$n$-$T$-torsionfree modules

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Abstract: As a generalization of the Auslander-Reiten transpose, Xi introduced and studied a more general transpose, called the relative transpose (or, $T$-transpose). Based on this notion, the notion of relative $n$-torsionfree modules (or, $n$-$T$-torsionfree modules) is introduced in this paper, which is a generalization of the $n$-torsionfree modules introduced by Auslander and Bridger. We show that relative $n$-torsionfree modules have many similar properties of $n$-torsionfree modules.

Key words: $n$-torsionfree modules, $n$-$T$-torsionfree module, $n$-spherical, $T$-grade

1. Introduction and preliminaries

It is well known that the Auslander–Reiten theory is very important for representation theory of Artin algebra and homological algebra. The transpose plays an important role in this theory. The transpose was studied by many authors. For example, let $C$ be a semidualizing $R$-bimodule, a transpose $\text{Tr}_C M$ of an $R$-module $M$ with respect to $C$ was introduced in [6]. Later, Geng [5] used $\text{Tr}_C M$ to further develop the generalized Gorenstein dimension with respect to $C$ in the two-sided Noetherian setting. Especially, he generalized the Auslander–Bridger formula to the generalized Gorenstein dimension case. The dual of Auslander transpose was studied in [7], and the relative transpose of an $R$-module was considered in [8].

Auslander and Bridger introduced $n$-torsionfree modules and obtained an approximation theory for finitely generated modules when $n$-syzygy modules and $n$-torsionfree modules coincide in [1]. Tang and Huang [7] introduced and studied the cotranspose of modules with respect to a semidualizing module $C$, and using it, they introduced $n$-$C$-cotorsionfree modules and showed that $n$-$C$-cotorsionfree modules have many dual properties of $n$-torsionfree modules.

Based on [8], we introduce the notion of $n$-$T$-torsionfree modules. It turns out that many important results on the $n$-$C$-torsion module are still true in this paper. We mainly prove the following two conclusions:

**Theorem 1.1** Let $T$ be self-orthogonal (i.e. $\text{Ext}_A^{ij+1}(T,T) = 0$). Assume that $M$ has an $\text{add}(T)$-resolution and $n \geq 1$. Then the following statements are equivalent:

1. $\Omega_T^n(M)$ is $n$-$T$-torsionfree
2. There exists an exact sequence $0 \rightarrow L \rightarrow N \rightarrow M \rightarrow 0$ with $N$ $n$-$T$-spherical and $\text{add}(T)$-$pd (L) \leq n - 1$.

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Theorem 1.2 Assume that $M$ has an $\text{add}(T)$-resolution and $n \geq 1$. Then $\Omega^i_T(M)$ is $i$-$T$-torsionfree for all $1 \leq i \leq n$ if and only if $T$-grade $\text{Ext}^i(M, T) \geq i - 1$ for all $1 \leq i \leq n$.

We note that the above theorems extend two interesting theorems proved by Auslander–Bridger [1].

Let $A$ be an Artin $R$-algebra, that is, $R$ is a commutative Artin ring and $A$ is an $R$-algebra which is finitely generated as an $R$-module. The category of finitely generated left $A$-modules will be denoted by $A$-mod. Throughout this paper, we assume that all modules are always finitely generated.

This paper is organized as follows. In Section 2, we introduce the definition of $n$-$T$-torsionfree modules as a generalization of $n$-torsionfree modules and give some characterizations of these modules (Theorem 2.7). In particular, the proof of Theorem 1.1 (i.e. Theorem 2.9 in this section) is presented. In Section 3, we give the definition of $T$-grade and prove Theorem 1.2 (i.e. Theorem 3.3 in this section).

Let $\mathcal{X}$ be a subcategory of $A$-mod and $M$ be a left $A$-module. A homomorphism $f: X \to M$ with $X \in \mathcal{X}$ is called a right $\mathcal{X}$-approximation (or, $\mathcal{X}$-precover) of $M$ if the induced morphism $\text{Hom}(X', f)$ is surjective for all $X' \in \mathcal{X}$. Dually, a homomorphism $f: M \to X$ with $X \in \mathcal{X}$ is called a left $\mathcal{X}$-approximation (or, $\mathcal{X}$-preenvelope) of $M$ if the induced morphism $\text{Hom}(f, X')$ is surjective for all $X' \in \mathcal{X}$ in $[2, 3]$. An $\mathcal{X}$-resolution of $M$ is an exact sequence

$$\cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with $X_i \in \mathcal{X}$ for all $i \geq 0$. In addition, if the exact sequence is $\text{Hom}(\mathcal{X}, -)$-exact, then the exact sequence is called a proper $\mathcal{X}$-resolution of $M$. Dually, we can define $\mathcal{X}$-coresolution and proper $\mathcal{X}$-coresolution. We say that $M$ has $\mathcal{X}$-projective dimension $M \leq m$, denoted by $\mathcal{X}$-$\text{pd} (M) \leq m$, if there is an $\mathcal{X}$-resolution of $M$ of the form $0 \to X_m \to \cdots \to X_1 \to X_0 \to M \to 0$. Let $T$ be a module in $A$-mod. We denote by $B$ the endomorphism algebra of $T$, thus $T$ is a $A$-$B$ bimodule in the natural manner. Throughout this paper, we shall fix such a triple $(A, T, B)$. Denoted by $\text{Pre}(A_T)$ the class whose objects are those left $A$-modules $M$ which possesses an exact sequence of the form $T_1 \to T_0 \to M \to 0$ with $T_0, T_1 \in \text{add}(T)$, here $\text{add}(T)$ stands for the additive category generated by $T$. For simplicity, we will denote the functor $\text{Hom}(\neg, T)$ by $(\neg)^*$.

2. $n$-$T$-torsionfree modules

In this section, we introduce the definition of $n$-$T$-torsionfree module and give a characterization of $n$-$T$-torsionfree modules (Theorem 2.7) and the relationship between $n$-$T$-torsionfree modules and $n$-$T$-spherical modules (Theorem 2.9). Firstly, we recall the definition of transpose [4] and relative transpose [8].

Let $P_1 \to P_0 \to M \to 0$ be a minimal projective presentation of $M$. Applying the functor $\text{Hom}_A(\neg, A)$, we obtain an exact sequence of right $A$-modules

$$0 \longrightarrow \text{Hom}_A(M, A) \longrightarrow \text{Hom}_A(P_0, A) \xrightarrow{f} \text{Hom}_A(P_1, A) \longrightarrow C \longrightarrow 0$$

We denote the Cokernel of $f$ by $\text{Tr} M$ and call it the transpose of $M$, i.e. $C = \text{Tr} M$.

Let $M$ be a left $A$-module in $\text{Pre}(A_T)$. Then we have an exact sequence

$$T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0$$
Applying $\text{Hom}_A(-, T)$ to exact sequence above, we have an exact sequence in mod-$B$:

$$
0 \longrightarrow M^* \longrightarrow T_0^* \longrightarrow T_1^* \longrightarrow \Sigma_T(M) \longrightarrow 0,
$$

where $\Sigma_T(M)$ stands for the cokernel of $\text{Hom}(f_1, T)$. We call the $\Sigma_T(M)$ the transpose of $M$ with respect to $T$, or $T$-transpose of $M$. Note the $T$-transpose is a right $B$-module and depends on the exact sequence above.

**Theorem 2.1** [8, Theorem 3.9] If $M$ lies in $\text{Pre}_A(T)$, then we have an exact sequence

$$
0 \longrightarrow \text{Ext}_B^1(\Sigma_T(M), T) \longrightarrow M \overset{\alpha_M}{\longrightarrow} M^{**} \longrightarrow \text{Ext}_B^2(\Sigma_T(M), T) \longrightarrow 0,
$$

where $\alpha_M$ is the natural homomorphism, given by $m \mapsto (f \mapsto f(m))$.

We introduce the following definition of $n$-$T$-torsionfree modules.

**Definition 2.2** Let $M$ be a finitely generated left $A$-module in $\text{Pre}_A(T)$. Then $M$ is called $n$-$T$-torsionfree if $\text{Ext}_B^i(\Sigma_T(M), T) = 0$ for all $1 \leq i \leq n$. If $\text{Ext}_B^i(\Sigma_T(M), T) = 0$ for all $i \geq 1$, then $M$ is $\infty$-$T$-torsionfree.

**Remark 2.3**

1. If $T = A$, then $n$-$T$-torsionfree module coincide with $n$-torsionfree;
2. If $M$ is in $\text{add}_A(T)$, then $M$ is $\infty$-$T$-torsionfree. This is very useful for the rest of the discussion;
3. If $M$ is $n$-$T$-torsionfree, then $M$ is $m$-$T$-torsionfree for any $m \leq n$.

The following lemma will be used frequently in this paper.

**Lemma 2.4** Let $M$ be a finitely generated left $A$-module in $\text{Pre}_A(T)$. Then $M$ is $n$-$T$-torsionfree if and only if $\alpha_M$ is an isomorphism and $\text{Ext}_B^1(M^*, T) = 0$ for all $1 \leq i \leq n - 2$.

**Proof** ($\Rightarrow$) Assume that $M$ is $n$-$T$-torsionfree, then $\alpha_M$ is an isomorphism by Theorem 2.1 and Definition 2.2. By dimension shifting for the exact sequence (2), we can obtain that $\text{Ext}_B^i(\Sigma_T(M), T) \cong \text{Ext}_B^{i-2}(M^*, T)$ for all $i \geq 3$. And $\text{Ext}_B^i(\Sigma_T(M), T) = 0$ for all $1 \leq i \leq n$ since $M$ is $n$-$T$-torsionfree, thus, $\text{Ext}_B^1(M^*, T) = 0$ for all $1 \leq i \leq n - 2$.

($\Leftarrow$) By the assumption, we have $\text{Ext}_B^i(\Sigma_T(M), T) \cong \text{Ext}_B^{i-2}(M^*, T) = 0$ for all $3 \leq i \leq n$, but $\text{Ext}_B^{1,2}(\Sigma_T(M), T) = 0$ by Theorem 2.1. Thus, the proof is completed.

**Proposition 2.5** Let the exact sequence $0 \to X \to Y \to Z \to 0$ be $\text{Hom}(-, T)$-exact and $Z$ be $n$-$T$-torsionfree. Then $X$ is $n$-$T$-torsionfree if and only if $Y$ is $n$-$T$-torsionfree.

**Proof** Applying $\text{Hom}(-, T)$ to the exact sequence $0 \to X \to Y \to Z \to 0$, we can obtain a new exact sequence $0 \to Z^* \to Y^* \to X^* \to 0$ since this exact sequence is $\text{Hom}(-, T)$-exact. In a similar way, we can obtain a new exact sequence $0 \to X^{**} \to Y^{**} \to Z^{**}$. Consider the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X^{**} & \longrightarrow & Y^{**} & \overset{f}{\longrightarrow} & Z^{**} & \longrightarrow & \text{Ext}_B^1(X^*, T)
\end{array}
$$
Since $Z$ is $n$-$T$-torsionfree, we have that $\alpha_Z$ is an isomorphism by Lemma 2.4. Thus $\alpha_X$ is an isomorphism if and only if $\alpha_Y$ is an isomorphism by Snake lemma. Now it is enough to prove that $\operatorname{Ext}^1_B(X^*, T) = 0$ if and only if $\operatorname{Ext}^1_B(Y^*, T) = 0$ for all $1 \leq i \leq n - 2$ by Lemma 2.4.

$(\Rightarrow)$ It follows from the long exact sequence theorem and Lemma 2.4.

$(\Leftarrow)$ We only prove that $\operatorname{Ext}^1_B(X^*, T) = 0$ by the long exact sequence theorem and Lemma 2.4. Consider the exact sequence

$$
0 \rightarrow X^{**} \rightarrow Y^{**} \rightarrow Z^{**} \rightarrow \operatorname{Ext}^1_B(X^*, T) \rightarrow \operatorname{Ext}^1_B(Y^*, T)
$$

From the commutative diagram above, it is easy to verify that $f$ is surjective, but $\operatorname{Ext}^1_B(Y^*, T) = 0$ since $Y$ is $n$-$T$-torsionfree. Thus, $\operatorname{Ext}^1_B(X^*, T) = 0$.

**Lemma 2.6** Let $M$ be in $\operatorname{Pre}(AT)$, then the following conclusions hold:

1. $M$ is $1$-$T$-torsionfree if and only if $M$ admits an injective $\operatorname{add}(AT)$-preenvelope.
2. $M$ is $2$-$T$-torsionfree if and only if there is an exact sequence $0 \rightarrow M \rightarrow T_0 \rightarrow T_1$, where $T_0$ and $T_1$ are in $\operatorname{add}(AT)$ and this exact sequence is $\operatorname{Hom}_A(-, AT)$-exact.

**Proof** (1), $(\Rightarrow)$ Assume that $M$ is 1-$T$-torsionfree, so $\alpha_M$ is an injection by Theorem 2.1. Note that there is an exact sequence $B^{(X)} \rightarrow M^* \rightarrow 0$ for some set $X$. By functor $\operatorname{Hom}_B(-, T_B)$, we obtain a new exact sequence $0 \rightarrow M^{**} \rightarrow \operatorname{Hom}_B(B^{(X)}, T_B) \cong T^X$. Note that $X$ is a finite set. Thus, we obtain a monomorphism $f\alpha_M: M \rightarrow T^X \cong T^{(X)}$ since $f$ and $\alpha_M$ are injective. Hence, we obtain an monomorphic $\operatorname{add}(AT)$-preenvelope: $M \rightarrow T^{(Y)}$ with $Y = \operatorname{Hom}_A(M, T)$ finite set and $v$ evaluation map.

$(\Leftarrow)$ Assume that $M$ admits an injective $\operatorname{add}(AT)$-preenvelope, then we have an exact sequence $0 \rightarrow M \rightarrow T_0$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & M \\
& \downarrow{\alpha_M} & \downarrow{\alpha_{T_0}} \\
0 & \rightarrow & M^{**} \\
& \downarrow{\alpha_{T_0}} & \alpha_{T_0} \\
& \downarrow{\alpha_{T_0}} & \alpha_{T_0} \\
& \downarrow{\alpha_{T_0}} & \alpha_{T_0} \\
0 & \rightarrow & T_0
\end{array}
$$

It follows from Snake lemma and $\alpha_{T_0}$ is an isomorphism that $\alpha_M$ is a monomorphism. i.e. $M$ is 1-$T$-torsionfree.

(2), $(\Rightarrow)$ Suppose that $M$ is 2-$T$-torsionfree, then we can obtain an exact sequence $0 \rightarrow M \rightarrow T_0 \rightarrow C \rightarrow 0$ with $\operatorname{Hom}_A(-, T)$-exact by (1). Now we only prove that $C$ is 1-$T$-torsionfree by (1) again. We consider the following commutative diagram with exact rows:

$$
\begin{array}{ccc}
0 & \rightarrow & M \\
& \downarrow{\alpha_M} & \downarrow{\alpha_{T_0}} \\
0 & \rightarrow & M^{**} \\
& \downarrow{\alpha_{T_0}} & \alpha_{T_0} \\
0 & \rightarrow & T_0^{**} \\
& \downarrow{\alpha_{T_0}} & \alpha_{T_0} \\
& \downarrow{\alpha_{T_0}} & \alpha_{T_0} \\
0 & \rightarrow & C^{**} \\
& \downarrow{\alpha_{T_0}} & \alpha_{T_0} \\
0 & \rightarrow & T_0^{**} \\
& \downarrow{\alpha_{T_0}} & \alpha_{T_0} \\
& \downarrow{\alpha_{T_0}} & \alpha_{T_0} \\
0 & \rightarrow & C
\end{array}
$$

Since $M$ is 2-$T$-torsionfree, $\alpha_M$ is an isomorphism by Lemma 2.4. It follows from Snake lemma that $\alpha_C$ is monomorphic. i.e. $C$ is 1-$T$-torsionfree.
Theorem 2.7 Let $M$ be in $\text{Pre}(\mathcal{A}T)$ and $n \geq 1$. Then $M$ is $n$-$T$-torsionfree if and only if there exists an exact sequence

$$0 \longrightarrow M \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1},$$

where $T_i$ are in $\text{add}(\mathcal{A}T)$ for any $0 \leq i \leq n-1$ and this exact sequence is $\text{Hom}_\mathcal{A}(-,-T)$-exact.

Proof. We proceed by induction on $n$. By Lemma 2.6, the cases $n \leq 2$ is clear. Suppose that $n \geq 3$ and that the conclusion holds for the case $n-1$.

($\Rightarrow$) There is an exact sequence $0 \rightarrow M \xrightarrow{f} T_0 \rightarrow M_1 \rightarrow 0$ with $f$ an $\text{add}(T)$-preenvelope by Lemma 2.6. Then we have a new exact sequence $0 \rightarrow M_1^* \rightarrow T_0^* \rightarrow M^* \rightarrow 0$. Note that $T_0^*$ is a projective $\mathcal{B}$-module. By dimension shifting, we have $\text{Ext}^i_\mathcal{B}(M_1^*, T) \cong \text{Ext}^{i+1}_\mathcal{B}(M^*, T)$ for all $i \geq 1$. Since $M$ is $n$-$T$-torsionfree, $\alpha_M$ is an isomorphism and $\text{Ext}^i_\mathcal{B}(M^*, T) = 0$ for all $1 \leq i \leq n-2$ by Lemma 2.4. We consider the following commutative with exact rows:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & M & \longrightarrow & T_0 & \longrightarrow & T_1 & \longrightarrow & \cdots & \longrightarrow & T_{n-1} & \longrightarrow & 0 \\
& & \alpha_M & & \alpha_{T_0} & & \alpha_{M_1} & & & & & \\
0 & \longrightarrow & M^{**} & \longrightarrow & T_0^{**} & \longrightarrow & T_1^{**} & \longrightarrow & T_{n-1} & \longrightarrow & 0
\end{array}
$$

Since $\alpha_{T_0}$ and $\alpha_M$ are isomorphisms, $\alpha_{M_1}$ is also an isomorphism by the Five Lemma. Note that $\text{Ext}^i_\mathcal{B}(M_1^*, T) \cong \text{Ext}^{i+1}_\mathcal{B}(M^*, T) = 0$ for all $1 \leq i \leq n-3$, so $M_1$ is $(n-1)$-$T$-torsionfree by Lemma 2.4. By the induction hypothesis, there exists an exact sequence, $0 \rightarrow M_1 \rightarrow T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_{n-1}$. So combining it with the exact sequence $0 \rightarrow M \rightarrow T_0 \rightarrow M_1 \rightarrow 0$, we can get the exact sequence we desired.

($\Leftarrow$) Set $M_1$ to be the cokernel of $M \rightarrow T_0$. By the induction hypothesis, $M_1$ is $(n-1)$-$T$-torsionfree.

We have $\text{Ext}^i_\mathcal{B}(M^*, T) \cong \text{Ext}^{i-1}_\mathcal{B}(M_1^*, T) = 0$ for all $2 \leq i \leq n-2$ by Lemma 2.4. Consider the following commutative diagram:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & M & \longrightarrow & T_0 & \longrightarrow & T_1 & \longrightarrow & \cdots & \longrightarrow & T_{n-1} & \longrightarrow & 0 \\
& & \alpha_M & & \alpha_{T_0} & & \alpha_{M_1} & & & & & \\
0 & \longrightarrow & M^{**} & \longrightarrow & T_0^{**} & \longrightarrow & T_1^{**} & \longrightarrow & \cdots & \longrightarrow & T_{n-1} & \longrightarrow & 0
\end{array}
$$

Since $\alpha_{T_0}$ and $\alpha_{M_1}$ are isomorphisms, $\alpha_M$ is also an isomorphism by the Five Lemma. It follows from the commutative diagram above that $f$ is surjective; thus, $\text{Ext}^1_\mathcal{B}(M^*, T) = 0$. So $M$ is $n$-$T$-torsionfree by Lemma 2.4 again. 

\[\square\]

Corollary 2.8 Let $M$ be in $\text{Pre}(\mathcal{A}T)$. The following statements are equivalent:

1. $M$ is 1-$T$-torsionfree;
2. there is an exact sequence $0 \rightarrow M \rightarrow T_0 \rightarrow N \rightarrow 0$ with $T_0 \in \text{add}(T)$ and $\text{Ext}^1_\mathcal{A}(N, T) = 0$;
3. there exists a monomorphic $\text{add}(T)$-preenvelope of $M$. 

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Assume that $M$ has an add($T$)-resolution, that is, there is an exact sequence

$$\cdots \to T_n \xrightarrow{f_n} T_{n-1} \xrightarrow{f_{n-1}} \cdots \to T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0 \tag{\dagger}$$

with $T_i \in \text{add}(T)$ for all $i \geq 0$. $\Omega^n_T(M) = \text{Im} f_i$ is called an $n$-th $T$-syzygy of $M$ for any $i \geq 0$. In particular, put $\Omega^0_T(M) = M$. In the following part of this section, we always assume that $M$ has an add($T$)-resolution.

A module $M$ is called $n$-$T$-spherical if $\text{Ext}^i_A(M; T) = 0$ for all $1 \leq i \leq n$, and $M$ is called $\infty$-$T$-spherical if it is $n$-$T$-spherical for all $n \geq 1$.

**Theorem 2.9** If $T$ is self-orthogonal and $M$ has an add($T$)-resolution (\dagger), then the following statements are equivalent:

1. $\Omega^n_T(M)$ is $n$-$T$-torsionfree
2. There exists an exact sequence $0 \to L \to N \to M \to 0$ such that $N$ is $n$-$T$-spherical and add($T$)-pd ($L) \leq n - 1$.

**Proof** (1) $\Rightarrow$ (2) Suppose that $\Omega^n_T(M)$ is $n$-$T$-torsionfree. By Theorem 2.7, there is an exact sequence $0 \to \Omega^n_T(M) \to T^0 \to L \to 0$, where $T^0 \in \text{add}(T)$, $L$ is $(n - 1)$-$T$-torsionfree and Ext$^1(L, T) = 0$. Consider the push-out of $\Omega^n_T(M) \to T^0$ and $\Omega^n_T(M) \to T_{n-1}$:

$$
\begin{array}{ccccccccc}
0 & & 0 & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & \Omega^n_T(M) & & T^0 & & L & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & T_{n-1} & & N_0 & & L & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^{n-1}_T(M) & & \cdots & & \Omega^{n-1}_T(M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
$$

When $n = 1$, it follows from the second row in diagram above that $\text{Ext}^1_A(N_0, T) = 0$ since $\text{Ext}^1(L, T) = 0 = \text{Ext}^1(T_{n-1}, T)$. Hence, the conclusion follows from the middle column. For the cases of $n \geq 2$. Since $\text{Ext}^1_A(L, T) = 0$, the second row in diagram above is Hom($-, T$)-exact. Then $N_0$ is $(n - 1)$-$T$-torsionfree by Proposition 2.5, since $L$ is $(n - 1)$-$T$-torsionfree. By Theorem 2.7, there is an exact sequence $0 \to N_0 \to T^1 \to V_0 \to 0$, where $T^1 \in \text{add}(T)$, $V_0$ is $(n - 2)$-$T$-torsionfree and $\text{Ext}^1_A(V_0, T) = 0$. Consider the push-out of $N_0 \to T^1$ and $N_0 \to \Omega^{n-1}_T(M)$:
Note that $\text{Ext}^1(N_0, T) = 0$. Thus $\text{Ext}^2(V_0, T) = 0$ by dimension shifting applied to the second row in the diagram above. From the second column, we have $\text{add}(T)\text{-pd} (L_0) \leq 1$. Consider the push-out of $\Omega_T^{n-1}(M) \to L_0$ and $\Omega_T^{n-1}(M) \to T_{n-2}$:

When $n = 2$, it is easy to show that $N_1$ is 2-$T$-spherical from the second row. Then the middle column in the diagram above is just the exact sequence we desired.

Now, we assume that $n \geq 3$. Then $N_1$ is $(n - 2)$-$T$-torsionfree by Proposition 2.5, since $V_0$ is $(n - 2)$-$T$-torsionfree. By Theorem 2.7, there is an exact sequence $0 \to N_1 \to T'_1 \to V_1 \to 0$, where $T'_1 \in \text{add}(T)$, $V_1$ is $(n - 3)$-$T$-torsionfree and $\text{Ext}_A^1(V_1, T) = 0$. Repeating the above discussion, and so on, we can obtain the exact sequence we desired.

(2) $\Rightarrow$ (1) Since $\text{add}(T)\text{-pd} (L) \leq n - 1$, we have the following exact sequence:

$0 \to T_{n-1}' \overset{f_{n-1}'}{\longrightarrow} \cdots \overset{f_1'}{\longrightarrow} T_1' \overset{f_0'}{\longrightarrow} L \to 0.$

with $T'_i \in \text{add}(T)$ for all $0 \leq i \leq n - 1$. Set $\text{Im} f_i' = L_i$ for all $0 \leq i \leq n - 1$. It is clear that $\text{Ext}_i^i(T, L_j) = 0$ for all $i \geq 1$ and $0 \leq j \leq n - 1$ since $T$ is self-orthogonal. Consider the pull-back of $T_0 \to M$ and $N \to M$:
Since $\text{Ext}^1(T, L) = 0$, we have that the second column is split in the diagram above and $H_0 \cong L \oplus T_0$. Hence, we can obtain a new short exact sequence $0 \to L_1 \to T_0' \oplus T_0 \to L \oplus T_0 \to 0$. Consider the pull-back of $T_0' \oplus T_0 \to L \oplus T_0$ and $\Omega^1_T(M) \to L \oplus T_0$:

Then consider further the pull-back of $N_1 \to \Omega^1_T(M)$ and $T_1 \to \Omega^1_T(M)$:
Since $\text{Ext}^1(T, L_1) = 0$, we have $H_1 \cong L_1 \bigoplus T_1$. Similar to the discussion above, we can get some exact sequences $0 \to N_i \to T_{i-1}' \bigoplus T_{i-1} \to N_{i-1} \to 0$ with $1 \leq i \leq n$ and $N_0 = N$. By dimension shifting, we have $\text{Ext}^{1 \leq j \leq n-1}(N_1, T) = 0$ for all $1 \leq i \leq n-1$. There is an exact sequence $0 \to N_{i-1}' \to (T_{i-1}' \bigoplus T_{i-1})' \to N_i' \to 0$. Next, we will prove that $N_i$ is $i$-$T$-torsionfree.

When $i=1$, we consider the following natural commutative diagram:

$$
\begin{array}{cccccc}
0 & \to & N_1 & \to & T_0' \bigoplus T_0 & \to & N_0 & \to & 0 \\
& & \downarrow \alpha_{N_1} & & \downarrow \alpha_{T_0}' \oplus \tau_0 & & \downarrow \alpha_{N_0} & \\
0 & \to & N_1'' & \to & (T_0' \bigoplus T_0)'' & \to & N_0'' & \\
\end{array}
$$

It follows from Snake lemma that $\alpha_{N_1}$ is injective, i.e. $N_1$ is $1$-$T$-torsionfree.

When $i=2$, we consider the following natural commutative diagram:

$$
\begin{array}{cccccc}
0 & \to & N_2 & \to & T_1' \bigoplus T_1 & \to & N_1 & \to & 0 \\
& & \downarrow \alpha_{N_2} & & \downarrow \alpha_{T_1}' \oplus \tau_1 & & \downarrow \alpha_{N_1} & \\
0 & \to & N_2'' & \to & (T_1' \bigoplus T_1)'' & \to & N_1'' & \\
\end{array}
$$

We have proved that $\alpha_{N_1}$ is injective, it follows from Snake lemma that $\alpha_{N_2}$ is an isomorphism, i.e. $N_2$ is $2$-$T$-torsionfree.

For the case $i=3$, we consider the following natural commutative diagram:

$$
\begin{array}{cccccc}
0 & \to & N_3 & \to & T_2' \bigoplus T_2 & \to & N_2 & \to & 0 \\
& & \downarrow \alpha_{N_3} & & \downarrow \alpha_{T_2}' \oplus \tau_2 & & \downarrow \alpha_{N_2} & \\
0 & \to & N_3'' & \to & (T_2' \bigoplus T_2)'' & \to & N_2'' & \to & \text{Ext}^1_B(N_3^*, T) & \to & 0 \\
\end{array}
$$

It follows from the diagram above that $\alpha_{N_3}$ is an isomorphism and $\text{Ext}^1_B(N_3^*, T) = 0$. Thus, $N_3$ is $3$-$T$-torsionfree by Lemma 2.4. Iterating the argument above, we can finally get that $N_n$ is $n$-$T$-torsionfree. It is clear to see that $\Omega_T^k(M) \cong N_n$. Thus $\Omega_T^k(M)$ is $n$-$T$-torsionfree. 

**Proposition 2.10** If $T$ is self-orthogonal and $\Omega_T^k(M)$ is $\infty$-$T$-torsionfree for some $n \geq 1$, then there exists an exact sequence $0 \to L \to N \to M \to 0$ such that $N \in \infty$-$T$-torsionfree and $\text{add}(T)$-$pd$ $(L) \leq n - 1$.

**Proof** We will prove the result by induction on $n$.

When $n = 1$, since $\Omega_T^1(M)$ is $\infty$-$T$-torsionfree, there exists an exact sequence $0 \to \Omega_T^1(M) \to T_0' \to X_0 \to 0$, where $T_0' \in \text{add}(T)$, $X_0$ is $\infty$-$T$-torsionfree and $\text{Ext}^1(X_0, T) = 0$ by Theorem 2.7. Consider the push-out of $\Omega_T^1(M) \to T_0$ and $\Omega_T^1(M) \to T_0'$:
It is easy to show that the middle row in the diagram above is just the exact sequence we desired.

Now suppose that \( n \geq 2 \). Since \( \Omega^n_T(M) = \Omega^{n-1}_T(\Omega^1_T(M)) \), we obtain an exact sequence \( 0 \to L' \to N' \to \Omega^1_T(M) \to 0 \) with \( N' \)-\( T \)-torsionfree and \( \text{add}(T) - \text{pd} (L') \leq n - 2 \) by induction hypothesis. And there is an exact sequence \( 0 \to N' \to T'_0 \to N'' \to 0 \) with \( N'' \)-\( T \)-torsionfree and \( T'_0 \in \text{add}(T) \) and \( \text{Ext}^1(N'', T) = 0 \) by Theorem 2.7. Consider the push-out of \( N' \to T'_0 \) and \( N' \to \Omega^1_T(M) \):

It follows from the middle row in the diagram above that \( \text{add}(T) - \text{pd} (L) \leq n - 1 \). We consider the push-out of \( \Omega^1_T(M) \to T_0 \) and \( \Omega^1_T(M) \to L \):
Since $\text{Ext}^1(N'', T) = 0$, we have that the second column in this diagram is $\text{Hom}(-, T)$-exact. From Proposition 2.5, we get that $N$ is $\infty$-$T$-torsionfree. Thus, the middle row in the diagram above is just desired.

3. $T$-grade

In this section, let $M$ be in $A$-mod, we give the definition of $T$-grade and mainly show that $\Omega^l_T(M)$ is $i$-$T$-torsionfree for all $1 \leq i \leq n$ if and only if $T$-grade $\text{Ext}^i(M, T) \geq i - 1$ for any $1 \leq i \leq n$.

Assume that $M$ has an $\text{add}(T)$-resolution,

$$0 \longrightarrow T^n \longrightarrow T^{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

with $T_i \in \text{add}(T)$ for all $i \geq 0$. Applying $\text{Hom}(-, T)$ to the exact sequence:

$$0 \longrightarrow \Omega^n_T(M) \longrightarrow T^{n-1}_n \longrightarrow \Omega^{n-1}_T(M) \longrightarrow 0,$n

we can obtain the following exact sequence

$$0 \longrightarrow (\Omega^{n-1}_T(M))^* \longrightarrow T^{n-1}_n \overset{f}{\longrightarrow} (\Omega^n_T(M))^* \longrightarrow \text{Ext}^1(\Omega^{n-1}_T(M), T) \longrightarrow 0.$n

It is easy to show that $\text{Ext}^1(\Omega^{n-1}_T(M), T) \cong \text{Ext}^n(M, T)$. Set $Q = \text{Im}f$. We get two new exact sequences

$$0 \longrightarrow (\Omega^{n-1}_T(M))^* \longrightarrow T^{n-1}_n \longrightarrow Q \longrightarrow 0. \quad (1)$$

and

$$0 \longrightarrow Q \longrightarrow (\Omega^n_T(M))^* \longrightarrow \text{Ext}^n(M, T) \longrightarrow 0. \quad (2)$$

Applying the functor $\text{Hom}(-, T)$ to the exact sequence (1), we have the following commutative diagram:

$$0 \longrightarrow \Omega^n_T(M) \longrightarrow T^{n-1}_n \longrightarrow \Omega^{n-1}_T(M) \longrightarrow 0$$

$$\begin{array}{c}
0 \longrightarrow Q^* \longrightarrow T^{n-1}_n \end{array} \overset{h}{\longrightarrow} (\Omega^{n-1}_T(M))^* \longrightarrow \text{Ext}^1(Q, T) \longrightarrow 0 \quad (3)$$

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Similarly, applying the functor Hom(−, T) to the exact sequence (2), we have the following commutative diagram with exact row:

\[
\begin{array}{ccc}
\Omega_T^1(M) & \longrightarrow & T_0 \\
\alpha_{\Omega_T^1(M)} & \downarrow & \alpha_{T_0} \\
(\Omega_T^1(M))^* & \longrightarrow & T_0^* \\
\end{array}
\]

(4)

From the left square in diagram (3), it is easy to verify that the square in diagram (4) is commutative.

**Lemma 3.1** Assume that \(M\) has an add(T)-resolution (§), then the following conclusions hold:

1. \(\Omega_T^1(M)\) is 1-T-torsionfree
2. For any \(n \geq 2\), \(
\text{Coker}(\alpha_{\Omega_T^1(M)}) \cong \text{Hom}_A(\text{Ext}_A^n(M, T), T).
\)

**Proof**
1. We have the following commutative diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & \Omega_T^1(M) \\
& & \alpha_{\Omega_T^1(M)} \\
& & \alpha_{T_0} \\
& (\Omega_T^1(M))^* & \longrightarrow \ T_0^* \\
\end{array}
\]

It follows that \(\alpha_{\Omega_T^1(M)}\) is injective, i.e. \(\Omega_T^1(M)\) is 1-T-torsionfree.

2. If \(n \geq 2\), then \(\alpha_{\Omega_T^{n-1}(M)}\) is injective by (1). Hence \(g\) is an isomorphism in diagram (3) since \(\alpha_{T_{n-1}}\) is an isomorphism. It follows from the diagram (3.4) by the Snake lemma that \(\text{Coker}(\alpha_{\Omega_T^n(M)}) \cong \text{Hom}_A(\text{Ext}_A^n(M, T), T)\).

**Definition 3.2** Let \(N\) be in \(A\)-mod. the \(T\)-grade of \(N\) with respect to \(T\), denoted by \(T\)-grade \(N\), is defined to be the integer \(n=\inf\{i|\text{Ext}_i^1(N, T) \neq 0\}\), and \(\infty\) if such integer doesn’t exist.

**Theorem 3.3** Assume that \(M\) has an add(T)-resolution (§) and \(n \geq 1\). Then \(\Omega_T^i(M)\) is i-T-torsionfree for all \(1 \leq i \leq n\) if and only if \(T\)-grade \(\text{Ext}_i^1(M, T) \geq i - 1\) for all \(1 \leq i \leq n\).

**Proof** We will prove the result by induction on \(n\).

For the case of \(n = 1\), the conclusion follows from Lemma 3.1.

Suppose that \(n = 2\). Then \(\Omega_T^2(M)\) is 2-T-torsionfree if and only if \(\alpha_{\Omega_T^2(M)}\) is an isomorphism. By Lemma 3.1 (1), \(\alpha_{\Omega_T^2(M)}\) is injective. So \(\Omega_T^2(M)\) is 2-T-torsionfree if and only if \(\alpha_{\Omega_T^2(M)}\) is surjective, but

\[
\text{Coker}(\alpha_{\Omega_T^2(M)}) \cong \text{Hom}_A(\text{Ext}_A^1(M, T), T)
\]

by Lemma 3.1 (2). Hence, \(\Omega_T^2(M)\) is 2-T-torsionfree if and only if \(\text{Hom}_A(\text{Ext}_A^1(M, T), T) = 0\), i.e., \(T\)-grade \(\text{Ext}_2^1(M, T) \geq 1\). Now we assume that \(n \geq 3\).

\((\Rightarrow)\) Assume that \(\Omega_T^i(M)\) is \(i\)-T-torsionfree for all \(1 \leq i \leq n\), we only need to prove that \(T\)-grade \(\text{Ext}_i^n(M, T) \geq n - 1\). By Lemma 3.1 (2), we have that \(0 = \text{Coker}(\alpha_{\Omega_T^n(M)}) \cong \text{Hom}_A(\text{Ext}_A^1(M, T), T)\). Applying
the functor \( \text{Hom}(-, T) \) to the exact sequence (2), we get the following new exact sequence:

\[
0 \longrightarrow (\text{Ext}^n(M, T))^* \longrightarrow (\Omega^n_T(M))^* \overset{i^*}{\longrightarrow} Q^* \longrightarrow \text{Ext}^1(\text{Ext}^n(M, T), T) \longrightarrow \text{Ext}^1((\Omega^n_T(M))^*, T).
\]

By induction hypothesis, we know that \( \alpha_{\Omega^{n-1}_T(M)} \) is an isomorphism. So \( g \) in diagram (3) is also an isomorphism, thus, \( i^* \) in diagram (4) is surjective. Since \( \text{Ext}^1((\Omega^n_T(M))^*, T) = 0 \) by Lemma 2.4, we have \( \text{Ext}^1(\text{Ext}^n(M, T), T) = 0 \) from the exact sequence above. Hence, \( T \)-grade \( \text{Ext}^n(M, T) \geq 2 \). Consider the two exact sequences (1) and (2), we have

\[
0 = \text{Ext}^i((\Omega^{n-1}_T(M))^*, T) \cong \text{Ext}^{i+1}(Q, T)
\]

for all \( 1 \leq i \leq n - 3 \) by the assumption and Lemma 2.4. By dimension shifting, we obtain that \( \text{Ext}^i(Q, T) \cong \text{Ext}^{i+1}(\text{Ext}^n(M, T), T) \) for \( 1 \leq i \leq n - 3 \) but \( \text{Ext}^{n-2}(Q, T) \neq \text{Ext}^{n-1}(\text{Ext}^n(M, T), T) \). Hence, \( \text{Ext}^i(\text{Ext}^n(M, T), T) = 0 \) for any \( 3 \leq j \leq n - 2 \). For the case of \( j = 2 \), by the assumption, we have that \( \alpha_{\Omega^{n-1}_T(M)} \) is an isomorphism.

It follows that \( h \) in diagram (3) is surjective, and \( \text{Ext}^1(Q, T) = 0 \). So \( 0 = \text{Ext}^1(Q, T) \cong \text{Ext}^2(\text{Ext}^n(M, T), T) \).

Consequently, \( \text{Ext}^k(\text{Ext}^n(M, T), T) = 0 \) for all \( 0 \leq k \leq n - 2 \), i.e. \( T \)-grade \( \text{Ext}^n(M, T) \geq n - 1 \).

(\( \Leftarrow \)) Assume that the assertion holds for the case \( n - 1 \). i.e. if \( T \)-grade \( \text{Ext}^n(M, T) \geq i - 1 \) for all \( 1 \leq i \leq n - 1 \), then \( \Omega^n_T(M) \) is \( i \)-\( T \)-torsionfree for all \( 1 \leq i \leq n - 1 \). Suppose that \( T \)-grade \( \text{Ext}^n(M, T) \geq i - 1 \) for all \( 1 \leq i \leq n \), it suffices to show that \( \Omega^n_T(M) \) is \( n \)-\( T \)-torsionfree by induction hypothesis. Note that \( \alpha_{\Omega^{n-1}_T(M)} \) is an isomorphism by Lemma 2.4. It follows that \( g \) is an isomorphism and \( \text{Ext}^1(Q, T) = 0 \) in diagram (3). Because \( T \)-grade \( \text{Ext}^n(M, T) \geq n - 1 \), \( i^* \) is an isomorphism in the diagram (4); thus, \( \alpha_{\Omega^n_T(M)} \) is an isomorphism by Snake lemma.

Next, we only need to prove that \( \text{Ext}^i((\Omega^n_T(M))^*, T) = 0 \) for all \( 1 \leq i \leq n - 2 \) by Lemma 2.4. From the exact sequence (1), we have that

\[
0 = \text{Ext}^i((\Omega^{n-1}_T(M))^*, T) \cong \text{Ext}^{i+1}(Q, T)
\]

for all \( 1 \leq i \leq n - 3 \) by the assumption and Lemma 2.4. Since \( T \)-grade \( \text{Ext}^n(M, T) \geq n-1 \), we have that \( \text{Ext}^j(\text{Ext}^n(M, T), T) = 0 \) for any \( 1 \leq j \leq n-2 \), and that \( \text{Ext}^j(Q, T) \cong \text{Ext}^j((\Omega^n_T(M))^*, T) \) for \( 1 \leq j \leq n - 3 \) from the exact sequence (2). Consequently, \( \text{Ext}^j((\Omega^n_T(M))^*, T) = 0 \) for \( 2 \leq j \leq n - 3 \). It follows from the assumption and Lemma 2.4 that \( \text{Ext}^{n-3}((\Omega^{n-1}_T(M))^*, T) = 0 \), so we have that \( 0 = \text{Ext}^{n-3}((\Omega^{n-1}_T(M))^*, T) \cong \text{Ext}^{n-2}(Q, T) \) from the exact sequence (1). Thus, \( \text{Ext}^{n-2}((\Omega^n_T(M))^*, T) = 0 \) from the exact sequence (2) since \( \text{Ext}^{n-2}(\text{Ext}^n(M, T), T) = 0 \). In former portion, we proved \( \text{Ext}^1(Q, T) = 0 \), so we have that \( 0 = \text{Ext}^1(Q, T) \cong \text{Ext}^1((\Omega^n_T(M))^*, T) \) from the exact sequence (2). Thus \( \text{Ext}^i((\Omega^n_T(M))^*, T) = 0 \) for all \( 1 \leq i \leq n - 2 \). \( \square \)

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References