On $\lambda$-pseudo $q$-bi-starlike functions

Prakash KAMBLE$^1$, Mallikarjun SHRIGAN$^{2,*}$, Şahsene ALTINKAYA$^3$$^*$

$^1$Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, Maharashtra State, India
$^2$Department of Mathematics, Dr. D. Y. Patil School of Engineering and Technology, Pune, Maharashtra State, India
$^3$Department of Mathematics, Faculty of Science, Uludağ University, Bursa, Turkey

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Abstract: Making use of the $\lambda$-pseudo-$q$-differential operator, we aim to investigate a new, interesting class of bi-starlike functions in the conic domain. Furthermore, we obtain certain sharp bounds of the Fekete–Szegö functional for functions belonging to this class.

Key words: Fekete–Szegö inequality, bi-starlike functions, $q$-differential operator

1. Introduction

Let $\mathcal{A}$ denote the family of functions analytic in the open unit disk

$$\mathcal{U} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$$

and given by the following Taylor–Maclaurin series:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

(1.1)

We denote by $\mathcal{S}$ the class of starlike functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$ (e.g., see [1, 4, 5, 9, 11]).

Let $\mathcal{S}^*(\beta)$ be the usual subclass of starlike functions $\mathcal{S}$ of order $\beta$, $0 \leq \beta < 1$, so that $f \in \mathcal{S}^*(\beta)$ if and only if, for $z \in \mathcal{U}$,

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta.$$

For $\alpha > 0$, let $\mathcal{B}(\alpha)$ denote the class of Bazilevič functions defined in the open unit disk $\mathcal{U}$, normalized by the condition $f(0) = f'(0) - 1 = 0$, and such that, for $z \in \mathcal{U}$,

$$\text{Re} \left( f'(z) \left( \frac{zf(z)}{z} \right)^{\alpha-1} \right) > 0.$$
The class $B(\alpha)$ reduces to the starlike function and bounded turning function whenever $\alpha = 0$ and $\alpha = 1$, respectively. This class is extended to $B(\alpha, \beta)$, which satisfies the geometric condition
\[
\operatorname{Re} \left( \frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}} \right) > \beta,
\]
where $\alpha$ is a nonnegative real number and $0 \leq \beta < 1$. This class of functions was intensively studied by Singh [18] and considered subsequently by London and Thomas [14]. Recently, Babalola [3] introduced a new subclass $L(\alpha)$ of $\lambda$-pseudo-starlike functions of order $\beta$ satisfying the geometric condition
\[
\operatorname{Re} \left( \frac{z(f'(z))^\lambda}{f(z)} \right) > \beta, \quad (z \in \mathbb{U}, 0 \leq \beta < 1, \lambda \geq 1).
\]
We note that, if $\lambda = 1$, we have the class of starlike functions of order $\beta$, which in this context is $1$-pseudo-starlike functions of order $\beta$. If $\beta = 0$, we simply write $L_\lambda$ instead of $L_\lambda(0)$. For $\lambda = 2$, we note that functions in $L_2(\beta)$ are defined by
\[
\operatorname{Re} \left( f'(z) \frac{zf'(z)}{f(z)} \right) > \beta, \quad (z \in \mathbb{U}),
\]
which is a product combination of geometric expression for bounded turning and starlike functions, an interesting analytic presentation on univalent functions in the open unit disk $\mathbb{U}$. Joshi et al. [8] defined the subclasses $S^\lambda(k, \alpha)$ and $S^\lambda(k, \beta)$ of bi-univalent functions associated with $\lambda$-bi-pseudo-starlike functions in the unit disk $\mathbb{U}$. Recently, Altinkaya and Özkan [2] introduced the subclasses $L_\lambda(\beta)$ and $L_\lambda(\beta, \phi)$ of Sălăgean type $\lambda$-pseudo-starlike functions. For these function classes, they found upper bounds for the initial coefficients as well as Fekete–Szegö inequalities.

**Definition 1.1.** Let $P$ be analytic and normalized Carathéodory functions with positive real part in $\mathbb{U}$. Let $P(p_k)(0 \leq k < \infty)$ denote the family of functions $p$, such that $p \in P$ and $p \prec P$ in $\mathbb{U}$, where $p_k$ maps the unit disk conformally onto the domain $\Omega_k$ such that $1 \in \Omega_k$ and $\partial \Omega_k$ is defined by
\[
\partial \Omega_k = \{ u + iv : u^2 = k^2(u-1)^2 + k^2v^2 \}.
\]
Moreover, $\Omega_k$ is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, and parabolic for $k = 1$ and it covers the right half plane when $k = 0$. The extremal functions of class $P(p_k)(0 \leq k < \infty)$ were presented and investigated by Kanas et al. in [12] and [13]. Obviously,
for $k = 0$, we have
\[
p_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + 2z^4 + ...,
\]
for $k = 1$, we have
\[
p_1(z) = 1 + \frac{2}{\pi^2} \log^2 \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right),
\]
and for $0 < k < 1$ and $A = A(k) = (2/\pi)\arccos k$, we have
\[
p_k(z) = 1 + \frac{2}{1 - k^2} \sinh^2 \left( A(k) \arctanh \sqrt{z} \right).
\]


By virtue of
\[ p(z) = \frac{zf'(z)}{f(z)} \prec p_k(z) \]
or
\[ p(z) = 1 + \frac{zf''(z)}{f'(z)} \prec p_k(z), \]
and the properties of domains, we have
\[ \text{Re}(p(z)) > \text{Re}(p_k(z)) > \frac{k}{k+1}. \]

The \( q \)-differential operator plays a vital role in the theory of geometric function theory. The various subclasses of the normalized analytic function class \( A \) have been studied from different viewpoints. Both \( q \)-calculus and fractional calculus provide important tools that have been used in order to investigate various subclasses of \( A \). Historically speaking, the firm footing of the usage of \( q \)-calculus in the context of geometric function theory was provided and \( q \)-hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, [19, p. 347 et seq.]). Ismail et al. [6] introduced the class of generalized complex functions via \( q \)-calculus on some subclasses of analytic functions. Recently, Purohit and Raina [16] investigated applications of the fractional \( q \)-calculus operator to define new classes of functions that are analytic in unit disk \( U \) (see, for details, [7], [10], and [20]–[23]).

For \( 0 < q < 1 \), the \( q \)-derivative of a function \( f \in A \) given by (1.1) is defined as follows:
\[ D_q f(z) = \frac{f(qz) - f(z)}{(q - 1)z} \quad (z \neq 0), \]
and \( D_q f(0) = f'(0), D_q^2 f(z) = D_q(D_q f(z)) \). From (1.1), we deduce that
\[ D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \]
where
\[ [k]_q = \frac{1 - q^k}{1 - q}. \]

As \( q \to 1^- \), \([k]_q \to k\). For a function \( g(z) = z^k \), we observe that
\[ D_q(g(z)) = D_q(z^k) = \frac{1 - q^k}{1 - q} z^{k-1} = k z^{k-1}, \]
\[ \lim_{q \to 1^-} (D_q(g(z))) = k z^{k-1} = g'(z), \]
where \( g' \) is the ordinary derivative.
We define the Sălăgean $q$-differential operator (also refer to [10]) using the $q$-differential operator as follows:

\[
\begin{align*}
D_0^q f(z) &= f(z), \\
D_1^q f(z) &= z D_q f(z), \\
D_n^q f(z) &= z D_q (D_{n-1}^q f(z)), \\
D_n^q f(z) &= z + \sum_{k=2}^{\infty} [k]_q^n a_k z^k \quad (n \in \mathbb{N}_0, z \in \mathbb{U}).
\end{align*}
\]  

(1.5)

We note that $\lim_{q \to 1^{-}}$

\[
D_n^q f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0, z \in \mathbb{U}).
\]  

(1.6)

**Definition 1.2** Let $0 \leq k < 1, \lambda \geq 1, n \in \mathbb{N}_0, 0 < q < 1$. For $p_k(z)$ as defined in Definition 1.1, the function $f$ given by (1.1) belongs to $S_{\lambda,k}^q(p_k)$ if

\[
\left( \frac{z[|D_q^n f(z)|^\lambda]}{(D_q^n f)^z} \right) \prec p_k(z) \quad (z \in \mathbb{U}).
\]  

(1.7)

Let $\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots (c_1 > 0)$ be an analytic function with positive real part on $\mathbb{U}$.

**Definition 1.3** For $\lambda \geq 1, 0 < q < 1$, we say a function $f$ given by (1.1) belongs to the class $S_{\lambda,q}^q(\phi)$ if it satisfies the quasi-subordination condition

\[
\left( \frac{z[|D_q^n f(z)|^\lambda]}{(D_q^n f)^z} \right) \prec_q \phi(z) - 1 \quad (z \in \mathbb{U}).
\]  

(1.8)

In order to derive our main results, we use the following lemma.

**Lemma 1.4** [15] Let $w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \ldots \in \mathbb{U}$ such that $|w(z)| < 1$ in $\mathbb{U}$. If $t$ is a complex number, then

\[
|w_2 + tw_1^2| \leq \max\{1, |t|\}.
\]

The inequality is sharp for the function $w(z) = z$ or $w(z) = z^2$.

In this paper, motivated by the earlier work of Babalola [3] and Altinkaya and Özkan [2], we introduce a new approach for studying a subclass of $\lambda$-pseudo bi-starlike functions using the $q$-differential operator and estimate the Fekete–Szegö body of the coefficient using subordination [17].
2. Main results

We investigate \(|a_3 - \sigma a_2^2|\) for the function \(f \in A\) for the class \(S_{\lambda,k}^q(p_k)\) associated with conical domains.

**Theorem 2.1** Let \(0 \leq k < 1, \lambda \geq 1, 0 < q < 1\) and \(p_k(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots\) defined in Definition 1.1. If the function \(f\) given by (1.1) belongs to \(S_{\lambda,k}^q(p_k)\), then for any complex \(\sigma\) we have

\[
|a_3 - \sigma a_2^2| \leq \frac{p_1}{(3\lambda - 1)[3]_q} \max \left\{ 1, \frac{p_2}{p_1} + \frac{p_1(4\lambda - 1 - 2\lambda^2)[2]_q^{2n} - \sigma p_1(3\lambda - 1)[3]_q^n}{(2\lambda - 1)[2]_q^n} \right\}.
\] (2.1)

**Proof** By (1.7), we have

\[
\frac{z[(D_q^n f)z]^{\lambda}}{(D_q^n f)z} = p_k(z) \quad (z \in \mathbb{U}).
\] (2.2)

We note that

\[
z[(D_q^n f)z]^{\lambda} = z + 2\lambda [2]_q^n a_2 z^2 + (3\lambda[3]_q^n a_3 + 2\lambda(\lambda - 1)a_2^2[2]_q^{2n}) z^3 + \ldots
\] (2.3)

and

\[
p_k(w(z))(D_q^n f)(z) = z + (p_1 w_1 + [2]_q^n a_2) z^2 + (p_1 w_2 + p_2 w_1^2 + [2]_q^n a_2 p_1 w_1 + [3]_q^n a_3) z^3 + \ldots.
\] (2.4)

Comparing coefficients of (2.2), (2.3), and (2.4), we obtain

\[
a_2 = \frac{p_1 w_1}{(2\lambda - 1)[2]_q^n}
\] (2.5)

and

\[
a_3 = \frac{p_1 w_2}{(3\lambda - 1)[3]_q^n} + \frac{p_2 w_1^2}{(3\lambda - 1)[3]_q^n} + \frac{(4\lambda - 1 - 2\lambda^2) p_1 w_1^2}{(3\lambda - 1)(2\lambda - 1)^2[3]_q^n}.
\] (2.6)

Hence, by (2.5) and (2.6), we get the following:

\[
a_3 - \sigma a_2^2 = \frac{p_1}{(3\lambda - 1)[3]_q^n} \left( (w_2 + \vartheta w_1^2) ,
\right.
\]

where

\[
\vartheta = \left( \frac{p_2}{p_1} + \frac{p_1(4\lambda - 1 - 2\lambda^2)[2]_q^{2n} - \sigma p_1(3\lambda - 1)[3]_q^n}{(2\lambda - 1)[2]_q^n} \right).
\] (2.7)

Using Lemma 1.4 and equation (2.7), we yield (2.1). This completes the proof. \(\square\)

**Corollary 2.2** Let \(f \in S_{\lambda,k}^q(p_k)\), then

\[
|a_2| = \frac{p_1}{(2\lambda - 1)[2]_q^n}
\] (2.8)

and

\[
|a_3| \leq \frac{p_1}{(3\lambda - 1)[3]_q^n} \max \left\{ 1, \frac{p_2}{p_1} + \frac{p_1(4\lambda - 1 - 2\lambda^2)}{(2\lambda - 1)^2[2]_q^n} \right\},
\] (2.9)

where \(0 \leq k < 1, \lambda \geq 1, 0 < q < 1\).
For the class of functions $f \in S^q_{\lambda, \varphi}(\phi)$, we can prove the following:

**Theorem 2.3** Let $\lambda \geq 1, 0 < q < 1$. If the function $f$ given by (1.1) belongs to $S^q_{\lambda, \varphi}(\phi)$, then for any complex $\sigma$ we have

$$|a_3 - \sigma a_2^2| \leq \frac{1}{3\lambda[3]^n_q} \left( c_1 + \max \left\{ c_1, \frac{2\lambda(2 - \lambda)[2]_{q}^{2n} - 3\sigma \lambda[3]_{q}^{n}}{4\lambda^2[2]_{q}^{2n}} \right\} |c_1| + |c_2| \right). \quad (2.10)$$

**Proof** If $f \in S^q_{\lambda, \varphi}(\phi)$, then

$$\frac{z[(D^n_q f)z]'}{(D^n_q f)z} = \varphi(z)(\phi(z) - 1) \quad (z \in U). \quad (2.11)$$

We have

$$z[(D^n_q f)z]'^\lambda = z + 2\lambda[2]_{q}^{n}a_2z^2 + (3\lambda[3]_{q}^{n}a_3 + 2\lambda(\lambda - 1)[2]_{q}^{2n}a_2^2)z^3 + \cdots$$

and

$$\varphi(z)(\phi(z) - 1)(D^n_q f)(z) = c_1A_0w_1z^2 + (c_1A_1w_1 + A_0(c_1w_2 + c_2w_1^2 + [2]_{q}^{n}c_1A_0w_1a_2))z^3 + \cdots. \quad (2.12)$$

From (2.11) and (2.12), it is easily seen that

$$a_2 = \frac{c_1A_0w_1}{2\lambda[2]_{q}^{n}}, \quad \text{(2.13)}$$

$$a_3 = \frac{c_1A_1w_1}{3\lambda[3]_{q}^{n}} + \frac{c_1A_0w_2}{3\lambda[3]_{q}^{n}} + \frac{A_0}{3\lambda[3]_{q}^{n}} \left( c_2 - \frac{(2 - \lambda)c_1^2A_0}{2\lambda} \right) w_1^2, \quad \text{(2.14)}$$

and

$$|a_3 - \sigma a_2^2| \leq \frac{1}{3\lambda[3]^n_q} \left\{ |c_1A_1w_1| + |c_1A_0\Psi| \right\}, \quad \text{(2.15)}$$

where

$$\Psi = \left\{ w_2 - \left( \frac{(2 - \lambda)c_1A_0}{2\lambda} + \frac{3\lambda c_1A_0w_1^2\sigma[3]_{q}^{n}}{4\lambda^2[2]_{q}^{2n}} - \frac{c_2}{c_1} \right) w_1^2 \right\}. \quad \text{(2.16)}$$

Since $\varphi$ is analytic in $U$, using the inequalities $|A_n| \leq 1$ and $|w_1| \leq 1$, we get

$$|a_3 - \sigma a_2^2| \leq \frac{c_1}{3\lambda[3]_{q}^{n}} \left[ 1 + |\Phi| \right], \quad \text{(2.17)}$$

where

$$\Phi = \left| w_2 - \left( - \frac{c_2}{c_1} - \left( \frac{(2 - \lambda)c_1}{2\lambda} + \frac{3\sigma \lambda[3]_{q}^{n}c_1}{4\lambda^2[2]_{q}^{2n}} \right) c_1 \right) w_1^2 \right|. \quad \text{(2.18)}$$

Applying Lemma 1.4 and equation (2.18) yields result (2.10). \qed
Corollary 2.4 Let \( f \in S^q_{\lambda p}(\phi) \), then

\[
|a_2| \leq \frac{c_1 A_0}{2\lambda |2|^n q} \tag{2.19}
\]

and

\[
|a_3| \leq \frac{1}{3\lambda |3|^n q} \left( (c_1 + \max \left( c_1, \frac{(2 - \lambda)c_1^2}{2\lambda} + |c_2| \right) \right), \tag{2.20}
\]

where \( \lambda \geq 1, 0 < q < 1 \).

References


