An inequality on diagonal $F$-thresholds over standard-graded complete intersection rings

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Abstract: In a recent paper, De Stefani and Núñez-Betancourt proved that for a standard-graded $F$-pure $k$-algebra $R$, its diagonal $F$-threshold $c(R)$ is always at least $-a(R)$, where $a(R)$ is the $a$-invariant. In this paper, we establish a refinement of this result in the setting of complete intersection rings.

Key words: Frobenius power, socle, $F$-threshold, $F$-pure threshold, $a$-invariant

1. Introduction and notations
Let $R$ be a commutative Noetherian ring in prime characteristic $p > 0$. Let $q$ be a power of $p$. For an ideal $I$ in $R$, let $I^{[q]}$ be the $q$th bracket power of $I$, that is, $I^{[q]} := \{a^q : a \in I\}$. For a pair of ideals $a$ and $J$ in $R$ such that $a \subseteq \sqrt{J}$, define $\nu_2^J(q) = \max\{r \in \mathbb{N} | a^r \not\subseteq J^{[q]}\}$. Recently, De Stefani, Núñez-Betancourt, and Pérez [5] proved that the limit of $\{\nu_2^J(q)/q\}$ as $q \to \infty$ always exists. Such a limit, denoted by $c'(a)$, is called the $F$-threshold of the pair $(a, J)$. We mention [1–8, 12] for details of some work done regarding this invariant. In the case of a local ring $(R, m)$ with an $m$-primary ideal $J$, the $F$-threshold $c'(m)$ is also called the diagonal $F$-threshold (for simplicity, the diagonal $F$-threshold $c^m(m)$ of $R$ is denoted by $c(R)$). For an ideal $a$ in a regular local ring $(R, m)$, the $F$-threshold $c^m(a)$ coincides with the $F$-pure threshold of $a$ (see [12, Remark 1.5]), that is,

$$c^m(a) = \sup\{s \in R_{\geq 0} | \text{the pair } (R, a^s) \text{ is } F\text{-pure}\}$$

In particular, for an element $f \in m$ in a regular local ring $(R, m)$, the $F$-threshold $c^m((f))$ coincides with the $F$-pure threshold of $f$. We use $\text{fpt}(f)$ to denote this invariant. The theories on $F$-threshold and $F$-pure threshold are motivated by the relations between $\text{fpt}(f)$ and $\text{lct}(f)$, where $\text{lct}(f)$ is the log canonical threshold, a notion developed in birational geometry. More precisely, if $R$ is a local ring in characteristic 0 and $R_p = R \otimes \mathbb{Z}/p\mathbb{Z}$, then

$$\text{fpt}(f_p) \leq \text{lct}(f), \text{ for all } p,$$

and

$$\lim_{p \to \infty} \text{fpt}(f_p) = \text{lct}(f).$$
Suppose $R$ is also a standard-graded algebra over a field $k$ in prime characteristic $p$, then the diagonal $F$-threshold of an $m$-primary ideal $J$ is easily seen to be

$$c^J(m) = \lim_{q \to \infty} \frac{t.s.d(R/J^{[q]})}{q}$$

where $t.s.d(R/J^{[q]})$ is the top socle degree of the Artinian algebra $R/J^{[q]}$.

Let $a(R)$ be the $a$-invariant of $R$. It was proved in [11] that when $R$ is a complete intersection ring or a Gorenstein $F$-pure ring, the inequality

$$c^J(m) \geq t.s.d(R/J) - a(R)$$

holds. In particular, by taking $J = m$, we have

$$c(R) \geq -a(R).$$

Hirose, Watanabe, and Yoshida conjectured that this latter inequality holds for any $F$-pure ring [9], and this was recently settled in [4, Theorem 4.9].

2. Main result

The following is the main result which improves the inequality $c(R) \geq -a(R)$ for the case of standard-graded complete intersection rings.

**Theorem 2.1** Let $a = (f_1, \cdots, f_t)$ be a homogeneous ideal of the standard-graded polynomial ring $S = k[x_1, \cdots, x_n]$, where $f_1, \cdots, f_t$ form a homogeneous $S$-sequence. Let $|f_i|$ be the degree of $f_i$, $i = 1, \cdots, t$. Let $m = (x_1, \cdots, x_n)$ be the maximal ideal of $S$. Let $R$ be the complete intersection ring $k[x_1, \cdots, x_n]/(f_1, \cdots, f_t)$. Let $fpt(f)$ denote the $F$-pure threshold for a polynomial $f \in S$. Then the following inequality holds

$$c(R) \geq -a(R) + \sum_{i} |f_i| \left(1 - fpt(f_i)\right)$$

In particular, if $c(R) = -a(R)$, then $fpt(f_i) = 1$ for all $i = 1, \cdots, t$.

Before we prove this theorem, we recall the following observation

**Lemma 2.2** (see [10], Observation 1.4) Let $R$ be an Artinian Gorenstein graded $k$-algebra with socle degree $\delta$, and $J$ a homogeneous ideal of $R$. If the socle degree of $R/J$ are $d_1$, then the degrees of the minimal generators of $(0 : J)$ are $\delta - d_1$.

**Proof** [Proof of Theorem 2.1] Let $I$ be an $m$-primary reducible ideal of $R$. Let $J$ be the pre-image of $I$ in $S$, so that $J$ contains $a$. Applying Lemma 2.2 to the Gorenstein algebra $S/J^{[q]}$, and use socdeg to denote the (unique) socle degree of a Gorenstein Artinian algebra, one has that $socdeg(S/J^{[q]}) = t.s.d(S/(J^{[q]} + a)) + M$ where $M$ is the smallest degree of the minimal generators of $(J^{[q]} : a)/J^{[q]}$. 

1373
Since \( J \) has finite projective dimension, \( \text{socdeg}(S/J^{\alpha}) - a(S) = q(\text{socdeg}(S/J) - a(S)) \). Therefore,

\[
\text{t. s. } d(R/I^{\alpha}) = \text{t. s. } d(S/(J^{\alpha} + \mathfrak{a})) = \text{socdeg}(S/J^{\alpha}) - M = q(\text{socdeg}(S/J)) - (q - 1)a(S) - M
\]

We need to estimate an upper bound for \( M \). Let \( h \) denote \( \nu_R^J(q) := \max \{ r \in \mathbb{N} : a^r \not\subseteq J^{\alpha} \} \). Since \( a^{h+1} \subseteq J^{\alpha} \), \( a^h \) contains an element of the form \( f_1^{\alpha_1} \cdots f_t^{\alpha_t} \) with \( \alpha_1 + \cdots + \alpha_t = h \), whose image in \( (J^{\alpha})/J^{\alpha} \) is nonzero. Hence,

\[
M \leq \sum_{i=1}^t |f_i| \alpha_i
\]

On the other hand, let \( \beta_i(q) \) denote \( \nu_{f_i}^J(q) := \max \{ t \in \mathbb{N} : f_i^t \not\subseteq J^{\alpha} \} \). It is obvious that \( \alpha_i \leq \beta_i(q) \), \( \forall i \).

It follows that

\[
M \leq \sum_{i=1}^t |f_i| \beta_i(q)
\]

On the other hand, the \( a \)-invariant \( a(R) = \sum_{i=1}^t |f_i| - n \) since \( R \) is a complete intersection ring, so

\[
\text{t. s. } d(R/I^{\alpha}) = q(\text{socdeg}(S/J)) - (q - 1)(-n) - M \geq q(\text{socdeg}(S/J)) - (q - 1)(-n) - \sum_{i=1}^t |f_i| \beta_i(q)
\]

\[
= q(\text{socdeg}(S/J)) - (q - 1)(-n) - q \sum_{i=1}^t |f_i| \beta_i(q)/q
\]

\[
= q(\text{socdeg}(S/J)) - n + qn - q(\sum_{i=1}^t |f_i|) + q(\sum_{i=1}^t |f_i|) - q \sum_{i=1}^t |f_i| \beta_i(q)/q
\]

\[
= q(\text{socdeg}(S/J)) - n + qn - q(\sum_{i=1}^t |f_i|) + q \sum_{i=1}^t |f_i| (1 - \beta_i(q)/q)
\]

\[
= q(\text{socdeg}(S/J)) - n + q(-a(R)) + q \sum_{i=1}^t |f_i| (1 - \beta_i(q)/q)
\]

Dividing both sides by \( q \) and taking the limit, we have

\[
c^m(I) \geq \text{socdeg}(R/I)) - a(R) + \sum_{i=1}^t |f_i| \left(1 - c^t(f_i)\right)
\]

The desired inequality is then obtained by taking \( I \) to be the maximal ideal of \( R \), and \( J = \mathfrak{m} \).  

\[\square\]
Remark 2.3 It is easy to see from the proof above that the condition \( c(R) = -a(R) \) also forces \( c^m(a) = t = \sum_{i=1}^t \text{fpt}(f_i) \). However, we do not know what else can be derived from this.

Let \( k \) be a field in characteristic 0. Let \( R = k[x_1, \cdots, x_n]/(f_1, \cdots, f_t) \) where \( f_1, \cdots, f_t \) form a homogeneous regular sequence in \( k[x_1, \cdots, x_n] \). For a prime number \( p \), let \( R_p = R \otimes \mathbb{Z} (\mathbb{Z}/p\mathbb{Z}) \).

Corollary 2.4 If \( \lim_{p \to \infty} c(R_p) = -a(R) \), then the log canonical thresholds \( \text{lct}(f_i) = 1 \) for all \( i = 1, \cdots, t \).

Proof This follows from Theorem 2.1 and (2) immediately.

3. An example on diagonal hypersurface rings

We use the following computation of Vraciu [13] to study an example of the diagonal hypersurface case.

Theorem 3.1 [13, Theorem 4.2] Let \( R = k[x_1, \cdots, x_{n+1}]/(x_1^a + \cdots + x_{n+1}^a) \) where \( k \) is a field of characteristic \( p \) and \( a \) is a positive integer not divisible by \( p \). Then

\[
c(R) = n + 1 - aM
\]

where \( M \) is equal to

\[
\min \left\{ \left[ \frac{(n+1)\kappa - n + 1}{2} \right] \cdot \frac{1}{p^{e_0}} + \frac{(n+1)s}{ap^{e_0}}, \left[ \frac{(n+1)\kappa - n + 2}{2} \right] \cdot \frac{1}{p^{e_0}} + \frac{ns}{ap^{e_0}}, \right. \\
\left. \left[ \frac{(n+1)\kappa + 1}{2} \right] \cdot \frac{1}{p^{e_0}} + \frac{s}{ap^{e_0}}, \left[ \frac{(n+1)\kappa + 2}{2} \right] \cdot \frac{1}{p^{e_0}}, \frac{1}{p^{e_0-1}} \right\}
\]

where \( e_0 \) is the smallest exponent such that \( p^{e_0} \geq a \), \( \kappa = \left\lfloor \frac{p^{e_0}}{a} \right\rfloor \), and \( s = p^{e_0} - \kappa a \) is the remainder of \( p^{e_0} \) modulo \( a \).

The following example then follows from the above theorem immediately by taking \( p \to \infty \).

Example 3.2 Let \( R = k[x_1, \cdots, x_{n+1}]/(x_1^a + \cdots + x_{n+1}^a) \), where \( k \) is a field in characteristic 0. Let \( R_p = R \otimes \mathbb{Z} (\mathbb{Z}/p\mathbb{Z}) \). Then the limit diagonal \( F \)-threshold

\[
\lim_{p \to \infty} c(R_p) = \begin{cases} 
\left\lfloor \frac{n+1}{2} \right\rfloor, & \text{if } \frac{n+1}{2} \leq a \\
\frac{n+1}{2} - a, & \text{if } 2a \leq n
\end{cases}
\]

Notice that the \( a \)-invariant \( a(R) = -(n+1-a) \), which is characteristic-free, we obtain by Corollary 2.4 that

\[
\text{lct}(x_1^a + \cdots + x_{n+1}^a) = 1,
\]

provided \( n \geq 2a \).
References


[13] Vraciu A. On the degrees of relations on $x_1^{d_1}, \ldots, x_n^{d_n}, (x_1+\cdots+x_n)^{d_{n+1}}$ in positive characteristic. Journal of Algebra 2015; 423 (1): 916-949. doi: 10.1016/j.jalgebra.2014.11.004