f + i, \quad y_m^{(i)} = y_m f + i, \quad i \in \{0, 1, \ldots, f - 1\} \), which are \( r \) independent solutions of the system

Note that one can get system (2.13) by taking \( k_1 \) and \( l_1 \), respectively, instead of \( k \) and \( l \) in system (2.12).

From now on, we assume that the greatest common divisor of \( k \) and \( l \) is equal to 1; that is, \( \gcd(k, l) = 1 \).

By substituting the formulas (2.6)–(2.7) into (2.10)–(2.11), we obtain the formulas for well-defined solutions of system (2.12) when \( \gcd(k, l) = 1 \).
where belongs to the forbidden set of the initial values for system \((x, y, z, t)\) given by the set

\[
\mathcal{F} = \bigcup_{m \in \mathbb{N}_0} \bigcup_{i=0}^{k-1} \left\{ (x_{-k-i}, \ldots, x_{-1}, y_{-k-i}, \ldots, y_{-1}) \in \mathbb{R}^{2(k+1)} : x_{i-k}y_{i-k-i} = \frac{1}{c_m}, y_{i-k}x_{i-k-i} = \frac{1}{d_m}, \right\}
\]

where

\[
c_m := -\sum_{j=0}^{m} \frac{b_{kj+i}}{a_{kj+i}} \prod_{l=0}^{j-1} \frac{1}{a_{kl+i}} \neq 0, \quad d_m := -\sum_{j=0}^{m} \frac{\beta_{kj+i}}{\alpha_{kj+i}} \prod_{l=0}^{j-1} \frac{1}{\alpha_{kl+i}} \neq 0 \bigcup \bigcup_{j=1}^{k+1} \left\{ (x_{-k-i}, \ldots, x_{-1}, y_{-k-i}, \ldots, y_{-1}) \in \mathbb{R}^{2(k+1)} : x_{-j} = 0, \ y_{-j} = 0 \right\}.
\]

\[\text{(2.15)}\]

**Proof**  At the beginning of Section 2, we have acquired that the set

\[
\bigcup_{j=1}^{k+1} \left\{ (x_{-k-i}, \ldots, x_{-1}, y_{-k-i}, \ldots, y_{-1}) \in \mathbb{R}^{2(k+1)} : x_{-j} = 0, \ y_{-j} = 0 \right\}
\]

belongs to the forbidden set of the initial values for system \((1.6)\). Now we assume that \(x_n \neq 0\) and \(y_n \neq 0\). Note that system \((1.6)\) is undefined when the conditions \(a_n + b_n x_{n-y_n-k} = 0\) or \(\alpha_n + \beta_n y_{n-k} x_{n-k-i} = 0\), that is, \(x_{n-y_n-k} = -\frac{\alpha}{\beta_n}\) or \(y_{n-k} x_{n-k-i} = -\frac{\alpha}{\beta_n}\), for some \(n \in \mathbb{N}_0\), are satisfied (here we consider that \(b_n \neq 0\) and \(\beta_n \neq 0\) for every \(n \in \mathbb{N}_0\)). From this and the substitution \(u_n = \frac{1}{x_n y_{n-i}}, \ v_n = \frac{1}{y_n x_{n-i}}\), we get

\[
u_{m-1+i} = -\frac{b_{km+i}}{a_{km+i}}, \ \nu_{m-1+i} = -\frac{\beta_{km+i}}{\alpha_{km+i}}\]

for some \(m \in \mathbb{N}_0\) and \(i \in \{0, 1, \ldots, k-1\}\). Hence, we can determine the forbidden set of the initial values for system \((1.6)\) by using the substitution \(u_n = \frac{1}{x_n y_{n-i}}, \ v_n = \frac{1}{y_n x_{n-i}}\). Now we consider the functions

\[
f_{km+i}(t) := a_{km+i} + b_{km+i}, \ \ g_{km+i}(t) := \alpha_{km+i} + \beta_{km+i}, \ \ m \in \mathbb{N}_0, \ i \in \{0, 1, \ldots, k-1\},
\]

which correspond to the equations of \((2.3)\). From \((2.16)\) and \((2.17)\), we can write

\[
u_{km+i} = f_{km+i} \circ f_{km+i} \circ \cdots \circ f_{1} (u_{i-k}),
\]

\[
u_{km+i} = g_{km+i} \circ g_{km+i} \circ \cdots \circ g_{1} (v_{i-k}),
\]

where \(m \in \mathbb{N}_0\), and \(i \in \{0, 1, \ldots, k-1\}\). By using \((2.16)\) and implicit forms of \((2.18)-(2.19)\), and considering \(f_{km+i}^{-1}(0) = -\frac{b_{km+i}}{a_{km+i}}, \ g_{km+i}^{-1}(0) = -\frac{\beta_{km+i}}{\alpha_{km+i}}\), for \(m \in \mathbb{N}_0\) and \(i \in \{0, 1, \ldots, k-1\}\), we have

\[
u_{i-k} = f_{i-k} \circ \cdots \circ f_{1} (u_{i-k})\]

\[
u_{i-k} = g_{i-k} \circ \cdots \circ g_{1} (v_{i-k}),
\]

where \(f_{km+i}^{-1}(0) = t - \frac{b_{km+i}}{a_{km+i}}, \ g_{km+i}^{-1}(0) = t - \frac{\beta_{km+i}}{\alpha_{km+i}}, \ m \in \mathbb{N}_0, \ i \in \{0, 1, \ldots, k-1\}\). From \((2.20)\), we obtain

\[
u_{i-k} = -\sum_{j=0}^{m} \frac{b_{kj+i}}{a_{kj+i}} \prod_{l=0}^{j-1} \frac{1}{a_{kl+i}}, \ \nu_{i-k} = -\sum_{j=0}^{m} \frac{\beta_{kj+i}}{\alpha_{kj+i}} \prod_{l=0}^{j-1} \frac{1}{\alpha_{kl+i}}
\]

for some \(m \in \mathbb{N}_0\) and \(i \in \{0, 1, \ldots, k-1\}\). This means that if one of the conditions in \((2.20)\) holds, then the \(m\)th iteration or \((m+1)\)th iteration in system \((1.6)\) can not be calculated. \(\square\)
3. The study of condition \( k = 2, l = k \)

In this section we give the asymptotic behavior of the solutions of system (2.12) when \( k = 2, l = k \). In this case, the system becomes

\[
x_n = \frac{x_{n-2}y_{n-k-2}}{y_{n-k} (a + bx_{n-2}y_{n-k-2})}, \quad y_n = \frac{y_{n-2}x_{n-k-2}}{x_{n-k} (\alpha + \beta y_{n-2}x_{n-k-2})}, \quad n \in \mathbb{N}_0. \tag{3.1}
\]

In (2.11), if we employ the formulas given in (2.6) and (2.7) for the case \( k = 2, l = k \), then the solution of system (3.1) is given by

\[
x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{(1-a) (v_{-1-j_1} + \beta (p(2t+1) + s - t + j_1))}{(1-a) (b + a^{p(2t+1)} + s - t + j_1) (u_{j_1-2} (1-a) - b)}, \tag{3.2}
\]

\[
y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{(1-a) (b + \alpha p^{(2t+1)} + s - t + j_1) (u_{-1-j_1} (1-a) - b)}{(1-a) (\beta + \alpha p^{(2t+1)} + s + 1) (v_{j_1-2} (1-a) - \beta)}, \tag{3.3}
\]

if \( a \neq 1 \neq \alpha \),

\[
x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{(1-a) (v_{-1-j_1} + \beta (p(2t+1) + s - t + j_1))}{b + a^{p(2t+1)} + s - t + j_1} (u_{j_1-2} (1-a) - b), \tag{3.4}
\]

\[
y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{b + \alpha p^{(2t+1)} + s - t + j_1}{} (v_{j_1-2} (1-a) - \beta), \tag{3.5}
\]

if \( a \neq 1, \alpha = 1 \),

\[
x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{\beta + \alpha p^{(2t+1)} + s - t + j_1}{} (v_{-1-j_1} (1-a) - \beta), \tag{3.6}
\]

\[
y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{1-a}{} (u_{-1-j_1} + b (p(2t+1) + s - t + j_1)) \beta + \alpha p^{(2t+1)} + s + 1) (v_{j_1-2} (1-a) - \beta), \tag{3.7}
\]

if \( a = 1, \alpha \neq 1 \), and

\[
x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{u_{-1-j_1} + b (p(2t+1) + s - t + j_1)}{} \frac{1-a}{} (u_{j_1-2} + b (p(2t+1) + s + 1) (1-a) - \beta), \tag{3.8}
\]

\[
y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{u_{-1-j_1} + b (p(2t+1) + s - t + j_1)}{} \frac{v_{j_1-2} + \beta (p(2t+1) + s + 1)}{}, \tag{3.9}
\]

if \( a = 1 = \alpha \), where \( m, t \in \mathbb{N}_0, j_1 \in \{0, 1\} \) and \( 2s + j_1 \in \{k - 2, k - 1, ..., 3k - 3\} \).

First, to present the \( 2k \)- and \( 4k \)-periodic solutions of system (3.1), we will give the following lemma.

**Lemma 3.1** Consider system (3.1). Then the next statements are true.
(a) If \( a \neq 1 \neq \alpha, b \neq 0 \neq \beta, \) and \( (1 - a)\beta = b(1 - \alpha) \) then system (3.1) has \( 2k \)-periodic solutions.

(b) If \( a \neq 1 \neq \alpha, b \neq 0 \neq \beta, \) and \( (1 - a)\beta = -b(1 - \alpha) \) then system (3.1) has \( 4k \)-periodic solutions.

**Proof** Let

\[
z_n = x_{n-2}y_{n-2} \quad \text{and} \quad w_n = y_{n-2}x_{n-2}, \quad n \in \mathbb{N}_0.
\]

Then from (3.1) we have

\[
z_{n+2} = \frac{z_n}{a + bz_n} \quad \text{and} \quad w_{n+2} = \frac{w_n}{\alpha + \beta w_n}, \quad n \in \mathbb{N}_0.
\]

(3.10)

If \( b \neq 0 \neq \beta, \) then system (3.10) has a unique equilibrium solution \((\bar{z}, \bar{w})\), which is different from \((0, 0)\); that is,

\[
z_n = \bar{z} = \frac{1 - a}{b} \neq 0, \quad w_n = \bar{w} = \frac{1 - \alpha}{\beta} \neq 0, \quad n \in \mathbb{N}_0.
\]

If \( \bar{z} = 0 \) or \( \bar{w} = 0, \) then system (3.1) does not have a well-defined solution. From (3.10), we get that

\[
x_{n-2} = \frac{1 - a}{by_{n-2}} = (1 - a)\beta x_{n-2}, \quad \text{and} \quad y_{n-2} = \frac{1 - a}{\beta y_{n-2}} = (1 - a)\beta y_{n-2}, \quad n \geq k,
\]

(3.11)

from which, along with the assumptions in (a)-(b), the results can be easily seen. \( \square \)

Now we investigate the asymptotic behavior of well-defined solutions of system (3.1) in detail according to the case of coefficients \( a \) and \( \alpha. \)

### 3.1. Case \( a \neq 1 \neq \alpha \)

Here we describe the asymptotic behavior of the solution of system (3.1) for the case \( a \neq 1 \neq \alpha, \) by employing the next notation:

\[
K_{j_1} := \frac{v_{-1-j_1}(1-a) - \beta}{a^{t+1-j_1}(u_{j_1-2}(1-a) - b)}, \quad L_{j_1} := \frac{u_{-1-j_1}(1-a) - b}{a^{t+1-j_1}(v_{j_1-2}(1-a) - \beta)},
\]

\[
\tilde{K}_{j_1} := \frac{(1-a)((-1)^{t+1}j_1v_{-1-j_1}(1-a) - \beta))}{a^{t+1-j_1}(1+a)(u_{j_1-2}(1-a) - b)}, \quad \tilde{L}_{j_1} := \frac{(1+a)(u_{-1-j_1}(1-a) - b)}{a^{t+1-j_1}(1-a)((-1)^{t+1}v_{j_1-2}(1+a) - \beta))},
\]

where \( k \in \mathbb{N}, m, t \in \mathbb{N}_0, \quad t = \left\lfloor \frac{k}{2} \right\rfloor, \quad j_1 \in \{0,1\}. \)

**Theorem 3.2** Suppose that \( a \neq 1 \neq \alpha, b \neq 0 \neq \beta \) and that \( (x_n, y_n)_{n \geq -k-2} \) is a well-defined solution of system (3.1). Then the next statements are true.

(a) If \( |a| > \max \{|a|, 1\} \) and \( v_{j_1-2} \neq \beta/(1-a) \neq v_{-1-j_1} \), for some \( j_1 \in \{0,1\}, \) then \( x_n \to \infty, y_n \to 0, \) as \( n \to \infty. \)

(b) If \( |a| > \max \{|a|, 1\} \) and \( u_{j_1-2} \neq b/(1-a) \neq u_{-1-j_1} \), for some \( j_1 \in \{0,1\}, \) then \( x_n \to 0, |y_n| \to \infty, \) as \( n \to \infty. \)
(c) If $\max\{|a|,|\alpha|\} < 1$ or $u_{j_1-2} = b/(1 - a) = u_{j_1-1}$ and $v_{j_1-2} = \beta/(1 - a) = v_{j_1-1}$, and $(1 - a)\beta < |(1 - \alpha)b|$, for some $j_1 \in \{0,1\}$, then $x_n \to 0$ and $|y_n| \to \infty$, as $n \to \infty$.

(d) If $\max\{|a|,|\alpha|\} < 1$ or $u_{j_1-2} = b/(1 - a) = u_{j_1-1}$ and $v_{j_1-2} = \beta/(1 - a) = v_{j_1-1}$, and $(1 - a)\beta > |(1 - \alpha)b|$, for some $j_1 \in \{0,1\}$, then $|x_n| \to \infty$ and $y_n \to 0$, as $n \to \infty$.

(e) If $\max\{|a|,|\alpha|\} < 1$ and $(1 - a)\beta = (1 - a)b$, then $(x_n,y_n)_{n \geq -k-2}$ converges to a not necessarily prime $2k$-periodic solution of system (3.1).

(f) If $u_{j_1-2} = b/(1 - a) = u_{j_1-1}$, $v_{j_1-2} = \beta/(1 - a) = v_{j_1-1}$, and $(1 - a)\beta = (1 - a)b$, for some $j_1 \in \{0,1\}$, then $(x_n,y_n)_{n \geq -k-2}$ is a not necessarily prime $2k$-periodic solution of system (3.1).

(g) If $\max\{|a|,|\alpha|\} < 1$ and $(1 - a)\beta = -(1 - a)b$, then $(x_n,y_n)_{n \geq -k-2}$ converges to a not necessarily prime $4k$-periodic solution of system (3.1).

(h) If $u_{j_1-2} = b/(1 - a) = u_{j_1-1}$, $v_{j_1-2} = \beta/(1 - a) = v_{j_1-1}$, and $(1 - a)\beta = -(1 - a)b$, for some $j_1 \in \{0,1\}$, then $(x_n,y_n)_{n \geq -k-2}$ is a not necessarily prime $4k$-periodic solution of system (3.1).

(i) If $a = \alpha$, $|a| > 1$ and $|K_{j_1}| < 1$, for some $j_1 \in \{0,1\}$, then $x_n \to 0$, as $n \to \infty$.

(j) If $a = \alpha$, $|a| > 1$ and $|L_{j_1}| < 1$, for some $j_1 \in \{0,1\}$, then $y_n \to 0$, as $n \to \infty$.

(k) If $a = \alpha$, $|a| > 1$ and $|K_{j_1}| > 1$, for some $j_1 \in \{0,1\}$, then $|x_n| \to \infty$, as $n \to \infty$.

(l) If $a = \alpha$, $|a| > 1$ and $|L_{j_1}| > 1$, for some $j_1 \in \{0,1\}$, then $|y_n| \to \infty$, as $n \to \infty$.

(m) If $a = \alpha$, $|a| > 1$ and $K_{j_1} = 1$, for some $j_1 \in \{0,1\}$, then the sequence $x_{2(2t+1)m+2s+j_1}$, for $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, is convergent.

(n) If $a = \alpha$, $|a| > 1$ and $L_{j_1} = 1$, for some $j_1 \in \{0,1\}$, then the sequence $y_{2(2t+1)m+2s+j_1}$, for $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, is convergent.

(o) If $a = \alpha$, $|a| > 1$ and $K_{j_1} = -1$, for some $j_1 \in \{0,1\}$, then the sequences $x_{4(2t+1)m+2s+j_1}$, $x_{4(2t+1)m+2(2t+1)+2s+j_1}$, for $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, are convergent.

(p) If $a = \alpha$, $|a| > 1$ and $L_{j_1} = -1$, for some $j_1 \in \{0,1\}$, then the sequences $y_{4(2t+1)m+2s+j_1}$, $y_{4(2t+1)m+2(2t+1)+2s+j_1}$, for $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, are convergent.

(q) If $a = -\alpha$, $|a| > 1$, $m$ is even, and $|K_{j_1}| < 1$, for some $j_1 \in \{0,1\}$, $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, then $x_{4(2t+1)m+2s+j_1} \to 0$, as $m \to \infty$.

(r) If $a = -\alpha$, $|a| > 1$, $m$ is even, and $|L_{j_1}| < 1$, for some $j_1 \in \{0,1\}$, $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, then $y_{4(2t+1)m+2s+j_1} \to 0$, as $m \to \infty$.

(s) If $a = -\alpha$, $|a| > 1$, $m$ is even, and $|K_{j_1}| > 1$, for some $j_1 \in \{0,1\}$, $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, then $|x_{4(2t+1)m+2s+j_1}| \to \infty$, as $m \to \infty$.  

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(t) If \( a = -\alpha, \ |a| > 1, \ m \) is even, and \( |\tilde{L}_{j_1}| > 1 \), for some \( j_1 \in \{0, 1\} \), \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\} \), then \( |y_{4(2t+1)m+2s+j_1}| \to \infty \), as \( m \to \infty \).

(u) If \( a = -\alpha, \ |a| > 1, \ m \) is even, and \( \tilde{K}_{j_1} = 1 \), for some \( j_1 \in \{0, 1\} \), \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\} \), then the sequence \( x_{4(2t+1)m+2s+j_1} \) is convergent.

(v) If \( a = -\alpha, \ |a| > 1, \ m \) is even, and \( \tilde{L}_{j_1} = 1 \), for some \( j_1 \in \{0, 1\} \), \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\} \), then the sequence \( y_{4(2t+1)m+2s+j_1} \) is convergent.

(w) If \( a = -\alpha, \ |a| > 1, \ m \) is even, and \( \tilde{K}_{j_1} = -1 \), for some \( j_1 \in \{0, 1\} \), \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\} \), then the sequences \( x_{8(2t+1)m+2s+j_1} \) and \( x_{8(2t+1)m+4(2t+1)+2s+j_1} \) are convergent.

(x) If \( a = -\alpha, \ |a| > 1, \ m \) is even, and \( \tilde{L}_{j_1} = -1 \), for some \( j_1 \in \{0, 1\} \), \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\} \), then the sequences \( y_{8(2t+1)m+2s+j_1} \) and \( y_{8(2t+1)m+4(2t+1)+2s+j_1} \) are convergent.

Proof Let

\[
p_m^{t,2s+j_1} = \frac{(1-a)((v_{-1-j_1}(1-\alpha)-\beta)a^{m(2t+1)+s-t+j_1}+\beta)}{(1-a)((u_{-j_1-2}(1-a)-b)a^{m(2t+1)+s+1}+b)},
\]

for \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\} \), \( j_1 \in \{0, 1\} \), and

\[
r_m^{t,2s+j_1} = \frac{(1-a)((u_{-1-j_1}(1-a)-b)a^{m(2t+1)+s-t+j_1}+b)}{(1-a)((v_{j_1-2}(1-\alpha)-\beta)a^{m(2t+1)+s+1}+\beta)},
\]

for \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\} \), \( j_1 \in \{0, 1\} \).

(a): Note that in this case

\[
\lim_{m \to \infty} |p_m^{t,2s+j_1}| = \infty, \quad \lim_{m \to \infty} r_m^{t,2s+j_1} = 0,
\]

for each \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\} \), \( j_1 \in \{0, 1\} \), from which, along with formulas (3.2) and (3.3), the result can be seen easily.

(b): Note that in this case

\[
\lim_{m \to \infty} p_m^{t,2s+j_1} = 0, \quad \lim_{m \to \infty} |r_m^{t,2s+j_1}| = \infty,
\]

for each \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\} \), \( j_1 \in \{0, 1\} \), from which, along with formulas (3.2) and (3.3), the result can be obtained easily.

(c)–(d): In this case we get

\[
\lim_{m \to \infty} |p_m^{t,2s+j_1}| = \frac{|(1-a)\beta|}{(1-a)b}, \quad \lim_{m \to \infty} |r_m^{t,2s+j_1}| = \frac{|(1-a)b|}{(1-a)\beta},
\]

for each \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\} \), \( j_1 \in \{0, 1\} \), from which, along with the assumptions in (c) and (d), the statements easily follow.
(e): Employing the Taylor expansion for \((1+x)^{-1}\) on the interval \((-\epsilon, \epsilon)\), where \(\epsilon > 0\), we have, for sufficiently large \(m\) and each \(2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}\), \(j_1 \in \{0, 1\}\),

\[
p^{t, 2s+j_1}_m = \frac{(1 - a) (\beta + \alpha^{m(2t+1)+s-t+j_1} (v_{-1-j_1}(1 - \alpha) - \beta))}{(1 - a) (b + \alpha^{m(2t+1)+s-t+j_1}(u_{-j_1-2}(1 - a) - b))} \]

\[
= \frac{(1 - a) \beta}{(1 - a)b} \left( 1 + \alpha^{m(2t+1)+s-t+j_1} \left( \frac{v_{-1-j_1}(1 - \alpha) - \beta}{\beta} \right) \right) - \frac{a^{m(2t+1)+s+1} (u_{j_1-2}(1 - a) - b)}{b} + \mathcal{O}\left( a^{2m(2t+1)} \right),
\]

(3.14)

and

\[
r^{t, 2s+j_1}_m = \frac{(1 - a) (b + \alpha^{m(2t+1)+s-t+j_1} (u_{-1-j_1}(1 - a) - b))}{(1 - a) (\beta + \alpha^{m(2t+1)+s+t+1}(v_{-1-j_1}(1 - \alpha) - \beta))} \]

\[
= \frac{(1 - a) b}{(1 - a)\beta} \left( 1 + \alpha^{m(2t+1)+s-t+j_1} \left( \frac{u_{-1-j_1}(1 - \alpha) - b}{\beta} \right) \right) - \frac{a^{m(2t+1)+s+1} (v_{j_1-2}(1 - \alpha) - \beta)}{\beta} + \mathcal{O}\left( a^{2m(2t+1)} \right).
\]

(3.15)

Employing (3.14)–(3.15) in (3.2)–(3.3) and the assumption \(\max \{|a|, |\alpha|\} < 1\), the convergence of the sequences \(x_{2(2t+1)m+2s+j_1}\) and \(y_{2(2t+1)m+2s+j_1}\) for \(2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}\), \(j_1 \in \{0, 1\}\), can be seen easily and consequently \((x_n, y_n)_{n \geq -k-2}\) converges to a not necessarily prime \(2k\)-periodic solution of system (3.1).

(f): From (3.2) and (3.3), we get

\[
p^{t, 2s+j_1}_m = \frac{(1 - a) \beta}{(1 - a)b} = 1, \quad r^{t, 2s+j_1}_m = \frac{(1 - a)b}{(1 - a)\beta} = 1,
\]

for every \(m \in \mathbb{N}_0\), if \(2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}\), \(j_1 \in \{0, 1\}\). That is, \(x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)}\) and \(y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)}\), for each \(2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}\), \(j_1 \in \{0, 1\}\), from which the result can be obtained easily.
(g): From (3.12) and (3.13), in this case, we have

\[
p_{m}^{t,2s+j_1} = \frac{(1 - a) \left( \beta + a^{m(2t+1)+s-t+j_1} (v_{-1-j_1} (1 - a) - \beta) \right)}{(1 - a) \left( b + a^{m(2t+1)+s+1} (u_{j_1-2} (1 - a) - b) \right)}
\]

\[
= \frac{(1 - a) \beta \left( 1 + a^{m(2t+1)+s-t+j_1} (v_{-1-j_1} (1 - a) - \beta) \right)}{(1 - a) b \left( 1 + a^{m(2t+1)+s+1} (u_{j_1-2} (1 - a) - b) \right)}
\]

\[
= - \left( 1 + a^{m(2t+1)+s-t+j_1} (v_{-1-j_1} (1 - a) - \beta) \right) - a^{m(2t+1)+s+1} (u_{j_1-2} (1 - a) - b) + O \left( a^{2m(2t+1)} \right),
\]

(3.16)

for sufficiently large \( m \) if \( 2s + j_1 \in \{ k - 2, k - 1, \ldots, 3k - 3 \}, j_1 \in \{ 0, 1 \}, \) and

\[
r_{m}^{t,2s+j_1} = \frac{(1 - a) \left( b + a^{m(2t+1)+s-t+j_1} (u_{-1-j_1} (1 - a) - b) \right)}{(1 - a) \left( \beta + a^{m(2t+1)+s+1} (v_{j_1-2} (1 - a) - \beta) \right)}
\]

\[
= \frac{(1 - a) b \left( 1 + a^{m(2t+1)+s-t+j_1} (u_{-1-j_1} (1 - a) - b) \right)}{(1 - a) \beta \left( 1 + a^{m(2t+1)+s+1} (v_{j_1-2} (1 - a) - \beta) \right)}
\]

\[
= - \left( 1 + a^{m(2t+1)+s-t+j_1} (u_{-1-j_1} (1 - a) - b) \right) - a^{m(2t+1)+s+1} (v_{j_1-2} (1 - a) - \beta) + O \left( a^{2m(2t+1)} \right).
\]

(3.17)

for sufficiently large \( m \) if \( 2s + j_1 \in \{ k - 2, k - 1, \ldots, 3k - 3 \}, j_1 \in \{ 0, 1 \}. \) Employing (3.16)-(3.17) in (3.2)-(3.3), Taylor expansion for \((1 + x)^{-1}\) on the interval \((-\epsilon, \epsilon)\), where \( \epsilon > 0 \) and the assumption \( \max \{|a|, |\alpha|\} < 1 \), the convergence of the sequences \( x_{4(2t+1)m+2s+j_1}, y_{1(2t+1)m+2s+j_1} \) for \( 2s + j_1 \in \{ k - 2, k - 1, \ldots, 5k - 3 \}, j_1 \in \{ 0, 1 \} \) easily follows, from which it follows that \((x_n, y_n)_{n \geq k-2}\) converges to \( a \) not necessarily prime 4k-periodic solution of system (3.1).

(h): From (3.12) and (3.13), in this case, we get

\[
p_{m}^{t,2s+j_1} = \frac{(1 - a) \beta}{(1 - a) b} = -1, \quad r_{m}^{t,2s+j_1} = \frac{(1 - a) b}{(1 - a) \beta} = -1,
\]

for every \( m \in \mathbb{N}_0 \), if \( 2s + j_1 \in \{ k - 2, k - 1, \ldots, 3k - 3 \}, j_1 \in \{ 0, 1 \}. \) That is, \( x_{2(2t+1)m+2s+j_1} = (-1)^{m+1} x_{2s+j_1 - 2(2t+1)} \) and \( y_{2(2t+1)m+2s+j_1} = (-1)^{m+1} y_{2s+j_1 - 2(2t+1)} \) for each \( 2s + j_1 \in \{ k - 2, k - 1, \ldots, 3k - 3 \}, j_1 \in \{ 0, 1 \}, \) from which the result can be obtained easily.

(i)-(l): From (3.12) and (3.13), in this case, we get

\[
\lim_{m \to \infty} p_{m}^{t,2s+j_1} = \lim_{m \to \infty} \frac{(v_{-1-j_1} (1 - a) - \beta) + a^{m(2t+1)+s-t+j_1} \frac{\beta}{b} (u_{j_1-2} (1 - a) - b) + a^{m(2t+1)+s+1} \frac{\beta}{b} a^{1-j_1} + a^{m(2t+1)+s+1} \frac{\beta}{b}}{a^{m(2t+1)+s+1} \frac{\beta}{b}} = K_{j_1},
\]

(3.18)
for $2s + j_1 \in \{k - 3, k - 2, \ldots, 3k - 4\}$, $j_1 \in \{0, 1\}$, 

$$
\lim_{m \to \infty} r_{2s+1}^{(2s+1)} = \lim_{m \to \infty} \frac{(u_{-1-j_1}(1-a) - b) + \beta a^{m(2t+1)+s-t+j_1}}{(v_{j_1-2}(1-a) - \beta) a^{m(2t+1)+s-t+j_1} + \beta} = L_{j_1},
$$

for $2s + j_1 \in \{k - 3, k - 2, \ldots, 3k - 4\}$, $j_1 \in \{0, 1\}$. Employing (3.18)–(3.19) in (3.2)–(3.3) and the assumptions in (i)–(l), the results easily follow.

(m): From (3.12) and (3.13), we have

$$
p_{2s+1}^{(2s+1)} = \frac{(v_{-1-j_1}(1-a) - \beta a^{m(2t+1)+s-t+j_1} + \beta}{(u_{j_1-2}(1-a) - b) a^{m(2t+1)+s+1} + b}
$$

$$
= \frac{(v_{-1-j_1}(1-a) - \beta a^{m(2t+1)+s-t+j_1} + \beta}{(u_{j_1-2}(1-a) - b) a^{m(2t+1)+s+1} + b}
$$

$$
= 1 + \frac{\beta}{a^{m(2t+1)+s-t+j_1}} + \frac{b}{a^{m(2t+1)+s+1} (u_{j_1-2}(1-a) - b)} + O\left(\frac{1}{a^{2m(2t+1)}}\right),
$$

for sufficiently large $m$ if $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$, from which the convergence of the sequence $x_{2(2t+1)m+2s+j_1}$, for $j_1 \in \{0, 1\}$, can be seen easily.

(n): From (3.12) and (3.13), we get

$$
r_{2s+1}^{(2s+1)} = \frac{(u_{-1-j_1}(1-a) - b) a^{m(2t+1)+s-t+j_1} + b}{(v_{j_1-2}(1-a) - \beta a^{m(2t+1)+s+1} + \beta}
$$

$$
= \frac{(u_{-1-j_1}(1-a) - b) a^{m(2t+1)+s-t+j_1} + b}{(v_{j_1-2}(1-a) - \beta a^{m(2t+1)+s+1} + \beta}
$$

$$
= 1 + \frac{b}{a^{m(2t+1)+s-t+j_1}} + \frac{\beta}{a^{m(2t+1)+s+1} (v_{j_1-2}(1-a) - \beta)} + O\left(\frac{1}{a^{2m(2t+1)}}\right),
$$

for sufficiently large $m$ if $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$, from which the convergence of the sequence $y_{2(2t+1)m+2s+j_1}$, for $j_1 \in \{0, 1\}$, can be obtained easily.
(o): From (3.12), in this case, we have

\[
p_{m}^{t, 2s+j_1} = \frac{(v_{-1-j_1}(1-a) - \beta)a^{m(2t+1)+s-t+j_1} + \beta}{(u_{j_1-2}(1-a) - b)a^{m(2t+1)+s+1} + b}
\]

\[
= \frac{(v_{-1-j_1}(1-a) - \beta)a^{m(2t+1)+s-t+j_1}}{(u_{j_1-2}(1-a) - b)a^{m(2t+1)+s+1} + b} \left( 1 + \frac{\beta}{a^{m(2t+1)+s-t+j_1}(v_{-1-j_1}(1-a) - \beta)} \right)
\]

\[
= - \left( 1 + \frac{\beta}{a^{m(2t+1)+s-t+j_1}(v_{-1-j_1}(1-a) - \beta)} \right) - \frac{b}{a^{m(2t+1)+s+1}(u_{j_1-2}(1-a) - b)}
\]

+ \mathcal{O} \left( \frac{1}{a^{2m(2t+1)}} \right),
\]

(3.22)

for sufficiently large \(m\) if \(2s+j_1 \in \{k-2, k-1, \ldots, 3k-3\}, \ j_1 \in \{0, 1\}\), from which the convergence of the sequences \(x_{4(2t+1)m+2s+j_1}, x_{4(2t+1)m+2(2t+1)+2s+j_1}\) for \(j_1 \in \{0, 1\}\) easily follows.

(p): From (3.13), in this case, we obtain

\[
r_{m}^{t, 2s+j_1} = \frac{(u_{-1-j_1}(1-a) - b)a^{m(2t+1)+s-t+j_1} + b}{(v_{j_1-2}(1-a) - \beta)a^{m(2t+1)+s+1} + \beta}
\]

\[
= \frac{(u_{-1-j_1}(1-a) - b)a^{m(2t+1)+s-t+j_1}}{(v_{j_1-2}(1-a) - \beta)a^{m(2t+1)+s+1} + \beta} \left( 1 + \frac{\beta}{a^{m(2t+1)+s-t+j_1}(u_{-1-j_1}(1-a) - b)} \right)
\]

\[
= - \left( 1 + \frac{\beta}{a^{m(2t+1)+s-t+j_1}(u_{-1-j_1}(1-a) - b)} \right) - \frac{b}{a^{m(2t+1)+s+1}(v_{j_1-2}(1-a) - \beta)}
\]

+ \mathcal{O} \left( \frac{1}{a^{2m(2t+1)}} \right),
\]

(3.23)

for sufficiently large \(m\) if \(2s+j_1 \in \{k-2, k-1, \ldots, 3k-3\}, \ j_1 \in \{0, 1\}\). Using (3.23) in (3.3) and the Taylor expansion for \((1+x)^{-1}\) on the interval \((-\epsilon, \epsilon)\), where \(\epsilon > 0\), the convergence of the sequences \(y_{4(2t+1)m+2s+j_1}, y_{4(2t+1)m+2(2t+1)+2s+j_1}\) for \(j_1 \in \{0, 1\}\) easily follows.

(q)–(t): To obtain these four results it is enough to note that the limits of the general terms (3.12)–(3.13) in (3.2)–(3.3) are equal to \(\hat{K}_{j_1}\) and \(\hat{L}_{j_1}\), respectively, and then employing the assumptions in (t)–(q), the results can be seen easily.

(u): From (3.12), in this case, we get

\[
p_{2m}^{t, 2s+j_1} = \frac{1 + \frac{\beta}{(-a)^{2m(2t+1)+s-t+j_1}(v_{-1-j_1}(1+a) - \beta)}}{1 + \frac{b}{a^{m(2t+1)+s+1}(u_{j_1-2}(1-a) - b)}}
\]

\[
= 1 + \frac{\beta}{(-a)^{2m(2t+1)+s-t+j_1}(v_{-1-j_1}(1+a) - \beta)} - \frac{b}{a^{2m(2t+1)+s+1}(u_{j_1-2}(1-a) - b)}
\]

+ \mathcal{O} \left( \frac{1}{a^{4m(2t+1)}} \right),
\]

(3.24)
From (3.13), in this case, we have

\[
\frac{r_{2m}^{t,2s+j_1}}{r_{2m}} = \frac{1 + \frac{b}{a^2m(2t+1)+s-t+j_1 (u_{-j_1}(1-a)-1)} + \beta}{1 + \frac{b}{a^2m(2t+1)+s-t+j_1 (v_{j_1-1}(1+a)-1)} + \beta}
\]

\[
= 1 + \frac{b}{a^2m(2t+1)+s-t+j_1 (u_{-j_1}(1-a)-1)} - \frac{b}{a^2m(2t+1)+s-t+j_1 (v_{j_1-1}(1+a)-1)} + \mathcal{O}\left(\frac{1}{a^4m(2t+1)}\right),
\]

(3.25)

for sufficiently large \( m \) if \( 2s + j_1 \in \{k-2,k-1,\ldots,3k-3\}, \ j_1 \in \{0,1\} \), from which the convergence of the sequence \( x_{4(2t+1)m+2s+j_1} \) for \( j_1 \in \{0,1\} \) easily follows.

From (3.12), in this case, we get

\[
\frac{p_{2m}^{t,2s+j_1}}{p_{2m}} = \frac{(1-a)(v_{-j_1}(1+a) - \beta)(-a)^{2m(2t+1)+s-t+j_1} + \beta}{(1+a)(u_{j_1-2}(1-a) - b)a^{2m(2t+1)+s+1} + b}
\]

\[
= \frac{(1-a)(v_{-j_1}(1+a) - \beta)(-a)^{2m(2t+1)+s-t+j_1} + \beta}{(1+a)(u_{j_1-2}(1-a) - b)a^{2m(2t+1)+s+1} + b}
\]

\[
= -\left(1 + \frac{\beta}{(-a)^{2m(2t+1)+s-t+j_1} (v_{-j_1}(1+a) - \beta)}\right) - \frac{\beta}{(u_{j_1-2}(1-a) - b)} + \mathcal{O}\left(\frac{1}{a^4m(2t+1)}\right),
\]

(3.26)

for sufficiently large \( m \) if \( 2s + j_1 \in \{k-2,k-1,\ldots,3k-3\}, \ j_1 \in \{0,1\} \). Using (3.26) in (3.2) and the Taylor expansion for \((1+x)^{-1}\) on the interval \((-\epsilon, \epsilon)\), where \( \epsilon > 0 \), the convergence of the sequences \( x_{8(2t+1)m+2s+j_1} \), \( x_{8(2t+1)m+4(2t+1)+2s+j_1} \), for \( j_1 \in \{0,1\} \), can be seen easily.

From (3.13), in this case, we have

\[
\frac{r_{2m}^{t,2s+j_1}}{r_{2m}} = \frac{1 + \frac{b}{a^2m(2t+1)+s-t+j_1 (u_{-j_1}(1-a)-1)} + \beta}{1 + \frac{b}{a^2m(2t+1)+s-t+j_1 (v_{j_1-1}(1+a)-1)} + \beta}
\]

\[
= \frac{(1+a)(u_{-j_1}(1-a) - b)a^{2m(2t+1)+s-t+j_1} + b}{(1-a)(v_{j_1-2}(1+a) - \beta)(-a)^{2m(2t+1)+s+1} + \beta}
\]

\[
= -\left(1 + \frac{\beta}{a^{2m(2t+1)+s-t+j_1} (u_{-j_1}(1-a) - b)}\right) - \frac{\beta}{(v_{j_1-2}(1+a) - \beta)} + \mathcal{O}\left(\frac{1}{a^4m(2t+1)}\right),
\]

(3.27)
for sufficiently large $m$ if $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$, from which the convergence of the sequences $y_8(2t+1)m+2s+j_1$, $y_8(2t+1)m+4(2t+1)+2s+j_1$, for $j_1 \in \{0, 1\}$, can be obtained easily.

\[\square\]

**Theorem 3.3** Assume that $a \neq 1 \neq b$, $a \neq 0 \neq \beta$, $a = -\alpha$, $|a| > 1$, $m, t, w, h \in \mathbb{N}_0$, $t = \left[\frac{k}{2}\right]$, $j_1 \in \{0, 1\}$, $m$ is odd, and $(x_n, y_n)_{n \geq k-2}$ is a well-defined solution of system (3.1). Then the next statements are true.

(a) If $\left|\frac{(1-a)(v_{-1-j_1}(1+a) - \beta)}{(1+a)^{t+1-j_1}(u_{j_1-2}(1-a) - b)}\right| < 1$ for some $j_1 \in \{0, 1\}$, then $x_{4(2t+1)m+4t+2+4w+j_1} \to 0$,

$x_{4(2t+1)m+4t+4w+4+j_1} \to 0$ as $m \to \infty$.

(b) If $\left|\frac{(1-a)(v_{-1-j_1}(1+a) - \beta)}{(1+a)^{t+1-j_1}(u_{j_1-2}(1-a) - b)}\right| > 1$ for some $j_1 \in \{0, 1\}$, then $|x_{4(2t+1)m+4t+2+4w+j_1}| \to \infty$,

$|x_{4(2t+1)m+4t+4w+4+j_1}| \to \infty$ as $m \to \infty$.

(c) If $\frac{(1-a)(v_{-1-j_1}(1+a) - \beta)}{(1+a)^{t+1-j_1}(u_{j_1-2}(1-a) - b)} = 1$ for some $j_1 \in \{0, 1\}$, then the sequences are $x_{4(2t+1)m+4t+2+4w+j_1}$,

$x_{4(2t+1)m+4t+4w+4+j_1}$ are convergent.

(d) If $\frac{(1-a)(v_{-1-j_1}(1+a) - \beta)}{(1+a)^{t+1-j_1}(u_{j_1-2}(1-a) - b)} = -1$ for some $j_1 \in \{0, 1\}$, then the sequences $x_{8(2t+1)m+4t+2+4w+j_1}$,

$x_{8(2t+1)m+12t+6+4w+j_1}$, $x_{8(2t+1)m+4t+4w+4+j_1}$, and $x_{8(2t+1)m+12t+4w+8+j_1}$ are convergent.

(e) If $\frac{(1-a)(u_{j_1-1}(1+a) - \beta)}{(1+a)^{t+1-j_1}(u_{j_1-2}(1+a) - b)} < 1$ for some $j_1 \in \{0, 1\}$, then $y_{4(2t+1)m+4t+2+4w+j_1} \to 0$ as $m \to \infty$.

(f) If $\frac{(1-a)(u_{j_1-1}(1+a) - \beta)}{(1+a)^{t+1-j_1}(u_{j_1-2}(1+a) - b)} > 1$ for some $j_1 \in \{0, 1\}$, then $|y_{4(2t+1)m+4t+2+4w+j_1}| \to \infty$ as $m \to \infty$.

(g) If $\frac{(1-a)(u_{j_1-1}(1+a) - \beta)}{(1+a)^{t+1-j_1}(u_{j_1-2}(1+a) - b)} = 1$ for some $j_1 \in \{0, 1\}$, then the sequence $y_{4(2t+1)m+4t+2+4w+j_1}$ is convergent.

(h) If $\frac{(1-a)(u_{j_1-1}(1+a) - \beta)}{(1+a)^{t+1-j_1}(u_{j_1-2}(1+a) - b)} = -1$ for some $j_1 \in \{0, 1\}$, then the sequences $y_{8(2t+1)m+4t+2+4w+j_1}$ and $y_{8(2t+1)m+12t+6+4w+j_1}$ are convergent.

(i) If $\frac{(a-1)(v_{-1-j_1}(1+a) - \beta)}{(1+a)^{t+1-j_1}(u_{j_1-2}(1-a) - b)} < 1$ for some $j_1 \in \{0, 1\}$, then $x_{4(2t+1)m+4t+2+4w+j_1} \to 0$,

$x_{4(2t+1)m+4t+4w+4+j_1} \to 0$ as $m \to \infty$.

(j) If $\frac{(a-1)(v_{-1-j_1}(1+a) - \beta)}{(1+a)^{t+1-j_1}(u_{j_1-2}(1-a) - b)} > 1$ for some $j_1 \in \{0, 1\}$, then $|x_{4(2t+1)m+4t+2+4w+j_1}| \to \infty$,

$|x_{4(2t+1)m+4t+4w+4+j_1}| \to \infty$ as $m \to \infty$.

(k) If $\frac{(a-1)(v_{-1-j_1}(1+a) - \beta)}{(1+a)^{t+1-j_1}(u_{j_1-2}(1-a) - b)} = 1$ for some $j_1 \in \{0, 1\}$, then the sequences $x_{4(2t+1)m+4t+2+4w+j_1}$,

$x_{4(2t+1)m+4t+4w+4+j_1}$ are convergent.
(l) If \( \frac{(a-1)(v_{n-1} + \beta)}{(1+a)q^{n-1} - (1-a) - b} \) = \(-1\) for some \( j_1 \in \{0, 1\} \), then the sequences \( x_{8(2t+1)m+4t+24w+j+1} \), \( x_{8(2t+1)m+12t+4w+j} \), \( x_{8(2t+1)m+4t+4w+4+j_1} \), \( x_{8(2t+1)m+12t+4w+8+j_1} \) are convergent.

(m) If \( \frac{(a-1)(v_{n-1} + \beta)}{(1+a)q^{n-1} - (1-a) - b} < 1 \) for some \( j_1 \in \{0, 1\} \), then \( y_{4(2t+1)m+4t+4w+4+j_1} \rightarrow 0 \) as \( m \rightarrow \infty \).

(n) If \( \frac{(a-1)(v_{n-1} + \beta)}{(1+a)q^{n-1} - (1-a) - b} > 1 \) for some \( j_1 \in \{0, 1\} \), then \( |y_{4(2t+1)m+4t+4w+4+j_1}| \rightarrow \infty \) as \( m \rightarrow \infty \).

(o) If \( \frac{(a-1)(v_{n-1} + \beta)}{(1+a)q^{n-1} - (1-a) - b} = 1 \) for some \( j_1 \in \{0, 1\} \), then the sequence \( y_{4(2t+1)m+4t+4w+4+j_1} \) is convergent.

(p) If \( \frac{(a-1)(v_{n-1} + \beta)}{(1+a)q^{n-1} - (1-a) - b} = -1 \) for some \( j_1 \in \{0, 1\} \), then the sequences \( y_{8(2t+1)m+4t+4w+4+j_1} \) and \( y_{8(2t+1)m+12t+4w+8+j_1} \) are convergent.

Proof Let \( a = -\alpha \), \(|a| > 1\), and \( m \) be odd. From (3.12)–(3.13) we get

\[
\hat{p}_{2m+1} = \frac{1 - a}{(1+a)q^{n-1} - (1-a) - b} \left( (v_{j_1-2} + \beta)(-1)^{2m+1}(2t+1)+s-t+j_1 + \frac{\beta}{a^{2m+1}q^{2t+1}s-1+j_1} \right),
\]

\[
\hat{r}_{2m+1} = \frac{1 + a}{(1-a)q^{n-1} - (1-a) - b} \left( (v_{j_1-2} + \beta)(-1)^{2m+1}(2t+1)+s+1 + \frac{\beta}{a^{2m+1}q^{2t+1}s+1+j_1} \right).
\]

Here, there are four cases to be considered.

- **s is even and s - t + j_1** is **odd**: In this case we have

\[
\hat{p}_{2w,2h+1} = \frac{1 - a}{(1+a)q^{n-1} - (1-a) - b} \left( (v_{j_1-2} + \beta)(-1)^{2m+1}(2t+1)+2h+1 + \frac{\beta}{a^{2m+1}q^{2t+1}s+1+j_1} \right),
\]

\[
\hat{r}_{2w,2h+1} = \frac{1 + a}{(1-a)q^{n-1} - (1-a) - b} \left( (v_{j_1-2} + \beta)(-1)^{2m+1}(2t+1)+2h+1 + \frac{\beta}{a^{2m+1}q^{2t+1}s+1+j_1} \right),
\]

where \( s = 2w \) and \( s - t + j_1 = 2h + 1 \), for \( w, h \in N_0 \). From (3.30)–(3.31), we have

\[
\lim_{m \to \infty} \hat{p}_{2w,2h+1} = \frac{1 - a}{(1+a)q^{n-1} - (1-a) - b} \left( (v_{j_1-2} + \beta)(-1)^{2m+1}(2t+1)+2h+1 + \frac{\beta}{a^{2m+1}q^{2t+1}s+1+j_1} \right),
\]

\[
\lim_{m \to \infty} \hat{r}_{2w,2h+1} = \frac{1 + a}{(1-a)q^{n-1} - (1-a) - b} \left( (v_{j_1-2} + \beta)(-1)^{2m+1}(2t+1)+2h+1 + \frac{\beta}{a^{2m+1}q^{2t+1}s+1+j_1} \right).
\]
• **s and s – t + j₁ are both even:** In this case we get

\[
\hat{p}^{2w,2h}_{2m+1} = \frac{(1-a) \left( (v_{-1-j_1}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2h} + \frac{b}{a(2m+1)(2t+1)+2h+2} \right)}{(1+a)a^{t+1-j_1} \left( (u_{j_1-2}(1-a) - b) + \frac{b}{a(2m+1)(2t+1)+2h+2} \right)}, \tag{3.34}
\]

\[
\hat{p}^{2w,2h}_{2m+1} = \frac{(1+a) \left( (u_{-1-j_1}(1-a) - b) + \frac{b}{a(2m+1)(2t+1)+2h+2} \right)}{(1-a)a^{t+1-j_1} \left( (v_{j_1-2}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2w+2} + \frac{b}{a(2m+1)(2t+1)+2w+2} \right)}, \tag{3.35}
\]

where \(s = 2w\) and \(s - t + j_1 = 2h\), for \(w, h \in \mathbb{N}_0\). From (3.34)–(3.35), we obtain

\[
\lim_{m \to \infty} \hat{p}^{2w,2h}_{2m+1} = \frac{(a-1)(v_{-1-j_1}(1+a) - \beta)}{(1+a)a^{t+1-j_1} \left( (u_{j_1-2}(1-a) - b) \right)}, \tag{3.36}
\]

\[
\lim_{m \to \infty} \hat{p}^{2w,2h}_{2m+1} = \frac{(1+a)(u_{-1-j_1}(1-a) - b)}{(1-a)a^{t+1-j_1} \left( (v_{j_1-2}(1+a) - \beta) \right)}, \tag{3.37}
\]

• **s and s – t + j₁ are both odd:** In this case we have

\[
\hat{p}^{2w+1,2h+1}_{2m+1} = \frac{(1-a) \left( (v_{-1-j_1}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2h+1} + \frac{b}{a(2m+1)(2t+1)+2h+2} \right)}{(1+a)a^{t+1-j_1} \left( (u_{j_1-2}(1-a) - b) + \frac{b}{a(2m+1)(2t+1)+2h+2} \right)}, \tag{3.38}
\]

\[
\hat{p}^{2w+1,2h+1}_{2m+1} = \frac{(1+a) \left( (u_{-1-j_1}(1-a) - b) + \frac{b}{a(2m+1)(2t+1)+2h+2} \right)}{(1-a)a^{t+1-j_1} \left( (v_{j_1-2}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2w+2} + \frac{b}{a(2m+1)(2t+1)+2w+2} \right)}, \tag{3.39}
\]

where \(s = 2w + 1\) and \(s - t + j_1 = 2h + 1\), for \(w, h \in \mathbb{N}_0\). From (3.38)–(3.39), we get

\[
\lim_{m \to \infty} \hat{p}^{2w+1,2h+1}_{2m+1} = \frac{(1-a)(v_{-1-j_1}(1+a) - \beta)}{(1+a)a^{t+1-j_1} \left( (u_{j_1-2}(1-a) - b) \right)}, \tag{3.40}
\]

\[
\lim_{m \to \infty} \hat{p}^{2w+1,2h+1}_{2m+1} = \frac{(1+a)(u_{-1-j_1}(1-a) - b)}{(1-a)a^{t+1-j_1} \left( (v_{j_1-2}(1+a) - \beta) \right)}, \tag{3.41}
\]

• **s is odd and s – t + j₁ is even:** In this case we get

\[
\hat{p}^{2w+1,2h}_{2m+1} = \frac{(1-a) \left( (v_{-1-j_1}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2h} + \frac{b}{a(2m+1)(2t+1)+2h+2} \right)}{(1+a)a^{t+1-j_1} \left( (u_{j_1-2}(1-a) - b) + \frac{b}{a(2m+1)(2t+1)+2h+2} \right)}, \tag{3.42}
\]

\[
\hat{p}^{2w+1,2h}_{2m+1} = \frac{(1+a) \left( (u_{-1-j_1}(1-a) - b) + \frac{b}{a(2m+1)(2t+1)+2h+2} \right)}{(1-a)a^{t+1-j_1} \left( (v_{j_1-2}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2w+2} + \frac{b}{a(2m+1)(2t+1)+2w+2} \right)}, \tag{3.43}
\]
where \( s = 2w + 1 \) and \( s - t + j_1 = 2h \), for \( w, h \in \mathbb{N}_0 \). From (3.42)-(3.43), we have

\[
\lim_{m \to \infty} \rho_{2m+1}^{2w+1,2h} = \frac{(a - 1)(v_{-1-j_1} (1 + a) - \beta)}{(1 + a)(a^{t+1-j_1} (u_{j_1-2} (1 - a) - b))},
\]

(3.44)

\[
\lim_{m \to \infty} \rho_{2m+1}^{2w+1,2h} = \frac{(1 + a)(u_{-1-j_1} (1 - a) - b)}{(a - 1)(a^{t+1-j_1} (v_{j_1-2} (1 + a) - \beta))}.
\]

(3.45)

From (3.32), (3.33), (3.36), (3.37), (3.40), (3.41), (3.44), and (3.45), the results can be seen easily.

\[\square\]

3.2. Case \( a \neq 1, \alpha = 1 \)

Here we introduce the asymptotic behavior of the solution of system (3.1) for the case \( a \neq 1 \) and \( \alpha = 1 \).

**Theorem 3.4** Suppose that \( k \in \mathbb{N} \), \( a \neq 1 \), \( \alpha = 1 \), \( b \neq 0 \neq \beta \), and \( (x_n, y_n)_{n \geq -k} \) is a well-defined solution of system (3.1). Then the next statements are true.

(a) If \( |a| > 1 \) and \( u_{j_1-2} \neq b/(1-a) \neq u_{-1-j_1} \), for some \( j_1 \in \{0, 1\} \), then \( x_{2(2t+1)m+2s+j_1} \to 0 \) and \( \lim |y_{2(2t+1)m+2s+j_1}| \to \infty \), as \( m \to \infty \).

(b) If \( |a| < 1 \) or \( a = -1 \) and \( u_{j_1-2} = b/(1-a) = u_{-1-j_1} \), for some \( j_1 \in \{0, 1\} \), then \( y_{2(2t+1)m+2s+j_1} \to 0 \) and \( \lim |x_{2(2t+1)m+2s+j_1}| \to \infty \), as \( m \to \infty \).

**Proof** Let

\[
\rho_{m}^{\ell,2s+j_1} = \frac{(1 - a)(v_{-1-j_1} + \beta (m (2t + 1) + s - t + j_1))}{b + a^m(2t+1)+s+1 (u_{j_1-2} (1 - a) - b)}
\]

for each \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\} \), \( j_1 \in \{0, 1\} \),

\[
\rho_{m}^{\ell,2s+j_1} = \frac{b + a^m(2t+1)+s-t+j_1 (u_{-1-j_1} (1 - a) - b)}{(1 - a)(v_{j1-2} + \beta (m (2t + 1) + s + 1))}
\]

for each \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\} \).

(a) Note that in this case

\[
\lim_{m \to \infty} \rho_{m}^{\ell,2s+j_1} = 0, \quad \lim_{m \to \infty} |\rho_{m}^{\ell,2s+j_1}| = \infty,
\]

for each \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\} \), from which the result can be obtained easily.

(b) In this case

\[
\lim_{m \to \infty} |\rho_{m}^{\ell,2s+j_1}| = \infty, \quad \lim_{m \to \infty} \rho_{m}^{\ell,2s+j_1} = 0,
\]

for each \( 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\} \), from which the result can be seen easily.

\[\square\]
3.3. Case $a = 1$, $\alpha \neq 1$

Here we study the asymptotic behavior of the solution of system (3.1) for the case $a = 1$ and $\alpha \neq 1$.

**Theorem 3.5** Suppose that $k \in \mathbb{N}$, $a = 1$, $\alpha \neq 1$, $b \neq 0 \neq \beta$, and $(x_n, y_n)_{n \geq -k-2}$ is a well-defined solution of system (3.1). Then the next statements hold.

(a) If $|\alpha| > 1$ and $v_{j_1-2} \neq \beta/(1-\alpha) \neq u_{-1-j_1}$, for some $j_1 \in \{0,1\}$, then $y_{2(2t+1)m+2s+j_1} \to 0$ and $|x_{2(2t+1)m+2s+j_1}| \to \infty$, as $m \to \infty$.

(b) If $|\alpha| < 1$ or $\alpha = -1$ and $v_{j_1-2} = \beta/(1-\alpha) = v_{-1-j_1}$, for some $j_1 \in \{0,1\}$, then $x_{2(2t+1)m+2s+j_1} \to 0$ and $|y_{2(2t+1)m+2s+j_1}| \to \infty$, as $m \to \infty$.

**Proof** Let

$$p_m^{t,2s+j_1} = \frac{\beta + \alpha^{m(2t+1)+s-t+j_1} (v_{-1-j_1} (1-\alpha) - \beta)}{(1-\alpha) (u_{j_1-2} + b (2t+1) + s + 1)}$$

for each $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, $j_1 \in \{0,1\}$,

$$p_m^{t,2s+j_1} = \frac{(1-\alpha) (u_{j_1+j_1} + b (2t+1) + s - t + j_1))}{\beta + \alpha^{m(2t+1)+s+t+1} (v_{j_1-2} (1-\alpha) - \beta)}$$

for each $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, $j_1 \in \{0,1\}$.

(a): Note that in this case

$$\lim_{m \to \infty} |p_m^{t,2s+j_1}| = \infty, \quad \lim_{m \to \infty} p_m^{t,2s+j_1} = 0,$$

for each $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}, j_1 \in \{0,1\}$, from which the result can be seen easily.

(b): In this case

$$\lim_{m \to \infty} p_m^{t,2s+j_1} = 0, \quad \lim_{m \to \infty} |p_m^{t,2s+j_1}| = \infty,$$

for each $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}, j_1 \in \{0,1\}$, from which the result can be seen easily.

\[ \square \]

3.4. Case $a = 1$, $\alpha = 1$

Here we describe the asymptotic behavior of solution to system (3.1) for the case $a = 1$ and $\alpha = 1$.

**Theorem 3.6** Suppose that $k \in \mathbb{N}$, $a = 1 = \alpha$, $b \neq 0 \neq \beta$, and $(x_n, y_n)_{n \geq -k-2}$ is a well-defined solution of system (3.1). Then the next statements hold.

(a) If $|\beta| < |b|$, then $x_n \to 0$ and $|y_n| \to \infty$, as $n \to \infty$.

(b) If $|\beta| > |b|$, then $y_n \to 0$ and $|x_n| \to \infty$, as $n \to \infty$.

(c) If $\beta = b$, $(t-j_1+1) \neq 0$, and

$$\frac{1}{t-j_1+1} (v_{-1-j_1} - u_{j_1-2}) = b,$$

then the sequence $x_{2(2t+1)m+2s+j_1}$ for $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, $j_1 \in \{0,1\}$, is convergent.
(d) If $\beta = b$ and $\frac{1}{b} (v_{t-j_1} - u_{j_1-2}) > t - j_1 + 1$, then $|x_n| \to \infty$, as $n \to \infty$.

(e) If $\beta = b$ and $\frac{1}{b} (v_{t-j_1} - u_{j_1-2}) < t - j_1 + 1$, then $x_n \to 0$, as $n \to \infty$.

(f) If $\beta = b$, $(t - j_1 + 1) \neq 0$ and $\frac{1}{t-j_1+1} (u_{t-j_1} - v_{j_1-2}) = b$, then the sequence $y_{2(2t+1)m+2s+j_1}$, for $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$, is convergent.

(g) If $\beta = b$ and $\frac{1}{b} (u_{t-j_1} - v_{j_1-2}) > t - j_1 + 1$, then $|y_n| \to \infty$, as $n \to \infty$.

(h) If $\beta = b$ and $\frac{1}{b} (u_{t-j_1} - v_{j_1-2}) < t - j_1 + 1$, then $y_n \to 0$, as $n \to \infty$.

(i) If $b = -\beta$, $(t - j_1 + 1) \neq 0$ and $b = -\frac{1}{t-j_1+1} (v_{t-j_1} + u_{j_1-2})$, then the sequences $x_{4(2t+1)m+2s+j_1}$, $x_{4(2t+1)m+4t+2+2s+j_1}$, for $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$, are convergent.

(j) If $b = -\beta$ and $\frac{1}{b} (v_{t-j_1} + u_{j_1-2}) + t - j_1 + 1 < 0$, then $|x_n| \to \infty$, as $n \to \infty$.

(k) If $b = -\beta$ and $\frac{1}{b} (v_{t-j_1} + u_{j_1-2}) + t - j_1 + 1 > 0$, then $x_n \to 0$, as $n \to \infty$.

(l) If $b = -\beta$, $(t - j_1 + 1) \neq 0$ and $b = \frac{1}{t-j_1+1} (u_{t-j_1} + v_{j_1-2})$, then the sequences $y_{4(2t+1)m+2s+j_1}$, $y_{4(2t+1)m+4t+2+2s+j_1}$, for $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$, are convergent.

(m) If $b = -\beta$ and $\frac{1}{b} (u_{t-j_1} + v_{j_1-2}) > t - j_1 + 1$, then $|y_n| \to \infty$, as $n \to \infty$.

(n) If $b = -\beta$ and $\frac{1}{b} (u_{t-j_1} + v_{j_1-2}) < t - j_1 + 1$, then $y_n \to 0$, as $n \to \infty$.

**Proof** (a)–(b): From Eq. (3.8) and Eq. (3.9) we get

$$x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{v_{t-j_1} + \beta (p(2t+1) + s - t + j_1)}{u_{j_1-2} + b (p(2t+1) + s + 1)}$$

$$= x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \beta \left( 1 + \frac{s-t+j_1 + \frac{v_{t-j_1}}{p(2t+1)}}{b \left( 1 + \frac{s+t+j_1 - \frac{u_{j_1-2}}{p(2t+1)}}{p(2t+1)} \right)} \right), \quad (3.46)$$

for each $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$,

$$y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{u_{t-j_1} + b (p(2t+1) + s - t + j_1)}{v_{j_1-2} + \beta (p(2t+1) + s + 1)}$$

$$= y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} b \left( 1 + \frac{s-t+j_1 + \frac{u_{t-j_1}}{p(2t+1)}}{\beta \left( 1 + \frac{s+t+j_1 - \frac{v_{j_1-2}}{p(2t+1)}}{p(2t+1)} \right)} \right), \quad (3.47)$$

for each $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$. From (3.46) and (3.47) the results can be seen easily.

(c)–(h): If $b = \beta$, from (3.46) and (3.47) and by employing the Taylor expansion for $(1 + x)^{-1}$ on the interval
(-\epsilon, \epsilon)$, where $\epsilon > 0$, we get, for sufficiently large $m$,

\[
x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \left( 1 + \frac{s-t+j_1+\frac{u-1-j_{1'}}{p(2t+1)}}{1 + \frac{s+1+u_{j_1-2}}{p(2t+1)}} \right) \\
= x_{2s+j_1-2(2t+1)} C_1(m_0) \prod_{p=m_0+1}^{m} \left( 1 + \frac{v_{s-1-j_1} - v_{j_1-2} - t + j_1 - 1}{p(2t+1)} + O\left( \frac{1}{p^2} \right) \right) \\
= x_{2s+j_1-2(2t+1)} C_1(m_0) \exp \left( \sum_{p=m_0+1}^{m} \left( \frac{v_{s-1-j_1} - v_{j_1-2} - t + j_1 - 1}{p(2t+1)} + O\left( \frac{1}{p^2} \right) \right) \right), \tag{3.48}
\]

for each $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\}$,

\[
y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \left( 1 + \frac{s-t+j_1+\frac{u-1-j_{1'}}{p(2t+1)}}{1 + \frac{s+1+u_{j_1-2}}{p(2t+1)}} \right) \\
= y_{2s+j_1-2(2t+1)} C_2(m_0) \prod_{p=m_0+1}^{m} \left( 1 + \frac{u_{s-1-j_1} - u_{j_1-2} - t + j_1 - 1}{p(2t+1)} + O\left( \frac{1}{p^2} \right) \right) \\
= y_{2s+j_1-2(2t+1)} C_2(m_0) \exp \left( \sum_{p=m_0+1}^{m} \left( \frac{u_{s-1-j_1} - u_{j_1-2} - t + j_1 - 1}{p(2t+1)} + O\left( \frac{1}{p^2} \right) \right) \right), \tag{3.49}
\]

for each $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\}$, where $C_1(m_0) = \prod_{p=0}^{m_0} \left( 1 + \frac{s-t+j_1+1}{p(2t+1)} \right)$ and $C_2(m_0) = \prod_{p=0}^{m_0} \left( 1 + \frac{s-t+j_1+1}{p(2t+1)} \right)$. From (3.48) and (3.49), the fact that $\sum_{p=m_0+1}^{m} \frac{1}{p} \to +\infty$, as $m \to \infty$,

and the convergence of the series $\sum_{p=m_0+1}^{+\infty} O\left( \frac{1}{p^2} \right)$, the statements can be seen easily.

(i)-(ii): If $-\beta = b$, from (3.46) and (3.47) by using the Taylor expansion for $(1 + x)^{-1}$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$, we get, for sufficiently large $m$,

\[
x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} (-1)^{m+1} \prod_{p=0}^{m} \left( 1 + \frac{s-t+j_1+\frac{u-1-j_{1'}}{p(2t+1)}}{1 + \frac{s+1+u_{j_1-2}}{p(2t+1)}} \right) \\
= x_{2s+j_1-2(2t+1)} (-1)^{m+1} C_1(m_1) \prod_{p=m_1+1}^{m} \left( 1 - \frac{v_{s-1-j_1} - v_{j_1-2} + t - j_1 + 1}{p(2t+1)} + O\left( \frac{1}{p^2} \right) \right) \\
= x_{2s+j_1-2(2t+1)} (-1)^{m+1} C_1(m_1) \exp \left( - \sum_{p=m_1+1}^{m} \left( \frac{v_{s-1-j_1} - v_{j_1-2} + t - j_1 + 1}{p(2t+1)} + O\left( \frac{1}{p^2} \right) \right) \right), \tag{3.50}
\]

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for each \(2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\}\),

\[
y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)}(-1)^{m+1} \prod_{p=0}^{m} \frac{1 + \frac{s-t+j_1+n-1-j_1}{p(2t+1)}}{1 + \frac{s+1-\nu_{j_1}+2}{p(2t+1)}}
\]

\[
= y_{2s+j_1-2(2t+1)}(-1)^{m+1} C_2(m_1) \prod_{p=1+1}^{m} \left( 1 + \frac{u_{j_1}+\nu_{j_1}-2}{p(2t+1)} - t + j_1 - 1 + O\left(\frac{1}{p^2}\right) \right)
\]

\[
= y_{2s+j_1-2(2t+1)}(-1)^{m+1} C_2(m_1) \exp \left( \sum_{p=m_1+1}^{m} \left( \frac{u_{j_1}+\nu_{j_1}-2}{p(2t+1)} - t + j_1 - 1 + O\left(\frac{1}{p^2}\right) \right) \right), (3.51)
\]

for each \(2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\}\), where \(C_1(m_1) = \prod_{p=0}^{m_1} \frac{1 + \frac{s-t+j_1+n-1-j_1}{p(2t+1)}}{1 + \frac{s+1-\nu_{j_1}+2}{p(2t+1)}}\) and

\[
C_2(m_1) = \prod_{p=0}^{m_1} \frac{1 + \frac{s-t+j_1+n-1-j_1}{p(2t+1)}}{1 + \frac{s+1-\nu_{j_1}+2}{p(2t+1)}}. \tag{3.50}
\]

From (3.50) and (3.51), the fact that \(\sum_{p=m_1+1}^{m} \frac{1}{p} \to +\infty\), as \(m \to \infty\),

and the convergence of the series \(\sum_{p=m_1+1}^{+\infty} O\left(\frac{1}{p^2}\right)\), the statements in (i)–(n) can be obtained easily. \(\square\)

3.5. Case \(a = -1, |\alpha| < 1\)

Here we introduce the asymptotic behavior of well-defined solutions of system (3.1) for the case \(a = -1\) and \(|\alpha| < 1\), by using the next two notations:

\[
\tilde{K}_{j_1} := \frac{2\beta}{(1-\alpha)(b+(-1)^{s+1}(2u_{j_1}-2-b))}, \quad \tilde{L}_{j_1} := \frac{(1-\alpha)(b+(-1)^{s-t+j_1}(2u_{j_1}-2-b))}{2\beta}.
\]

where \(m, t \in \mathbb{N}_0, j_1 \in \{0, 1\}\).

**Theorem 3.7** Suppose that \(a = -1, |\alpha| < 1, b \neq 0 \neq \beta, m\) is even, and \((x_n, y_n)_{n \geq -k-2}\) is a well-defined solution of system (3.1). Then the next statements are true.

(a) If \(|\tilde{K}_{j_1}| < 1\) for some \(j_1 \in \{0, 1\}\), \(2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}\), then \(x_{4(2t+1)m+2s+j_1} \to 0\) as \(m \to \infty\).

(b) If \(|\tilde{K}_{j_1}| > 1\) for some \(j_1 \in \{0, 1\}\), \(2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}\), then \(|x_{4(2t+1)m+2s+j_1}| \to \infty\) as \(m \to \infty\).

(c) If \(\tilde{K}_{j_1} = 1\) for some \(j_1 \in \{0, 1\}\), \(2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}\), then the sequence \(x_{4(2t+1)m+2s+j_1}\) is convergent.

(d) If \(\tilde{K}_{j_1} = -1\) for some \(j_1 \in \{0, 1\}\), \(2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}\), then the sequences \(x_{8(2t+1)m+2s+j_1}\), \(x_{8(2t+1)m+8t+4+2s+j_1}\) are convergent.
(e) If $|\bar{L}_{j_1}| < 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then $y_{4(2t+1)m+2s+j_1} \to 0$ as $m \to \infty$.

(f) If $|\bar{L}_{j_1}| > 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then $|y_{4(2t+1)m+2s+j_1}| \to \infty$ as $m \to \infty$.

(g) If $\bar{L}_{j_1} = 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then the sequence $y_{4(2t+1)m+2s+j_1}$ is convergent.

(h) If $\bar{L}_{j_1} = -1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then the sequences $y_{8(2t+1)m+2s+j_1}$, $y_{8(2t+1)m+8t+4+2s+j_1}$ are convergent.

Proof (a), (b) and (e), (f): By using (3.12) and (3.13) for the case $a = -1$ and $|\alpha| < 1$,

$$\lim_{m \to \infty} \frac{2(\beta + \alpha^{2m(2t+1)+s-t+j_1}(v_{-1-j_1}(1-\alpha) - \beta))}{(1-\alpha)(b + (-1)^{2m(2t+1)+s+1}(2u_{j_1-2} - b))} = \bar{L}_{j_1},$$

(3.52)

for each $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0,1\}$,

$$\lim_{m \to \infty} \frac{(1-\alpha)(b + (-1)^{2m(2t+1)+s-t+j_1}(2u_{-1-j_1} - b))}{2(\beta + \alpha^{2m(2t+1)+s-t+j_1}(v_{j_1-2}(1-\alpha) - \beta))} = \bar{L}_{j_1},$$

(3.53)

for each $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0,1\}$, from which, along with formulas (3.52) and (3.53), the results in (a), (b), (e), and (f) can be seen easily.

(c): From (3.12), we get that

$$\frac{2(\beta + \alpha^{2m(2t+1)+s-t+j_1}(v_{-1-j_1}(1-\alpha) - \beta))}{(1-\alpha)(b + (-1)^{2m(2t+1)+s+1}(2u_{j_1-2} - b))} = 1 + \frac{\alpha^{2m(2t+1)+s-t+j_1}(v_{-1-j_1}(1-\alpha) - \beta)}{\beta},$$

which holds for $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$ and $j_1 \in \{0,1\}$. By using Taylor expansion for $\ln(1+x)$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$, in the above equation, the statement easily follows.

(d): The result can be easily obtained from the relation

$$\frac{2(\beta + \alpha^{2m(2t+1)+s-t+j_1}(v_{-1-j_1}(1-\alpha) - \beta))}{(1-\alpha)(b + (-1)^{2m(2t+1)+s+1}(2u_{j_1-2} - b))} = - \frac{1 + \alpha^{2m(2t+1)+s-t+j_1}(v_{-1-j_1}(1-\alpha) - \beta)}{\beta},$$

which holds for $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0,1\}$ and Taylor expansion for $\ln(1+x)$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$.

(g): By using (3.13) for the case $a = -1$ and $|\alpha| < 1$, we can write

$$\frac{(1-\alpha)(b + (-1)^{2m(2t+1)+s-t+j_1}(2u_{-1-j_1} - b))}{2(\beta + \alpha^{2m(2t+1)+s+t+j_1}(v_{j_1-2}(1-\alpha) - \beta))} = 1 - \frac{\alpha^{2m(2t+1)+s-t+j_1}(v_{j_1-2}(1-\alpha) - \beta)(1 + O(1))}{\beta},$$

which holds for $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$ and $j_1 \in \{0,1\}$. By using Taylor expansion for $(1 + x)^{-1}$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$, in the above equation, the result can be seen easily.

(h): In this case by using (3.13) for the case $a = -1$ and $|\alpha| < 1$, we have

$$\frac{(1-\alpha)(b + (-1)^{2m(2t+1)+s-t+j_1}(2u_{-1-j_1} - b))}{2(\beta + \alpha^{2m(2t+1)+s+t+j_1}(v_{j_1-2}(1-\alpha) - \beta))} = - \left(1 - \frac{\alpha^{2m(2t+1)+s-t+j_1}(v_{j_1-2}(1-\alpha) - \beta)(1 + O(1))}{\beta}\right),$$

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which holds for $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$ and $j_1 \in \{0, 1\}$, from which, along with Taylor expansion for $(1 + x)^{-1}$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$, the statement follows.

**Theorem 3.8** Suppose that $a = -1$, $|\alpha| < 1$, $b \neq 0 \neq \beta$, $m, t, w, h \in \mathbb{N}_0$, $j_1 \in \{0, 1\}$, $m$ is odd, and $(x_n, y_n)_{n \geq k - 2}$ is a well-defined solution of system (3.1). Then the next statements are true.

(a) If $|\frac{\beta}{(1 - \alpha)u_{j_1 - 2}}| < 1$ for some $j_1 \in \{0, 1\}$, then $x_{4(2t+1)m+4t+2+4w+j_1} \to 0$ as $m \to \infty$.

(b) If $|\frac{\beta}{(1 - \alpha)u_{j_1 - 2}}| > 1$ for some $j_1 \in \{0, 1\}$, then $|x_{4(2t+1)m+4t+2+4w+j_1}| \to \infty$ as $m \to \infty$.

(c) If $\frac{\beta}{(1 - \alpha)u_{j_1 - 2}} = 1$ for some $j_1 \in \{0, 1\}$, then the sequence $x_{4(2t+1)m+4t+2+4w+j_1}$ is convergent.

(d) If $\frac{\beta}{(1 - \alpha)u_{j_1 - 2}} = -1$ for some $j_1 \in \{0, 1\}$, then the sequences $x_{8(2t+1)m+4t+2+4w+j_1}$, $x_{8(2t+1)m+12t+6+4w+j_1}$ are convergent.

(e) If $|\frac{(1 - \alpha)u_{j_1 - 2}}{\beta}| < 1$ for some $j_1 \in \{0, 1\}$, then $y_{4(2t+1)m+4t+2+4w+j_1} \to 0$, $y_{4(2t+1)m+4t+4w+j_1} \to 0$ as $m \to \infty$.

(f) If $|\frac{(1 - \alpha)u_{j_1 - 2}}{\beta}| > 1$ for some $j_1 \in \{0, 1\}$, then $|y_{4(2t+1)m+4t+2+4w+j_1}| \to \infty$, $|y_{4(2t+1)m+4t+4w+j_1}| \to \infty$ as $m \to \infty$.

(g) If $\frac{(1 - \alpha)u_{j_1 - 2}}{\beta} = 1$ for some $j_1 \in \{0, 1\}$, then the sequences $y_{4(2t+1)m+4t+2+4w+j_1}$, $y_{4(2t+1)m+4t+4w+j_1}$ are convergent.

(h) If $\frac{(1 - \alpha)u_{j_1 - 2}}{\beta} = -1$ for some $j_1 \in \{0, 1\}$, then the sequences $y_{8(2t+1)m+4t+2+4w+j_1}$, $y_{8(2t+1)m+12t+6+4w+j_1}$, $y_{8(2t+1)m+4t+4w+j_1}$, $y_{8(2t+1)m+12t+4w+j_1}$ are convergent.

(i) If $|\frac{\beta}{(1 - \alpha)(b-u_{j_1 - 2})}| < 1$ for some $j_1 \in \{0, 1\}$, then $x_{4(2t+1)m+4t+4w+j_1} \to 0$ as $m \to \infty$.

(j) If $|\frac{\beta}{(1 - \alpha)(b-u_{j_1 - 2})}| > 1$ for some $j_1 \in \{0, 1\}$, then $|x_{4(2t+1)m+4t+4w+j_1}| \to \infty$ as $m \to \infty$.

(k) If $\frac{\beta}{(1 - \alpha)(b-u_{j_1 - 2})} = 1$ for some $j_1 \in \{0, 1\}$, then the sequence $x_{4(2t+1)m+4t+4w+j_1}$ is convergent.

(l) If $\frac{\beta}{(1 - \alpha)(b-u_{j_1 - 2})} = -1$ for some $j_1 \in \{0, 1\}$, then the sequences $x_{8(2t+1)m+4t+4w+j_1}$, $x_{8(2t+1)m+12t+4w+j_1}$ are convergent.

(m) If $|\frac{(1 - \alpha)(b-u_{j_1 - 2})}{\beta}| < 1$ for some $j_1 \in \{0, 1\}$, then $y_{4(2t+1)m+4t+2+4w+j_1} \to 0$, $y_{4(2t+1)m+4t+4w+j_1} \to 0$ as $m \to \infty$.

(n) If $|\frac{(1 - \alpha)(b-u_{j_1 - 2})}{\beta}| > 1$ for some $j_1 \in \{0, 1\}$, then $|y_{4(2t+1)m+4t+2+4w+j_1}| \to \infty$, $|y_{4(2t+1)m+4t+4w+j_1}| \to \infty$ as $m \to \infty$.

(o) If $\frac{(1 - \alpha)(b-u_{j_1 - 2})}{\beta} = 1$ for some $j_1 \in \{0, 1\}$, then the sequences $y_{4(2t+1)m+4t+2+4w+j_1}$, $y_{4(2t+1)m+4t+4w+j_1}$ are convergent.
If \( \frac{(1-\alpha)(b-u_{j_1})}{\beta} = -1 \) for some \( j_1 \in \{0, 1\} \), then the sequences \( y_k(2t+1)m+4t+2+4w+j_1 \), \( y_k(2t+1)m+12t+6+4w+j_1 \), and \( y_k(2t+1)m+4t+4w+4+j_1 \) are convergent.

**Proof** Let \( a = -1, |\alpha| < 1 \), and \( m \) be odd. From (3.12)-(3.13) we have

\[
 \begin{align*}
 p_{2m+1}^{s-s-t+j_1} &= \frac{2((v_{-1}-j_1(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+s-t+j_1} + \beta)}{(1-\alpha)((2u_{j_1-2}-b)(-1)^{(2m+1)(2t+1)+s+1} + b)}, \\
 r_{2m+1}^{s-s-t+j_1} &= \frac{(1-\alpha)((2u_{-1}-j_1-b)(-1)^{(2m+1)(2t+1)+s-t+j_1} + b)}{2((v_{j_1-2}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+s+1} + \beta)}.
\end{align*}
\]

Here there are four cases to be considered.

- **s is even and \( s-t+j_1 \) is odd**: In this case we have

\[
 \begin{align*}
 p_{2m+1}^{2w,2h+1} &= \frac{2((v_{-1}-j_1(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2h+1} + \beta)}{(1-\alpha)((2u_{j_1-2}-b)(-1)^{(2m+1)(2t+1)+2w+1} + b)}, \\
 r_{2m+1}^{2w,2h+1} &= \frac{(1-\alpha)((2u_{-1}-j_1-b)(-1)^{(2m+1)(2t+1)+2h+1} + b)}{2((v_{j_1-2}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2w+1} + \beta)}.
\end{align*}
\]

where \( s = 2w \) and \( s-t+j_1 = 2h+1 \), for \( w, h \in \mathbb{N}_0 \). Taking the limit of both sides in equations (3.56) and (3.57), we get

\[
 \begin{align*}
 \lim_{m \to \infty} p_{2m+1}^{2w,2h+1} &= \lim_{m \to \infty} \frac{2((v_{-1}-j_1(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2h+1} + \beta)}{(1-\alpha)2u_{j_1-2}} = \frac{\beta}{(1-\alpha)u_{j_1-2}}, \\
 \lim_{m \to \infty} r_{2m+1}^{2w,2h+1} &= \lim_{m \to \infty} \frac{(1-\alpha)2u_{-1}-j_1}{2((v_{j_1-2}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2w+1} + \beta)} = \frac{(1-\alpha)u_{-1}-j_1}{\beta}.
\end{align*}
\]

- **s and \( s-t+j_1 \) are both even**: In this case we get

\[
 \begin{align*}
 p_{2m+1}^{2w,2h} &= \frac{2((v_{-1}-j_1(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2h} + \beta)}{(1-\alpha)((2u_{j_1-2}-b)(-1)^{(2m+1)(2t+1)+2w+1} + b)}, \\
 r_{2m+1}^{2w,2h} &= \frac{(1-\alpha)((2u_{-1}-j_1-b)(-1)^{(2m+1)(2t+1)+2h} + b)}{2((v_{j_1-2}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2w+1} + \beta)},
\end{align*}
\]

where \( s = 2w \) and \( s-t+j_1 = 2h \), for \( w, h \in \mathbb{N}_0 \). Taking the limit of both sides in equations (3.60) and (3.61), we have

\[
 \begin{align*}
 \lim_{m \to \infty} p_{2m+1}^{2w,2h} &= \lim_{m \to \infty} \frac{2((v_{-1}-j_1(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2h} + \beta)}{(1-\alpha)2u_{j_1-2}} = \frac{\beta}{(1-\alpha)u_{j_1-2}}, \\
 \lim_{m \to \infty} r_{2m+1}^{2w,2h} &= \lim_{m \to \infty} \frac{(1-\alpha)(2b-2u_{-1}-j_1)}{2((v_{j_1-2}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2w+1} + \beta)} = \frac{(1-\alpha)(b-u_{-1}-j_1)}{\beta}.
\end{align*}
\]
• \( s \) and \( s - t + j_1 \) are both odd: In this case we have

\[
p_{2m+1}^{2w+1,2h+1} = \frac{2 \left( (v_{-1-j_1} (1 - \alpha) - \beta) \alpha^{(2m+1)(2t+1)+2h+1} + \beta \right)}{(1 - \alpha) \left( (2u_{j_1-2} - b)(-1)^{(2m+1)(2t+1)+2w+2} + b \right)},
\]

(3.64)

\[
r_{2m+1}^{2w+1,2h+1} = \frac{(1 - \alpha) \left((2u_{-1-j_1} - b)(-1)^{(2m+1)(2t+1)+2h+1} + b \right)}{2 \left( (v_{j_1-2} (1 - \alpha) - \beta) \alpha^{(2m+1)(2t+1)+2w+2} + b \right)},
\]

(3.65)

where \( s = 2w + 1 \) and \( s - t + j_1 = 2h + 1, \) for \( w, h \in \mathbb{N}_0. \) Taking the limit of both sides in equations (3.64) and (3.65), we obtain

\[
\lim_{m \to \infty} p_{2m+1}^{2w+1,2h+1} = \lim_{m \to \infty} \frac{2 \left( (v_{-1-j_1} (1 - \alpha) - \beta) \alpha^{(2m+1)(2t+1)+2h+1} + \beta \right)}{(1 - \alpha) \left( (2b - 2u_{j_1-2}) \alpha^{(2m+1)(2t+1)+2w+2} + b \right)} = \frac{\beta}{(1 - \alpha) \left( b - u_{j_1-2} \right)},
\]

(3.66)

\[
\lim_{m \to \infty} r_{2m+1}^{2w+1,2h+1} = \lim_{m \to \infty} \frac{(1 - \alpha) \left( (2u_{-1-j_1} - b)(-1)^{(2m+1)(2t+1)+2h+1} + b \right)}{2 \left( (v_{j_1-2} (1 - \alpha) - \beta) \alpha^{(2m+1)(2t+1)+2w+2} + b \right)} = \frac{(1 - \alpha) \left( b - u_{-1-j_1} \right)}{\beta},
\]

(3.67)

• \( s \) is odd and \( s - t + j_1 \) is even: In this case we get

\[
p_{2m+1}^{2w+1,2h} = \frac{2 \left( (v_{-1-j_1} (1 - \alpha) - \beta) \alpha^{(2m+1)(2t+1)+2h} + \beta \right)}{(1 - \alpha) \left( (2u_{j_1-2} - b)(-1)^{(2m+1)(2t+1)+2w+2} + b \right)},
\]

(3.68)

\[
r_{2m+1}^{2w+1,2h} = \frac{(1 - \alpha) \left((2u_{-1-j_1} - b)(-1)^{(2m+1)(2t+1)+2h} + b \right)}{2 \left( (v_{j_1-2} (1 - \alpha) - \beta) \alpha^{(2m+1)(2t+1)+2w+2} + b \right)},
\]

(3.69)

where \( s = 2w + 1 \) and \( s - t + j_1 = 2h, \) for \( w, h \in \mathbb{N}_0. \) Similarly, taking the limit of both sides in equations (3.68) and (3.69), we have

\[
\lim_{m \to \infty} p_{2m+1}^{2w+1,2h} = \lim_{m \to \infty} \frac{2 \left( (v_{-1-j_1} (1 - \alpha) - \beta) \alpha^{(2m+1)(2t+1)+2h} + \beta \right)}{(1 - \alpha) \left( (2b - 2u_{j_1-2}) \alpha^{(2m+1)(2t+1)+2w+2} + b \right)} = \frac{\beta}{(1 - \alpha) \left( b - u_{j_1-2} \right)},
\]

(3.70)

\[
\lim_{m \to \infty} r_{2m+1}^{2w+1,2h} = \lim_{m \to \infty} \frac{(1 - \alpha) \left( (2b - 2u_{-1-j_1}) \alpha^{(2m+1)(2t+1)+2h} + b \right)}{2 \left( (v_{j_1-2} (1 - \alpha) - \beta) \alpha^{(2m+1)(2t+1)+2w+2} + b \right)} = \frac{(1 - \alpha) \left( b - u_{-1-j_1} \right)}{\beta},
\]

(3.71)

From (3.58), (3.59), (3.62), (3.63), (3.66), (3.67), (3.70), and (3.71), the results of the theorem can be seen easily.

\[\Box\]

3.6. Case \(|a| < 1, \alpha = -1\)

Here, by employing the next two notations,

\[
M_{j_1} := \frac{(1 - a) \left( \beta + (-1)^{s-t+j_1} (2v_{-1-j_1} - \beta) \right)}{2b}, \quad P_{j_1} := \frac{2b}{(1 - a) \left( \beta + (-1)^{s+1} (2v_{j_1-2} - \beta) \right)},
\]

where \( m, t \in \mathbb{N}_0, \ j_1 \in \{0, 1\}, \) for nonnegative even integer number \( m \) we can first give the following theorem, which can be proven like Theorem 3.7.
Theorem 3.9 Suppose that $\alpha = -1$, $|a| < 1$, $b \neq 0 \neq \beta$, $m$ is even, and $(x_n, y_n)_{n \geq -k-2}$ is a well-defined solution of system (3.1). Then the next statements hold.

(a) If $|M_{j_1}| < 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then $x_{4(2t+1)m+2s+j_1} \rightarrow 0$ as $m \to \infty$.

(b) If $|M_{j_1}| > 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then $|x_{4(2t+1)m+2s+j_1}| \rightarrow \infty$ as $m \to \infty$.

(c) If $M_{j_1} = 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then the sequence $x_{4(2t+1)m+2s+j_1}$ is convergent.

(d) If $M_{j_1} = -1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then the sequences $x_{8(2t+1)m+2s+j_1}$ are convergent.

(e) If $|P_{j_1}| < 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then $y_{4(2t+1)m+2s+j_1} \rightarrow 0$ as $m \to \infty$.

(f) If $|P_{j_1}| > 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then $|y_{4(2t+1)m+2s+j_1}| \rightarrow \infty$ as $m \to \infty$.

(g) If $P_{j_1} = 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then the sequence $y_{4(2t+1)m+2s+j_1}$ is convergent.

(h) If $P_{j_1} = -1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then the sequences $y_{8(2t+1)m+2s+j_1}$ are convergent.

Similarly, for nonnegative odd integer number $m$ we can give the next theorem, which can be proven like Theorem 3.8.

Theorem 3.10 Suppose that $\alpha = -1$, $|a| < 1$, $b \neq 0 \neq \beta$, $m, t, w \in \mathbb{N}_0$, $j_1 \in \{0,1\}$, $m$ is odd, and $(x_n, y_n)_{n \geq -k-2}$ is a well-defined solution of system (3.1). Then the next statements hold.

(a) If $\left|\frac{b}{(1-a)v_{j_1-2}}\right| < 1$ for some $j_1 \in \{0,1\}$, then $y_{4(2t+1)m+4t+2+4w+j_1} \rightarrow 0$ as $m \to \infty$.

(b) If $\left|\frac{b}{(1-a)v_{j_1-2}}\right| > 1$ for some $j_1 \in \{0,1\}$, then $|y_{4(2t+1)m+4t+2+4w+j_1}| \rightarrow \infty$ as $m \to \infty$.

(c) If $\frac{b}{(1-a)v_{j_1-2}} = 1$ for some $j_1 \in \{0,1\}$, then the sequence $y_{4(2t+1)m+4t+2+4w+j_1}$ is convergent.

(d) If $\frac{b}{(1-a)v_{j_1-2}} = -1$ for some $j_1 \in \{0,1\}$, then the sequences $y_{8(2t+1)m+4t+2+4w+j_1}$, $y_{8(2t+1)m+12t+6+4w+j_1}$ are convergent.

(e) If $\left|\frac{b}{(1-a)v_{j_1-1}}\right| < 1$ for some $j_1 \in \{0,1\}$, then $x_{4(2t+1)m+4t+2+4w+j_1} \rightarrow 0$, $x_{4(2t+1)m+4t+4w+4+j_1} \rightarrow 0$ as $m \to \infty$.

(f) If $\left|\frac{b}{(1-a)v_{j_1-1}}\right| > 1$ for some $j_1 \in \{0,1\}$, then $|x_{4(2t+1)m+4t+2+4w+j_1}| \rightarrow \infty$, $|x_{4(2t+1)m+4t+4w+4+j_1}| \rightarrow \infty$ as $m \to \infty$.

(g) If $\frac{b}{(1-a)v_{j_1-1}} = 1$ for some $j_1 \in \{0,1\}$, then the sequences $x_{4(2t+1)m+4t+2+4w+j_1}$, $x_{4(2t+1)m+4t+4w+4+j_1}$ are convergent.
(h) If \( \frac{(1-a)^{j_1-1}}{b} = -1 \) for some \( j_1 \in \{0,1\} \), then the sequences \( x_{8(2t+1)m+4t+2+4w+j_1} \), \( x_{8(2t+1)m+12t+6+4w+j_1} \), \( x_{8(2t+1)m+4t+4w+4+j_1} \), \( x_{8(2t+1)m+12t+4w+8+j_1} \) are convergent.

(i) If \( \frac{b}{(1-a)(\beta-v_{j_1-2})} < 1 \) for some \( j_1 \in \{0,1\} \), then \( y_{4(2t+1)m+4t+4w+4+j_1} \to 0 \) as \( m \to \infty \).

(j) If \( \frac{b}{(1-a)(\beta-v_{j_1-2})} > 1 \) for some \( j_1 \in \{0,1\} \), then \( |y_{4(2t+1)m+4t+4w+4+j_1}| \to \infty \) as \( m \to \infty \).

(k) If \( \frac{b}{(1-a)(\beta-v_{j_1-2})} = 1 \) for some \( j_1 \in \{0,1\} \), then the sequence \( y_{4(2t+1)m+4t+4w+4+j_1} \) is convergent.

(l) If \( \frac{b}{(1-a)(\beta-v_{j_1-2})} = -1 \) for some \( j_1 \in \{0,1\} \), then the sequences \( y_{8(2t+1)m+4t+4w+4+j_1} \), \( y_{8(2t+1)m+12t+4w+8+j_1} \) are convergent.

(m) If \( \frac{(1-a)(\beta-v_{1-j_1})}{b} < 1 \) for some \( j_1 \in \{0,1\} \), then \( x_{4(2t+1)m+4t+2+4w+j_1} \to 0 \), \( x_{4(2t+1)m+4t+4w+4+j_1} \to 0 \) as \( m \to \infty \).

(n) If \( \frac{(1-a)(\beta-v_{1-j_1})}{b} > 1 \) for some \( j_1 \in \{0,1\} \), then \( |x_{4(2t+1)m+4t+2+4w+j_1}| \to \infty \), \( |x_{4(2t+1)m+4t+4w+4+j_1}| \to \infty \) as \( m \to \infty \).

(o) If \( \frac{(1-a)(\beta-v_{1-j_1})}{b} = 1 \) for some \( j_1 \in \{0,1\} \), then the sequences \( x_{4(2t+1)m+4t+2+4w+j_1} \), \( x_{4(2t+1)m+4t+4w+4+j_1} \) are convergent.

(p) If \( \frac{(1-a)(\beta-v_{1-j_1})}{b} = -1 \) for some \( j_1 \in \{0,1\} \), then the sequences \( x_{8(2t+1)m+4t+2+4w+j_1} \), \( x_{8(2t+1)m+12t+6+4w+j_1} \), \( x_{8(2t+1)m+4t+4w+4+j_1} \), \( x_{8(2t+1)m+12t+4w+8+j_1} \) are convergent.

Theorem 3.11 Suppose that \( a = -1 = \alpha \), \( b \neq 0 \neq \beta \), \( k \) is odd, and \( (x_n, y_n)_{n \geq -k-2} \) is a well-defined solution of system (3.1). Then the next statements hold.

(a) If \( \frac{|x-y-k-2}{y-1-x-k-1} \frac{1}{bx-2y-k-2-1} | < 1 \), then \( x_{4n} \to 0 \), \( |y_{4n+3}| \to \infty \) as \( n \to \infty \).

(b) If \( \frac{|x-y-k-2}{y-1-x-k-1} \frac{1}{bx-2y-k-2-1} | > 1 \), then \( x_{4n} \to \infty \), \( y_{4n+3} \to 0 \) as \( n \to \infty \).

(c) If \( \frac{x-y-k-2}{y-1-x-k-1} \frac{1}{bx-2y-k-2-1} = 1 \), then \( x_{4n} \) and \( y_{4n+3} \) are 2k-periodic.

(d) If \( \frac{x-y-k-2}{y-1-x-k-1} \frac{1}{bx-2y-k-2-1} = -1 \), then \( x_{4n} \) and \( y_{4n+3} \) are 4k-periodic.

(e) If \( \frac{|x-y-k-2}{y-1-x-k-1} (\beta y_1 x_{k-1} - 1) | < 1 \), then \( x_{4n+2} \to 0 \), \( |y_{4n+1}| \to \infty \) as \( n \to \infty \).

(f) If \( \frac{|x-y-k-2}{y-1-x-k-1} (\beta y_1 x_{k-1} - 1) | > 1 \), then \( x_{4n+2} \to \infty \), \( y_{4n+1} \to 0 \) as \( n \to \infty \).

(g) If \( \frac{x-y-k-2}{y-1-x-k-1} (\beta y_1 x_{k-1} - 1) = 1 \), then \( x_{4n+2} \) and \( y_{4n+1} \) are 2k-periodic.

(h) If \( \frac{x-y-k-2}{y-1-x-k-1} (\beta y_1 x_{k-1} - 1) = -1 \), then \( x_{4n+2} \) and \( y_{4n+1} \) are 4k-periodic.

(i) If \( \frac{|x-y-k-2}{y-1-x-k-1} \frac{1}{bx-2y-k-2-1} | < 1 \), then \( x_{4n+1} \to 0 \), \( |y_{4n+2}| \to \infty \) as \( n \to \infty \).
(j) If \(|x_{n} + \frac{1}{y_{n} - k} - 1\| > 1\), then \(|x_{4n+1}| \to \infty, y_{4n+2} \to 0\) as \(n \to \infty\).

(k) If \(\frac{1}{y_{n} - k} - 1 = 1\), then \(x_{4n+1}\) and \(y_{4n+2}\) are 2k-periodic.

(l) If \(\frac{1}{y_{n} - k} - 1 = -1\), then \(x_{4n+1}\) and \(y_{4n+2}\) are 4k-periodic.

(m) If \(|\frac{1}{y_{n} - k} - 1 (\beta y_{n} - k - 2 - 1)\| < 1\), then \(x_{4n+3} \to 0\), \(|y_{4n}| \to \infty\) as \(n \to \infty\).

(n) If \(|\frac{1}{y_{n} - k} - 1 (\beta y_{n} - k - 2 - 1)\| > 1\), then \(|x_{4n+3}| \to \infty, y_{4n} \to 0\) as \(n \to \infty\).

(o) If \(\frac{1}{y_{n} - k} - 1 (\beta y_{n} - k - 2 - 1) = 1\), then \(x_{4n+3}\) and \(y_{4n}\) are 2k-periodic.

(p) If \(\frac{1}{y_{n} - k} - 1 (\beta y_{n} - k - 2 - 1) = -1\), then \(x_{4n+3}\) and \(y_{4n}\) are 4k-periodic.

**Proof** From (2.3), in this case, we have

\[u_{n} = -u_{n-2} + b = u_{n-4}, \quad v_{n} = -v_{n-2} + \beta = v_{n-4}, \quad n \in \mathbb{N}_{0}, \quad n \geq 2,\]  

(3.72)

which means that the sequences \(u_{n}\) and \(v_{n}\) are four-periodic, and consequently the sequences \(x_{n}y_{n-k}\) and \(y_{n}x_{n-k}\) are four-periodic. Hence, we have

\[x_{4n}y_{n-k} = x_{0}y_{-k},\]
\[x_{4n+1}y_{n-k+1} = x_{1}y_{-k+1},\]
\[x_{4n+2}y_{n-k+2} = x_{-2}y_{-k-2},\]
\[x_{4n+3}y_{n-k+3} = x_{-1}y_{-k-1},\]  

(3.73)

and

\[y_{4n}x_{n-k} = y_{0}x_{-k},\]
\[y_{4n+1}x_{n-k+1} = y_{1}x_{-k+1},\]
\[y_{4n+2}x_{n-k+2} = y_{-2}x_{-k-2},\]
\[y_{4n+3}x_{n-k+3} = y_{-1}x_{-k-1}.\]  

(3.74)

If \(k = 2t + 1\), for some \(t \in \mathbb{N}_{0}\), then from (3.1), (3.73), and (3.74) we have

\[x_{4n}y_{n-(2t+1)} = x_{0}y_{-(2t+1)},\]
\[x_{4n+1}y_{n-2t} = x_{1}y_{-2t},\]
\[x_{4n+2}y_{n-(2t-1)} = x_{-2}y_{-(2t+3)},\]
\[x_{4n+3}y_{n-(2t-2)} = x_{-1}y_{-(2t+2)},\]  

(3.75)

and

\[y_{4n}x_{n-(2t+1)} = y_{0}x_{-(2t+1)},\]
\[y_{4n+1}x_{n-2t} = y_{1}x_{-2t},\]
\[y_{4n+2}x_{n-(2t-1)} = y_{-2}x_{-(2t+3)},\]
\[y_{4n+3}x_{n-(2t-2)} = y_{-1}x_{-(2t+2)}.\]  

(3.76)
Hence,

\[ x_{4n}y_{4n-(2t+1)} = x_0y_{-(2t+1)} = \frac{x_{-2}y_{-(2t+3)}}{bx_{-2}y_{-(2t+3)} - 1}, \]
\[ x_{4n+1}y_{4n-2t} = x_1y_{-2t} = \frac{x_{-1}y_{-(2t+2)}}{bx_{-1}y_{-(2t+2)} - 1}, \]

(3.77)

and

\[ y_{4n}x_{4n-(2t+1)} = y_0x_{-(2t+1)} = \frac{y_{-2}x_{-(2t+3)}}{\beta y_{-2}x_{-(2t+3)} - 1}, \]
\[ y_{4n+1}x_{4n-2t} = y_1x_{-2t} = \frac{y_{-1}x_{-(2t+2)}}{\beta y_{-1}x_{-(2t+2)} - 1}. \]  

(3.78)

From (3.75)–(3.78) we obtain

\[ x_{4n} = \frac{x_{-2}y_{-(2t+3)}}{bx_{-2}y_{-(2t+3)} - 1} \frac{1}{y_{-1}y_{-(2t+1)}} = \frac{x_{-2}y_{-(2t+3)}}{y_{-1}x_{-(2t+2)} - 1} \frac{1}{bx_{-2}y_{-(2t+3)} - 1} x_{4n-(4t+2)}, \]

(3.79)

if \( t \) is even,

\[ x_{4n+1} = \frac{x_{-1}y_{-(2t+2)}}{bx_{-1}y_{-(2t+2)} - 1} \frac{1}{y_{-1}y_{-(2t+1)}} = \frac{x_{-1}y_{-(2t+2)}}{y_{-2}x_{-(2t+3)} - 1} \frac{1}{bx_{-1}y_{-(2t+2)} - 1} x_{4n+1-(4t+2)}, \]

(3.80)

if \( t \) is odd,

\[ x_{4n+2} = \frac{x_{-2}y_{-(2t+3)}}{y_{4n-(2t-1)}} = \frac{x_{-2}y_{-(2t+3)}}{y_{-1}x_{-(2t+2)} - 1} \left( \beta y_{-1}x_{-(2t+2)} - 1 \right) x_{4n+2-(4t+2)}, \]

(3.81)

if \( t \) is even,

\[ x_{4n+3} = \frac{x_{-1}y_{-(2t+2)}}{y_{4n-(2t-2)}} = \frac{x_{-1}y_{-(2t+2)}}{y_{-2}x_{-(2t+3)} - 1} \left( \beta y_{-2}x_{-(2t+3)} - 1 \right) x_{4n+3-(4t+2)}, \]

(3.82)

if \( t \) is odd,

\[ y_{4n} = \frac{y_{-2}x_{-(2t+3)}}{\beta y_{-2}x_{-(2t+3)} - 1} \frac{1}{x_{4n-(2t+1)}} = \frac{y_{-2}x_{-(2t+3)}}{x_{-1}y_{-(2t+2)} - 1} \frac{1}{\beta y_{-2}x_{-(2t+3)} - 1} y_{4n-(4t+2)}, \]

(3.83)

if \( t \) is even,

\[ y_{4n+1} = \frac{y_{-1}x_{-(2t+2)}}{\beta y_{-1}x_{-(2t+2)} - 1} \frac{1}{x_{4n-2t}} = \frac{y_{-1}x_{-(2t+2)}}{x_{-2}y_{-(2t+3)} - 1} \frac{1}{\beta y_{-1}x_{-(2t+2)} - 1} y_{4n+1-(4t+2)}, \]

(3.84)

if \( t \) is odd,

\[ y_{4n+2} = \frac{y_{-2}x_{-(2t+3)}}{x_{4n-(2t-1)}} = \frac{y_{-2}x_{-(2t+3)}}{x_{-1}y_{-(2t+2)} \left( bx_{-1}y_{-(2t+2)} - 1 \right)} y_{4n+2-(4t+2)}, \]

(3.85)

if \( t \) is even,

\[ y_{4n+3} = \frac{y_{-1}x_{-(2t+2)}}{x_{4n-(2t-2)}} = \frac{y_{-1}x_{-(2t+2)}}{x_{-2}y_{-(2t+3)} \left( bx_{-2}y_{-(2t+3)} - 1 \right)} y_{4n+3-(4t+2)}. \]

(3.86)

if \( t \) is odd. From relations (3.79)–(3.86), the results in this theorem can be easily seen. \( \square \)
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