A new comprehensive subclass of analytic bi-close-to-convex functions

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Abstract: In a very recent work, Şeker and Sümer Eker [On subclasses of bi-close-to-convex functions related to the odd-starlike functions. Palestine Journal of Mathematics 2017; 6: 215-221] defined two subclasses of analytic bi-close-to-convex functions related to the odd-starlike functions in the open unit disk $U$. The main purpose of this paper is to generalize and improve the results of Şeker and Sümer Eker (in the aforementioned study) defining a comprehensive subclass of bi-close-to-convex functions. Also, we investigate the Fekete-Szegö type coefficient bounds for functions belonging to this new class.

Key words: Analytic and univalent functions, bi-univalent functions, close-to-convex functions, starlike functions, subordination principle, Fekete-Szegö problem

1. Introduction

Let $A$ denote the family of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $S := \{f \in A : f \text{ is univalent in } U\}$.

The family of starlike functions of order $\beta$ $(0 \leq \beta < 1)$ shall be denoted by $S^*(\beta)$ and is defined by the conditions that $f \in A$ and

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \beta \quad (z \in U).$$

It is well known that

$$S^*(\beta) \subset S^*(0) = S^* \subset S.$$

A function $f \in A$ is said to be close-to-convex if there exists a function $g \in S^*$ such that the inequality

$$\Re \left( \frac{zf'(z)}{g(z)} \right) > 0 \quad (z \in U)$$

holds. We will denote the class which consists of all functions $f \in A$ that are close-to-convex by $K$.

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We have well-known inclusion relations:

\[ S^* \subset \mathcal{K} \subset S. \]

Gao and Zhou [2] introduced the subclass \( \mathcal{K}_s \) of close-to-convex analytic functions as follows:

**Definition 1** [2] Let the function \( f \) be analytic in \( U \) and normalized by the condition (1.1). We say that \( f \in \mathcal{K}_s \) if there exists a function \( g \in S^* (1/2) \) such that

\[
\Re \left( \frac{z^2 f'(z)}{-g(z)g(-z)} \right) > 0 \quad (z \in U).
\]

In recent years, the subclasses of close-to-convex functions are studied by several authors (see, for example, [3, 5, 11, 13–15]). Motivated by this works, Goyal and Singh [4] defined the general subclass of close-to-convex functions by using the principle of subordination (see [8]) as follows:

**Definition 2** [4] For a function \( \varphi \) with positive real part, a function \( f \in \mathcal{A} \) is said to be in the class \( \mathcal{K}_s (\lambda, \mu, \varphi) \) if it satisfies the following subordination condition:

\[
\frac{z^2 f'(z) + (\lambda - \mu + 2\lambda \mu) z^3 f''(z) + \lambda \mu z^4 f'''(z)}{-g(z)g(-z)} \prec \varphi(z) \quad (z \in U),
\]

where \( 0 \leq \mu \leq \lambda \leq 1 \) and \( g \in S^* (1/2) \).

**Remark 1** (i) For \( \mu = 0 \) and \( \varphi(z) = \frac{1 + A z}{1 + B z} \) \((-1 < B < A < 1)\), we get the class \( \mathcal{K}_s (\lambda, A, B) \) studied by Wang and Chen [13].

(ii) For \( \mu = \lambda = 0 \) and \( \varphi(z) = \frac{1 + \beta z}{1 - \alpha z} \) \((0 \leq \alpha \leq 1, 0 < \beta \leq 1)\), we get the class \( \mathcal{K}_s (\alpha, \beta) \) studied by Wang et al. [15].

(iii) For \( \mu = \lambda = 0 \) and \( \varphi(z) = \frac{1 + (1 - 2\beta) z}{1 - z} \) \((0 \leq \beta < 1)\), we get the class \( \mathcal{K}_s (\beta) \) studied by Kowalczyk and Leś-Bomba [5].

(iv) For \( \mu = \lambda = 0 \) and \( \varphi(z) = \frac{1 + z}{1 - z} \), we get the class \( \mathcal{K}_s \) defined in the Definition 1.

**Theorem 1.1** [1] (Koebe One-Quarter Theorem) The range of every function of class \( \mathcal{S} \) contains the disk of radius \( \{ w : |w| < \frac{1}{4} \} \).

Thus, by Theorem 1.1, every function \( f \in \mathcal{A} \) has an inverse \( f^{-1} \) defined by

\[
f^{-1} (f(z)) = z \quad (z \in U) \quad \text{and} \quad f \left( f^{-1} (w) \right) = w \quad \left( |w| < r_0 (f) ; \; r_0 (f) \geq \frac{1}{4} \right).
\]

For the inverse function \( F = f^{-1} \), we have:

\[
F(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_3^2 - 5a_2 a_3 + a_4) w^4 + \cdots \quad (1.3)
\]

**Definition 3** [7] If both the function \( f \) and its inverse function \( f^{-1} \) are univalent in \( U \), then the function \( f \) is called bi-univalent. We will denote the class which consists of functions \( f \) that are bi-univalent by \( \Sigma \).
In a recent paper, Şeker and Sümer Eker [12] defined new subclasses of the bi-univalent function class \( \Sigma \) given in Definitions 4 and 5 as follows:

**Definition 4** (see [12]) A function \( f \in A \) given by (1.1) is said to be in the class \( K^+_\Sigma(\alpha) \) if there exists a function

\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*(1/2), \quad G(w) = w + \sum_{n=2}^{\infty} c_n w^n \in S^*(1/2)
\]

and the following conditions are satisfied:

\[
f \in \Sigma \quad \text{and} \quad \left| \arg \left( \frac{-z^2 f'(z)}{g(z) g(-z)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ z \in \mathbb{U})
\]

and

\[
\left| \arg \left( \frac{-w^2 F'(w)}{G(w) G(-w)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ w \in \mathbb{U}),
\]

where the function \( F = f^{-1} \) is defined by (1.3).

**Theorem 1.2** (see [12]) Let the function \( f(z) \) given by (1.1) be in the class \( K^+_\Sigma(\alpha) \) \((0 < \alpha \leq 1)\), then

\[
|a_2| \leq \sqrt{\frac{\alpha (1 + 2\alpha)}{2 + \alpha}} \quad \text{and} \quad |a_3| \leq \frac{\alpha (3\alpha + 2) + 1}{3}
\]

**Definition 5** (see [12]) A function \( f \in A \) given by (1.1) is said to be in the class \( K^+_\Sigma(\beta) \) if there exists a function

\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*(1/2), \quad G(w) = w + \sum_{n=2}^{\infty} c_n w^n \in S^*(1/2)
\]

and the following conditions are satisfied:

\[
f \in \Sigma \quad \text{and} \quad \Re \left( \frac{-z^2 f'(z)}{g(z) g(-z)} \right) > \beta \quad (0 \leq \beta < 1, \ z \in \mathbb{U})
\]

and

\[
\Re \left( \frac{-w^2 F'(w)}{G(w) G(-w)} \right) > \beta \quad (0 \leq \beta < 1, \ w \in \mathbb{U}),
\]

where the function \( F = f^{-1} \) is defined by (1.3).

**Theorem 1.3** (see [12]) Let the function \( f(z) \) given by (1.1) be in the class \( K^+_\Sigma(\beta) \) \((0 \leq \beta < 1)\), then

\[
|a_2| \leq \sqrt{\frac{3 - 2\beta}{3}} \quad \text{and} \quad |a_3| \leq \frac{1 - \beta (5 - 3\beta) + 1}{3}
\]

Now we introduce the a new comprehensive subclass of \( A \) which includes Definitions 4 and 5.
Definition 6 For \(0 \leq \mu \leq \lambda \leq 1\) and a function \(\varphi\) with positive real part, a function \(f \in \Sigma\) given by (1.1) is said to be in the class \(\mathcal{K}_{\Sigma_{\mu}}(\lambda, \mu, \varphi)\) if there exist the functions
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*(1/2), \quad G(w) = w + \sum_{n=2}^{\infty} c_n w^n \in S^*(1/2)
\]
and the following conditions are satisfied:
\[
z^2 f'(z) + (\lambda - \mu + 2\lambda \mu) z^3 f''(z) + \lambda \mu z^4 f'''(z) \prec \varphi(z) \quad (z \in \mathbb{U}) \quad (1.4)
\]
and
\[
w^2 F'(w) + (\lambda - \mu + 2\lambda \mu) w^3 F''(w) + \lambda \mu w^4 F'''(w) \prec \varphi(w) \quad (w \in \mathbb{U}), \quad (1.5)
\]
where the function \(F = f^{-1}\) is defined by (1.3).

Remark 2 (i) For \(\mu = 0\), we have a new class \(\mathcal{K}_{\Sigma_{\mu}}(\lambda, \varphi)\) of bi-close-to-convex functions satisfying the conditions
\[
z^2 f'(z) + \lambda z^3 f''(z) \prec \varphi(z) \quad (z \in \mathbb{U})
\]
and
\[
w^2 F'(w) + \lambda w^3 F''(w) \prec \varphi(w) \quad (w \in \mathbb{U}).
\]
(ii) For \(\mu = \lambda = 0\), we have a new class \(\mathcal{K}_{\Sigma_{\mu}}(\varphi)\) of bi-close-to-convex functions satisfying the conditions
\[
z^2 f'(z) \prec \varphi(z) \quad (z \in \mathbb{U})
\]
and
\[
w^2 F'(w) \prec \varphi(w) \quad (w \in \mathbb{U}).
\]
(iii) In addition to the conditions given in (ii), if we set
\[
\varphi(z) = \left(\frac{1 + z}{1 - z}\right)^\alpha \quad (0 < \alpha \leq 1)
\]
or
\[
\varphi(z) = \left(\frac{1 + (1 - 2\beta) z}{1 - z}\right)^\beta \quad (0 \leq \beta < 1),
\]
then we have the classes \(\mathcal{K}_{\Sigma_{\mu}}^\alpha\) and \(\mathcal{K}_{\Sigma_{\mu}}^\beta\) defined in Definitions 4 and 5, respectively.

In the light of the work of Şeker and Sümer Eker [12], we obtain initial coefficient estimates for functions \(f \in A\) given by (1.1) belonging to the bi-close-to-convex function class \(\mathcal{K}_{\Sigma_{\mu}}(\lambda, \mu, \varphi)\) introduced in Definition 6 above. We obtain the improvements of results of Şeker and Sümer Eker [12] given in Theorems 1.2 and 1.3 as a result of our main theorem (Theorem 2.1.1). Also, we find Fekete-Szegö type coefficient bounds for the bi-close-to-convex functions \(f \in \mathcal{K}_{\Sigma_{\mu}}(\lambda, \mu, \varphi)\). The following lemmas will be required for proving our main results.
Lemma 1.4 [2] If \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^* (1/2) \), then

\[
\psi(z) = \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in S^* \subset S,
\]

where the coefficients of the odd-starlike function \( \psi \) satisfy the condition

\[
|B_{2n-1}| = \left| 2b_{2n-1} - 2b_2 b_{2n-2} + \cdots + 2(-1)^n b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2 \right| \leq 1 \quad (n \geq 2).
\]

Lemma 1.5 [9] Let the function \( h \) given by

\[
h(z) = 1 + \sum_{k=1}^{\infty} h_k z^k \quad (z \in \mathbb{U})
\]

be holomorphic in \( \mathbb{U} \) and the function \( q \) given by

\[
q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k \quad (z \in \mathbb{U})
\]

be convex in \( \mathbb{U} \). If

\[
h(z) \prec q(z) \quad (z \in \mathbb{U}),
\]

then

\[
|h_k| \leq |q_1| \quad (k = 1, 2, \ldots).
\]

Lemma 1.6 [17] Let \( k, l \in \mathbb{R} \) and \( z_1, z_2 \in \mathbb{C} \). If \( |z_1| < R \) and \( |z_2| < R \), then

\[
|(k + l) z_1 + (k - l) z_2| \leq \begin{cases} 
2R |k|, & |k| \geq |l| \\
2R |l|, & |k| < |l|
\end{cases}
\]

2. Initial coefficient estimates

Throughout this paper, we assume that \( 0 \leq \mu \leq \lambda \leq 1 \) and \( \varphi \) be a function with positive real part.

Theorem 2.1 If the function \( f(z) \) given by (1.1) be in the function class \( K_{\Sigma_x}(\lambda, \mu, \varphi) \), then

\[
|a_2| \leq \min \left\{ \frac{1 + |\varphi'(0)|}{2(1 + \lambda - \mu + 2\lambda\mu)}, \sqrt[3]{\frac{1 + |\varphi'(0)|}{3[1 + 2(\lambda - \mu) + 6\lambda\mu]}} \right\}
\]

and

\[
|a_3| \leq \frac{1 + |\varphi'(0)|}{3[1 + 2(\lambda - \mu) + 6\lambda\mu]}.
\]
Proof  Firstly, we will re-arrange the relations in (1.4) and (1.5) as follows:

\[
\begin{align*}
    h(z) &= \frac{z^2 f'(z) + (\lambda - \mu + 2\lambda \mu) z^3 f''(z) + \lambda \mu z^4 f'''(z)}{-g(z)g(-z)} \\
    &= \frac{z f'(z) + (\lambda - \mu + 2\lambda \mu) z^2 f''(z) + \lambda \mu z^3 f'''(z)}{-g(z)g(-z)} \\
    &= \frac{z f'(z) + (\lambda - \mu + 2\lambda \mu) z^2 f''(z) + \lambda \mu z^3 f'''(z)}{\psi(z)} \\
    &< \varphi(z) \quad (z \in \mathbb{U}) \quad (2.3)
\end{align*}
\]

and

\[
\begin{align*}
    p(w) &= \frac{w^2 F'(w) + (\lambda - \mu + 2\lambda \mu) w^3 F''(w) + \lambda \mu w^4 F'''(w)}{-G(w)G(-w)} \\
    &= \frac{w F'(w) + (\lambda - \mu + 2\lambda \mu) w^2 F''(w) + \lambda \mu w^3 F'''(w)}{-G(w)G(-w)} \\
    &= \frac{w F'(w) + (\lambda - \mu + 2\lambda \mu) w^2 F''(w) + \lambda \mu w^3 F'''(w)}{\Omega(w)} \\
    &< \varphi(w) \quad (w \in \mathbb{U}), \quad (2.4)
\end{align*}
\]

respectively, where

\[
\begin{align*}
    \psi(z) := \frac{-g(z)g(-z)}{z} \quad \text{and} \quad \Omega(w) := \frac{-G(w)G(-w)}{w}.
\end{align*}
\]

Here \( h \) and \( p \) are two functions with positive real part defined by

\[
\begin{align*}
    h(z) := 1 + h_1 z + h_2 z^2 + \cdots
\end{align*}
\]

and

\[
\begin{align*}
    p(w) := 1 + p_1 w + p_2 w^2 + \cdots,
\end{align*}
\]

respectively. The relations (2.4) and (2.6) imply by Lemma 1.5 that for all \( k = 1, 2, \ldots \),

\[
|h_k| \leq |\varphi'(0)| \quad (2.7)
\]

and

\[
|p_k| \leq |\varphi'(0)|. \quad (2.8)
\]

Furthermore, by Lemma 1.4, we have following equations:

\[
\begin{align*}
    \psi(z) &= \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in S^* \quad \text{and} \quad |B_{2n-1}| \leq 1, \quad (2.9)
\end{align*}
\]

\[
\begin{align*}
    \Omega(w) &= \frac{-G(w)G(-w)}{w} = w + \sum_{n=2}^{\infty} C_{2n-1} w^{2n-1} \in S^* \quad \text{and} \quad |C_{2n-1}| \leq 1. \quad (2.10)
\end{align*}
\]
Now, upon equating the coefficients in (2.3) and (2.5), we obtain

\[
2 (1 + \lambda - \mu + 2\lambda \mu) a_2 = h_1 \tag{2.11}
\]
\[
3 [1 + 2 (\lambda - \mu) + 6\lambda \mu] a_3 - B_3 = h_2 \tag{2.12}
\]
\[
-2 (1 + \lambda - \mu + 2\lambda \mu) a_2 = p_1 \tag{2.13}
\]
\[
3 [1 + 2 (\lambda - \mu) + 6\lambda \mu] (2a_2^2 - a_3) - C_3 = p_2. \tag{2.14}
\]

From (2.11) and (2.13), we get

\[ h_1 = -p_1 \]

and

\[
8 (1 + \lambda - \mu + 2\lambda \mu)^2 a_2^2 = h_1^2 + p_1^2. \tag{2.15}
\]

We thus find (by (2.7) – (2.10)) that

\[
|a_2| \leq \frac{|\varphi'(0)|}{2 (1 + \lambda - \mu + 2\lambda \mu)}. \tag{2.16}
\]

Furthermore, from the equalities (2.12) and (2.14), we find

\[
6 [1 + 2 (\lambda - \mu) + 6\lambda \mu] a_2^2 - B_3 - C_3 = h_2 + p_2. \tag{2.17}
\]

Consequently (by (2.7) – (2.10)), we have

\[
|a_2| \leq \sqrt{\frac{1 + |\varphi'(0)|}{3 [1 + 2 (\lambda - \mu) + 6\lambda \mu]}}. \tag{2.18}
\]

Hence, we get the desired result on the coefficient \(a_2\) as asserted in (2.1) from the inequalities (2.16) and (2.18).

Now, in order to obtain the bound on the coefficient \(a_3\), we subtract (2.14) from (2.12). We thus get

\[
3 [1 + 2 (\lambda - \mu) + 6\lambda \mu] (2a_3 - 2a_2^2) - B_3 + C_3 = h_2 - p_2
\]

or

\[
a_3 = a_2^2 + \frac{h_2 - p_2 + B_3 - C_3}{6 [1 + 2 (\lambda - \mu) + 6\lambda \mu]}. \tag{2.19}
\]

Upon substituting the value of \(a_2^2\) from (2.15) into (2.19), it follows that

\[
a_3 = \frac{h_1^2 + p_1^2}{8 (1 + \lambda - \mu + 2\lambda \mu)^2} + \frac{h_2 - p_2 + B_3 - C_3}{6 [1 + 2 (\lambda - \mu) + 6\lambda \mu]}.
\]

We thus find (by (2.7) – (2.10)) that

\[
|a_3| \leq \frac{|\varphi'(0)|^2}{4 (1 + \lambda - \mu + 2\lambda \mu)^2} + \frac{1 + |\varphi'(0)|}{3 [1 + 2 (\lambda - \mu) + 6\lambda \mu]}. \tag{2.20}
\]
On the other hand, upon substituting the value of $a_2$ from (2.17) into (2.19), it follows that

$$a_3 = \frac{h_2 + p_2 + B_3 + C_3}{6[1 + 2(\lambda - \mu) + 6\lambda\mu]} + \frac{h_2 - p_2 + B_3 - C_3}{6[1 + 2(\lambda - \mu) + 6\lambda\mu]} = \frac{h_2 + B_3}{3[1 + 2(\lambda - \mu) + 6\lambda\mu]}.$$

Consequently (by (2.7), (2.8), (2.9) and (2.10)), we have

$$|a_3| \leq \frac{1 + |\varphi'(0)|}{3[1 + 2(\lambda - \mu) + 6\lambda\mu]}.$$  

(2.21)

Combining (2.20) and (2.21), we get the desired result on the coefficient $a_3$ as asserted in (2.2).

Letting $\mu = 0$ in Theorem 2.1, we have the following consequence.

**Corollary 2.2** If the function $f(z)$ given by (1.1) be in the function class $K_{\Sigma}(\lambda, \varphi)$, then

$$|a_2| \leq \min \left\{ \frac{|\varphi'(0)|}{2(1 + \lambda)}, \sqrt{\frac{1 + |\varphi'(0)|}{3(1 + 2\lambda)}} \right\}$$

and

$$|a_3| \leq \frac{1 + |\varphi'(0)|}{3(1 + 2\lambda)}.$$

Letting $\lambda = 0$ in Corollary 2.2, we have the following consequence.

**Corollary 2.3** If the function $f(z)$ given by (1.1) be in the function class $K_{\Sigma}(\varphi)$, then

$$|a_2| \leq \min \left\{ \frac{|\varphi'(0)|}{2}, \sqrt{\frac{1 + |\varphi'(0)|}{3}} \right\}$$

and

$$|a_3| \leq \frac{1 + |\varphi'(0)|}{3}.$$

Setting

$$\varphi(z) = \left(\frac{1 + z}{1 - z}\right)^{\alpha} \quad (0 < \alpha \leq 1)$$

in Corollary 2.3, we have the following result.

**Corollary 2.4** If the function $f(z)$ given by (1.1) be in the function class $K_{\Sigma}^{\varphi}(\alpha)$, then

$$|a_2| \leq \alpha \quad \text{and} \quad |a_3| \leq \frac{1 + 2\alpha}{3}.$$

**Remark 3** Note that Corollary 2.4 is an improvement of the Theorem 1.2.

Setting

$$\varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1)$$

in Corollary 2.3, we have the following result.
Corollary 2.5 If the function $f(z)$ given by (1.1) be in the function class $K^s_2(\beta)$, then

$$|a_2| \leq 1 - \beta \quad \text{and} \quad |a_3| \leq \frac{3 - 2\beta}{3}.$$  

Remark 4 Note that Corollary 2.5 is an improvement of the Theorem 1.3.

3. Fekete-Szegő problem

Theorem 3.1 If the function $f(z)$ given by (1.1) be in the function class $K^s_2(\lambda, \mu, \varphi)$, then, for $\delta \in \mathbb{R}$,

$$|a_3 - \delta a_2^2| \leq \frac{1 + |\varphi'(0)|}{3[1 + 2(\lambda - \mu) + 6\lambda\mu]} \left\{ \begin{array}{ll} |1 - \delta| & , \delta \in (-\infty, 0] \cup [2, \infty) \\
1 & , \delta \in [0, 2] \end{array} \right.$$  

Proof By using the equality (2.19) in the proof of Theorem 2.1, we obtain

$$a_3 - \delta a_2^2 = a_2^2 + \frac{h_2 - p_2 + B_3 - C_3}{6[1 + 2(\lambda - \mu) + 6\lambda\mu]} - \delta a_2^2$$

$$= (1 - \delta) a_2^2 + \frac{h_2 - p_2 + B_3 - C_3}{6[1 + 2(\lambda - \mu) + 6\lambda\mu]}.$$

Upon substituting the value of $a_2^2$ from (2.17) into the above equality, it follows that

$$a_3 - \delta a_2^2 = (1 - \delta) \frac{h_2 + p_2 + B_3 + C_3}{6[1 + 2(\lambda - \mu) + 6\lambda\mu]} + \frac{h_2 - p_2 + B_3 - C_3}{6[1 + 2(\lambda - \mu) + 6\lambda\mu]}$$

$$= \frac{1}{6[1 + 2(\lambda - \mu) + 6\lambda\mu]} [(2 - \delta) (h_2 + B_3) - \delta (p_2 + C_3)].$$

Thus, by Lemma 1.6, we get desired estimate. \qed

Letting $\mu = 0$ in Theorem 3.1, we have the following consequence.

Corollary 3.2 If the function $f(z)$ given by (1.1) be in the function class $K^s_2(\lambda, \varphi)$, then, for $\delta \in \mathbb{R}$,

$$|a_3 - \delta a_2^2| \leq \frac{1 + |\varphi'(0)|}{3(1 + 2\lambda)} \left\{ \begin{array}{ll} |1 - \delta| & , \delta \in (-\infty, 0] \cup [2, \infty) \\
1 & , \delta \in [0, 2] \end{array} \right.$$  

Letting $\lambda = 0$ in Corollary 3.2, we have the following consequence.

Corollary 3.3 If the function $f(z)$ given by (1.1) be in the function class $K^s_2(\varphi)$, then, for $\delta \in \mathbb{R}$,

$$|a_3 - \delta a_2^2| \leq \frac{1 + |\varphi'(0)|}{3} \left\{ \begin{array}{ll} |1 - \delta| & , \delta \in (-\infty, 0] \cup [2, \infty) \\
1 & , \delta \in [0, 2] \end{array} \right.$$  

Setting

$$\varphi(z) = \left(\frac{1 + z}{1 - z}\right)\alpha \quad (0 < \alpha \leq 1)$$

in Corollary 3.3, we have the following result.

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Corollary 3.4 If the function $f(z)$ given by (1.1) be in the function class $K^3_s(\alpha)$, then, for $\delta \in \mathbb{R}$,

$$|a_3 - \delta a_2^2| \leq \frac{1 + 2\alpha}{3} \begin{cases} |1 - \delta|, & \delta \in (-\infty, 0] \cup [2, \infty) \\ 1, & \delta \in [0, 2] \end{cases}.$$ 

Setting

$$\varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1)$$

in Corollary 3.3, we have the following result.

Corollary 3.5 If the function $f(z)$ given by (1.1) be in the function class $K^3_s(\beta)$, then, for $\delta \in \mathbb{R}$,

$$|a_3 - \delta a_2^2| \leq \frac{3 - 2\beta}{3} \begin{cases} |1 - \delta|, & \delta \in (-\infty, 0] \cup [2, \infty) \\ 1, & \delta \in [0, 2] \end{cases}.$$ 

References


