Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 702 – 711
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doi:10.3906/mat-1808-74

A generalization of π-regular rings

Peter V. DANCHEV*
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria

Received: 14.08.2018 • **Accepted/Published Online:** 21.01.2019 • **Final Version:** 27.03.2019

Abstract: We introduce the class of so-called regularly nil clean rings and systematically study their fundamental characteristic properties accomplished with relationships among certain classical sorts of rings such as exchange rings, Utumi rings etc. These rings of ours naturally generalize the long-known classes of π-regular and strongly π-regular rings. We show that the regular nil cleanness possesses a symmetrization which extends the corresponding one for strong π-regularity that was visualized by Dischinger [10]. Likewise, our achieved results substantially improve on establishments presented in two recent papers by Danchev and Šter [8] and Danchev [6].

Key words: π-regular rings, Utimi rings, exchange rings, regularly nil clean rings, nil clean rings

1. Introduction and conventions

Everywhere in the text of the present paper, all our rings R are assumed to be associative, containing the identity element 1, which in general differs from the zero element 0 of R, and all proper subrings are unital (i.e. containing the same identity as that of the former ring). Our standard terminology and notation are mainly in agreement with those in [17]. For instance, to be more exact, $U(R)$ denotes the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R, $J(R)$ the Jacobson radical of R, and $C(R)$ the center of R. The newly defined notions will be explicitly stated herein.

Reminding these fundamentals unambiguously emphasizes that here we shall mainly deal with nilpotent and idempotent elements, which sometimes will be central and sometimes will be arbitrary. Likewise, here is the place to recall the classical result of Drazin from [11, Theorem 1] stating that all idempotents of a given ring are always central exactly when all its nilpotents are always central.

A retrospective plan of some major ring classes which are critical for our presentation to make the article more nearly self-contained and friendly to the reader is as follows: Concerned with some questions from the functional analysis, in a connection with the continuous geometry, in [19], a ring R was called *regular* (or presently, in the modern terminology, a *von Neumann regular*) if, for every $x \in R$, there exists $y \in R$ such that $x = xyx$ – this also is a successful generalization of the classical *Boolean* rings, that are rings for which $x^2 = x$. Regular rings have also the interesting property, a detailed analysis of which will lead to our major motivating omnibus, that is, Definition 1.1 stated below, namely that $x = xe = fx$ for $e = yx \in Rx \cap Id(R)$ and $f = xy \in xR \cap Id(R)$, so that $x(1-e) = (1-f)x = 0$.

On the other hand, *strongly regular* rings was defined in [1] as another enlargement of the Boolean rings

*Correspondence: danchev@math.bas.bg
2010 *AMS Mathematics Subject Classification:* 16U99; 16E50; 13B99

so that $x = x^2y$. It is principally known that strongly regular rings are themselves reduced regular rings in which all idempotents are central (see, e.g., [12] and [26]). In fact, since strongly regular rings are obviously free of nilpotent elements, $x = x^2y$ yields that $(x - xyx)^2 = 0$ and therefore $x = xyx$, as required.

Furthermore, for applicability purposes and as a common generalization of von Neumann regular rings, it was introduced by McCoy in [18] the class of *π-regular rings* as those rings R for which, for every $a \in R$, there exists a positive integer n such that $a^n \in a^nRa^n$. Important examples of such rings, besides the von Neumann regular ones, include Artinian and perfect rings, that are, left or right perfect. Moreover, the class of *strongly π-regular rings* was introduced by Kaplansky in [13] as rings R for which $a^n \in a^{n+1}R$. Dischinger proved in [10] that this concept is left-right symmetric, that is, $a^n \in a^{n+1}R \cap Ra^{n+1}$ and, consequently, it follows immediately that strongly π-regular rings are themselves π-regular (see [2] as well). Also, von Neumann regular rings in general are not strongly π-regular, but Artinian and perfect rings are so. Nevertheless, these two defined classes of rings do coincide when all idempotents are central.

On the other side, it was recently defined in [9] the class of *nil clean rings* whose elements are sums of a nilpotent and an idempotent; if these two elements commute, the rings are said to be *strongly nil clean*. This was extended respectively in [8] to the so-called *weakly nil clean rings* R for which, for each $a \in R$, there exist $e \in Id(R)$ and $q \in Nil(R)$ such that $a - q - e \in eRa$ (see [6] and [24]). The latter class is strictly contained in the class of *exchange* rings in terms of Goodearl-Nicholson (cf. [20]) which are rings R such that, for any $a \in R$, there exists $e \in Ra$ with $1 - e \in R(1 - a)$ – notice that this notion is left-right symmetric as something more than an ordinary symmetrization was obtained in [14].

Sixty years ago, in order to enlarge the above kinds of rings, Utumi [27] dealt with such rings R for which, for any $x \in R$, there exists $y \in R$ depending on x with the property $x - x^2y \in C(R)$. He proved a structural theorem for these rings, namely [27, Theorem 1]; this theorem shows that the symmetrization $x - yx^2 \in C(R)$ holds equivalently (see, for more account, Corollary in [27]) as well that in such rings, the containment $Nil(R) \subseteq C(R)$ is valid, where $Nil(R)$ forms a two-sided ideal of R. In view of these considerations, we call a ring R *an Utumi ring* if, for every $x \in R$, there exists $y \in R$ depending on x such that $x - x^2y \in Nil(R)$ (it is still unknown at this stage whether, for a reason of symmetry, $x - yx^2 \in Nil(R)$ holds or not). If, in addition, $xy = yx$, the ring R will be called *strongly Utumi*, whereas if $x - x^2y \in Nil(R) \cap C(R)$, the ring R will be called *centrally Utumi*. If $y = 1$ in the initial situation, i.e. $x^2 - x \in Nil(R)$ for all $x \in R$, then we exactly obtain the above-defined strongly nil clean rings which, subsuming, means that strongly nil clean rings are always Utumi's rings. In that aspect, strongly nil clean rings were completely characterized in [7] and [16] as those rings R for which $J(R)$ is nil and the quotient $R/J(R)$ is a Boolean ring. Since in terms of Definition 1.1 listed below Boolean rings are obviously regularly nil clean, according to Theorem 2.9 listed below, strongly nil clean rings are necessarily regularly nil clean. Certainly, on the other vein, it is well known that strongly nil clean rings are strongly π-regular (see, e.g., [9]) and thus they are π-regular, so that, in accordance with Proposition 2.1 stated below the implication follows once again. Some special cases of rings of the types presented above were also considered in [5].

In the style of the last paragraph, using the notations above, it is reasonably adequate to consider rings in which $x - xyx \in C(R)$ or $x - xyx \in C(R) \cap Nil(R)$ as well as the more restricted version in which $x(1-e) \in C(R)$ or $(1 - f)x \in C(R)$ with $e = yx \in Rx \cap Id(R)$ and $f = xy \in xR \cap Id(R)$. However, this will be the theme of some other research exploration, where a new approach might work.

Analyzing all stated above, our aim is to find some more special transversal between the above-mentioned classes of rings. With this goal, we shall find a new and very attractive class of rings, which are both exchange and Utumi rings and which are a common extension of the classical π-regular rings, possessing many valuable properties. Some close relationships are extracted as well.

Hence, the next concept is our key point of view.

Definition 1.1 *A ring R is said to be regularly nil clean if, for each $a \in R$, there exists $e \in Id(R) \cap Ra$ such that $a(1-e) = a - ae \in Nil(R)$. The element a will also be called regularly nil clean.*

If, in addition, $ae = ea$, then a and R are called strong regularly nil clean. If, however, the existing above idempotent e is unique, R is called unique regularly nil clean.

It is self-evident by virtue of a direct check that for idempotents, nilpotents, and units, a regularity of this type always holds. That is why the triangular matrix ring $\mathbb{T}_2(\mathbb{Z}_2)$ is strong regularly nil clean because it consists only of such elements as its unique nontrivial nilpotent has index at most 2 and as its unique nontrivial unit is an involution which is also a unipotent.

We shall show now that this notion is somewhat left-right symmetric, which will be our crucial tool in the sequel. Specifically, the following equivalent condition holds:

Proposition 1.2 *Regularly nil clean rings R are left-right symmetric in the sense that, for any $a \in R$, there exists $e \in Id(R) \cap Ra$ such that $(1-e)a = a - ea \in Nil(R)$.*

Proof It relies on the obvious fact that $(1-e)a$ is nilpotent precisely when $a(1-e)$ is nilpotent, as required. $\square$

Although the equivalence above between Definition 1.1 and Proposition 1.2, it is nevertheless interesting whether the inclusion $a - af \in Nil(R)$ for some $f \in aR \cap Id(R)$ amounts to our initial definition of regular nil cleanness. It is worthwhile noticing that the answer is absolutely positive as well as our above and subsequent "symmetric" results from Propositions 1.2 and 1.3 somewhat refine that of [10].

Proposition 1.3 *Regularly nil clean rings R are left-right symmetric in the sense that, for any $a \in R$, there exists $f \in Id(R) \cap aR$ such that $a(1-f) = a - af \in Nil(R)$ and such that $(1-f)a = a - fa \in Nil(R)$.*

Proof In view of Definition 1.1 above, we write that $a(1-e) = a - ae \in Nil(R)$ for some $e = ra \in Ra \cap Id(R)$ with $r \in R$. By a direct inspection, it follows that $f = (ar)^2 \in aR \cap Id(R)$. Also, one derives that $f = aer$ and that $fa = ae$. This allows us to obtain that $(1-f)a = a - fa = a - ae = a(1-e) \in Nil(R)$. That $a(1-f) \in Nil(R)$ follows now immediately. $\square$

At the beginning of some simple observations, it is pretty obvious that strongly regular rings are regularly nil clean. Also, it is not too hard to check that local rings with nil Jacobson radical are regularly nil clean; indeed, this follows according to the fact that any element is either invertible or nilpotent.

Our motivation for writing up this article was to promote a nontrivial generalization of the π-regular rings in the classes of exchange and Utumi rings. The ideas developed here somewhat shed further light on the complete characterization of large sorts of the aforementioned nil clean rings and they also give further strategy for obtaining interesting relationships between the studied classes.

2. Main results and examples

We start here with a series of preliminaries which is of importance for our further considerations. The first few of them show that there is an abundance of regularly nil clean rings in different aspects.

We are now ready to discover some characteristic properties of the newly defined class of regularly nil clean rings.

Proposition 2.1 *Every π-regular ring is regularly nil clean.*

Proof Letting R be a π-regular ring, fix $a \in R$, and then there is some $r \in R$ and $n \in \mathbb{N}$ such that $a^n = a^n r a^n$. Therefore, ra^n is obviously an idempotent in Ra. Since a direct check shows that $a^n(1-ra^n) = 0$ and $(1-ra^n)a(1-ra^n) = a(1-ra^n)$, it follows that $[a(1-ra^n)]^{n+1} = 0$, as required. □

A ring R is said to have *bounded index of nilpotence* if there exists a fixed natural number i such that $x^i = 0$ for every nilpotent $x \in R$. In [2], it was established that π-regular rings of bounded index of nilpotence are strongly π-regular. In that direction, it is a good chance to note here that our proof, in conjunction with [8, Proposition 3.10] and Proposition 2.1, actually provides us with a little more information, namely that a regularly nil clean ring of bounded index of nilpotence is strongly π-regular.

In the case of strong π-regularity, the last assertion could be slightly improved in a rather surprising way:

Proposition 2.2 *Strongly π-regular rings are strong regularly nil clean, and vice versa.*

Proof "**Necessity**". Let a be an element in the strongly π-regular ring R. Referring to [21, Proposition 1] (see also [3, Theorem 2.1]), one can write that $a^n = fw = wf$ for some $n \in \mathbb{N}$ and some $f \in Id(R)$, $w \in U(R)$ such that $af = fa$. Hence, it follows from this that $f = w^{-1}a^n \in Ra$ and that $(a-af)^n = [a(1-f)]^n = a^n(1-f) = wf(1-f) = 0$, as required.

"**Sufficiency**". Knowing that Proposition 1.2 holds, for any $a \in R$ assume that there is $e \in Id(R) \cap Ra$ such that $ae = ea$ and $(1-e)a \in Nil(R)$. Consequently, $[(1-e)a]^n = (1-e)a^n = 0$ for some $n \in \mathbb{N}$ whence $a^n = ea^n = ra^{n+1}$ for some existing $r \in R$, as needed. □

Recall that, mimicking [8], a ring R is termed *weakly nil clean* if, for each $a \in R$, there are $e \in Id(R)$ and $q \in Nil(R)$ such that $a - e - q \in eRa$. If, in addition, $eq = qe$, R is called *weakly nil clean with the strong property*. While regularly nil clean rings are always weakly nil clean (see the proof of Proposition 2.4 stated in what follows) and the weakly nil clean rings with central idempotents are themselves strongly π-regular (see [8]), that is, in virtue of Proposition 2.2, strong regularly nil clean, we can say even something more:

Proposition 2.3 *Weakly nil clean rings with the strong property are regularly nil clean.*

Proof In the presence of notations above, we write $a = e + q + exa$ for some $x \in R$. Denoting $u := 1 + q$, one sees that $ue = eu$ because $qe = eq$ and that $e + u - 1 = e + q = a - exa \in Ra$. Moreover, $e = u^{-1}eu = u^{-1}e(e + u - 1) \in Ra$ and thus with the application of Proposition 1.2, one infers that $(1-e)a = (1-e)q \in Nil(R)$, as q and e commute, which substantiates our assertion. □

Notice also that it is not obvious whether weakly nil clean rings having the strong property are strongly π-regular; just as aforementioned by [8, Proposition 3.8] weakly nil clean rings (and thus regularly nil clean rings owing to the subsequent Proposition 2.4) with central idempotents are always strongly π-regular. That is why it is interesting to find a weakly nil clean ring which is not regularly nil clean itself. If such a construction eventually exists, it should be quite curious.

Proposition 2.4 *Every regularly nil clean ring is an exchange ring.*

Proof We shall prove even a lot more, namely that regularly nil clean rings are themselves weakly nil clean in the sense of [8], as we already have stated above, namely imitating [8, Definition 2.1] a ring R is weakly nil clean if, for each $a \in R$, there are $e \in Id(R)$ and $q \in Nil(R)$ with $a - e - q \in eRa$. In fact, by what we have just proved in Proposition 1.2, for any element a from the regularly nil clean ring R there is $e \in Id(R) \cap Ra$ such that $a - ea \in Nil(R)$, say $a - ea = q$. Thus, $a - e - q = ea - e = ea - ee = ea - eda = e(1-d)a \in eRa$, for some $d \in R$, as expected.

Furthermore, since it was established in [8, Proposition 2.5] that weakly nil clean rings are exchange, we are done. □

Proposition 2.5 *Every regularly nil clean ring is an Utimi ring.*

Proof This follows immediately by application of Proposition 1.3 since in the notations from there it must be that $f = ax$ for some $x \in R$ and hence $a - af = a - a^2x \in Nil(R)$, as needed. □

Combining Propositions 2.1, 2.4, and 2.5, the next diagram showing the relationships between various classes of rings holds:

$$\pi\text{-regular} \Rightarrow \text{regularly nil clean} \Rightarrow \text{Utumi + exchange}.$$

Our next property demonstrates that the regular nil cleanness is closed for taking corners as it is preserved by the same token and for exchange rings (compare with [20]). As usual, $\mathbb{M}_n(R)$ will denote the full $n \times n$ matrix ring, where $n \in \mathbb{N}$.

Proposition 2.6 *If R is a regularly nil clean ring, then so is the corner ring eRe for any $e \in Id(R)$. In particular, if $\mathbb{M}_n(R)$ is regularly nil clean, then R is regularly nil clean.*

Proof Given an arbitrary element $ere \in eRe$ for some $r \in R$, it follows that $ere \in R$ and hence, there is an idempotent $f \in Rere \subseteq R$ such that $(1-f)ere$ is a nilpotent in R. But, this can be written as $ere - fere = (e-f)ere = (e-fe)ere = t$ for some $t \in Nil(R)$. However, this assures that $(e-efe)ere = et = ete \in Nil(eRe)$, as needed. In fact, we foremost observe that $fe = f$ and thus, by a simple check, that efe is an idempotent in eRe. Moreover, $te = t$ and hence, an induction guarantees that $(et)^m = et^m$ for all $m \in \mathbb{N}$, as expected.

As for the second part, it is well known that there is an idempotent $f \in \mathbb{M}_n(R)$ such that $R \cong f\mathbb{M}_n(R)f$. Thus, the first part applies to get the wanted implication. □

Although it has been long known that the center of an exchange ring need not to be again exchange, the following statement is somewhat surprising.

Proposition 2.7 *The center of a regularly nil clean ring is also regularly nil clean.*

Proof Let R be a regularly nil clean ring. We intend to prove that $C(R)$ remains regularly nil clean. For that purpose, given $c \in C(R)$, we write $c - ec = q$ for some $e \in Id(R) \cap (Rc)$ and $q \in Nil(R)$. Thus, there is $n \in \mathbb{N}$ such that $[(1-e)c]^n = (1-e)c^n = 0$.

We claim that e is a central idempotent, that is, $eR(1-e) = (1-e)Re = \{0\}$. In fact, for all $r \in R$, one sees that $er(1-e) \in Rc^n(1-e) = R(1-e)c^n = \{0\}$ because $e \in Rc$ implies that $e \in Rc^n$. So, $er = ere$ and, by a way of similarity, we also have that $re = ere$. Finally, $er = re$ illustrating that $e \in C(R)$ and hence, $q \in C(R)$.

What remains to be established is that $e \in C(R)c$. Indeed, write $e = bc$ for some $b \in R$. This forces that $e = bce = bec = (be)c = yc$, where $y = be = eb$. However, one observes that $y \in C(R)$, as required, since e being central ensures that $R = Re \oplus R(1-e)$ and thus, $yR(1-e) = (1-e)Ry = R(1-e)y = \{0\}$. □

As a parallel more conceptual confirmation of this fact, we can process like this: It was shown in the proof of Proposition 2.4 that R is weakly nil clean. However, [8, Proposition 3.19] implies that $C(R)$ is weakly nil clean and hence strongly π-regular. Finally, Proposition 2.2 tells us that $C(R)$ is regularly nil clean, as desired.

Lemma 2.8 *Suppose that R is a ring and $a \in R$. If there exists $e^2 = e = 1-f \in Ra$ such that faf is regularly nil clean in fRf, then a is regularly nil clean in R.*

Proof Writing $faf - hfaf = q$ for some $h \in Id(fRf)$ and $q \in Nil(fRf)$, we detect that $(1-f)q = q(1-f) = 0$ because $qf = fq = q$. Thus, defining $t = q + fae$, we directly verify that $t \in Nil(R)$. Letting $g = 1-(f-h) = e+h$, so $g \in Id(R)$ since $eh = he = 0$. Furthermore, one needs to check that $a - ga$ is a nilpotent in R. In doing that, we observe that $tf = qf = q = fq = ft$ and that $a - ga = (1-g)a = (f-h)a$. But $hf = h = fh$ whence, by assumption, $q := faf - haf = (f-h)af \in Nil(R)$. However, $(f-h)a(1-f) = (f-h)a - q$ insuring that $(f-h)a = q + (f-h)a(1-f)$ because $q(f-h) = qf - qh = q - qh = q(1-h)$. In fact, knowing that $(1-f)q = (1-f)(f-h) = (1-f)h = 0$, it follows by use of an induction that

$$[q + (f-h)a(1-f)]^k = q^k + q^{k-1}(f-h)a(1-f),$$

for all $k \in \mathbb{N}$, as previously claimed. □

The following, concerning a certain lifting of idempotents modulo a nil ideal, is critical for our next theorem: *If R is a ring with a nil ideal I, $c \in R$ and $c + I \in Id(R/I)$, then $c + I = e + I$ for some $e \in Id(R) \cap Rc$ with $ce = ec$.*

Hence, we now have all the information needed to prove our main reduction result.

Theorem 2.9 *A ring R is regularly nil clean if, and only if, $J(R)$ is nil and $R/J(R)$ is regularly nil clean. Moreover, in particular, if I is a nil ideal of R, then R is regularly nil clean if, and only if, R/I is regularly nil clean.*

Proof "$\Rightarrow$". Take an arbitrary element $z \in J(R)$. Then, by definition, there is $e \in Rz \cap Id(R)$ such that $z(1-e) \in Nil(R)$, but then $e \in J(R) \cap Id(R) = \{0\}$ and hence $e = 0$. Finally, $z \in Nil(R)$, as required. Moreover, since it is not too hard to verify that an epimorphic image of a regularly nil clean ring retains the same property, it follows that both $R/J(R)$ and R/I are regularly nil clean by taking into account the epimorphisms $R \to R/I \to R/J(R)$ because $I \subseteq J(R)$.

"$\Leftarrow$". We shall restrict our attention to the regularly nil clean factor-ring R/I, bearing in mind that $R/J(R) \cong R/I/J(R)/I = R/I/J(R/I)$. To show that R is also regularly nil clean, choose an arbitrary element $a \in R$. Hence, by assuming the validity of Definition 1.1, there exists $b \in R$ with $b+I \in Id(R/I) \cap (R/I)(a+I)$

such that $(a+I)(1+I-(b+I)) \in Nil(R/I)$. However, owing to the above-listed folklore fact on special lifting of idempotents, one observes that there is $r \in R$ with $b+I=(r+I)(a+I)=ra+I=e+I$ for some $e \in Id(R) \cap R(ra) \subseteq Id(R) \cap Ra$. Furthermore,

$$(a+I)(1+I-(e+I))=(a+I)((1-e)+I)=a(1-e)+I \in Nil(R/I)$$

and therefore, there exists $m \in \mathbb{N}$ such that $[a(1-e)]^m \in I \subseteq Nil(R)$ implying that $a(1-e) \in Nil(R)$, as required. $\square$

A long-standing question regarding whether or not a ring R is (strongly) π-regular, provided I is a nil ideal of R whose factor-ring R/I is strongly (von Neumann) regular. An unsuccessful proof attempt was made in [15, Theorem 4.8], where unfortunately there is a serious gap in the given proof. In the case of central idempotents, it was established in [4] that a ring R is (strongly) π-regular if and only if $Nil(R)=J(R)$ and $R/J(R)$ is (strongly) regular. Besides, [25, Examples 3.1 and 3.2] guarantee that there exists a nil clean ring which is not π-regular containing a nilpotent ideal I such that $I^2=\{0\}$ and the quotient R/I is (von Neumann) regular but *not* strongly π-regular. However, surprisingly, the so-constructed by Šter example of a nil clean ring will definitely be regularly nil clean (compare with Example 2.13 stated below).

So, what we can currently offer in order to resolve this outstanding problem, is the following statement which also strengthens [8, Proposition 3.6] and the corresponding result from [24].

Proposition 2.10 *Let R be a ring with an ideal I. Then the following two items are true:*

(1) If both I and R/I are π-regular, then R is regularly nil clean.

(2) If I is nil and R/I is π-regular, then R is regularly nil clean.

Proof (1) With Proposition 2.1 and Lemma 2.8 at hand, the evidence goes on the same argumentation as that in [8, Proposition 3.6], as we leave the details to the interested reader for a precise check.

(2) It follows by a plain combination of Proposition 2.1 and Theorem 2.9. $\square$

Notice that the rings of point (2) need not be in general π-regular as well. Also, (2) follows directly from (1) since any nilpotent is π-regular.

We now explore how the uniqueness of the existing idempotent will affect the structure of regularly nil clean rings. Unfortunately, as in [8, Proposition 4.1], we are unable to say something new.

Proposition 2.11 *A ring R is unique regularly nil clean if, and only if, R is strongly π-regular with central idempotents.*

Proof To prove left-to-right implication, in accordance with [8, Proposition 3.8], it suffices to show that all idempotents are central. In view of Definition 1.1 and Proposition 1.2, one writes that $e(1-e)=e(1-(e+(1-e)re))$ and that $(1-e)e=(1-(e+er(1-e)))e$, where it is readily checked that $e+(1-e)re$ and $e+er(1-e)$ are both idempotents. Therefore, $e+(1-e)re=e=e+er(1-e)$ giving that $re=ere=er$, as desired.

As for the proof of the right-to-left implication, consulting with [21, Proposition 1], let, for an arbitrary $a \in R$, there exist a natural number n such that $a^n=we=uf$ for some $w,u \in U(R)$ and $e,f \in Id(R) \cap C(R)$ as well as let simultaneously $(1-e)a,(1-f)a \in Nil(R)$ be fulfilled exploiting Proposition 1.2. We claim that

$e = f$. In fact, we see that $(1-f)a^m = 0$ for some $m \geq n$. Consequently, $(1-f)ew^{m-n} = 0$. Canceling by $w^{m-n} \in U(R)$, it must be that $(1-f)e = 0$, i.e. $e = fe$. By the same token, joining this with $(1-e)f = 0$, we get that $f = ef = fe = e$, as wanted, and the claim is sustained. □

The next comments are clarifications of the results obtained so far.

Remark 2.12 *All of our statements quoted above demonstrate that the newly introduced regularly nil clean rings are (perhaps properly) situated between the classes of π-regular rings and weakly nil clean rings as defined in [8]. We, thereby, conjecture that there will exist a weakly nil clean ring which is not regularly nil clean.*

We are now in a position to exhibit a few more examples pertaining to the suitability and the independence of the new definition of regularly nil clean rings. In sharp contrast to strongly nil clean rings, the following is true:

Example 2.13 *There exists a nil clean ring which is regularly nil clean but not π-regular.*

Proof Imitating [25, Examples 3.1,3.2], we consider the nil clean (von Neumann) regular ring $P = \prod_{n=1}^{\infty} \mathbb{M}_n(\mathbb{Z}_2)$ which is not strongly π-regular as well as the nil clean ring $R = \prod_{n=1}^{\infty} \mathbb{M}_n(\mathbb{Z}_4)$ which is not π regular. It was shown there that there is a nil ideal I of R such that $R/I \cong P$. We hereafter can apply Proposition 2.1 and Theorem 2.9 to conclude the claim. □

Another example of a regularly nil clean ring which is not π-regular can be extracted from [8, Examples 3.4 and 3.5] with the aid of some basic results on *semiperfect* rings from [22] and [23], respectively.

The next less exotic example than the preceding one will make the class of regularly nil clean rings more compelling.

Example 2.14 *There exists a regularly nil clean ring which is neither nil clean nor π-regular.*

Proof Based on the example by Rowen, whose construction of a semiperfect ring S is posed as in [8, Example 3.4] (see also [22, 23]), we can infer that this ring is regularly nil clean but hardly π-regular (cf. [22]), as expected.

Concerning nil cleanness, we found that the constructed ring S need not be also nil clean (see [9]). The reason for deducing this is that we can select $2 \notin Nil(S)$. □

It is also worthwhile to notice that [8, Corollary 3.12] establishes that nil clean rings having bounded index of nilpotency (respectively, nil clean rings of nil-clean index 2 as posed in [25, Proposition 4.3]) are themselves strongly π-regular.

3. Left-open problems

We end our work with the following six questions of some interest and importance:

Problem 3.1 *Does it follow that a ring R is regularly nil clean if and only if $\forall a \in R$, $\exists e \in (aRa) \cap Id(R)$: $a(1-e) \in Nil(R)$ and $(1-e)a \in Nil(R)$?*

Problem 3.2 *If R is a ring with $e \in Id(R)$ such that both eRe and $(1-e)R(1-e)$ are regularly nil clean rings, is it true that R is regularly nil clean, too?*

Problem 3.3 *Are (centrally) Utumi's rings exchange?*

Let us now remember that a ring R is said to have a *bounded index of nilpotence* provided that there is a fixed integer $k \geq 1$ such that $t^k = 0$ for all $t \in Nil(R)$. It was established in [8, Corollary 3.12] that a nil clean ring of bounded index of nilpotence is strongly π-regular. A similar notion of a *finite nil-clean index* was defined in [25, Definition 2.3] in the following manner: There exists an integer $k \geq 1$ such that for every $a \in R$ there is $e \in Id(R)$ with $(a-e)^k = 0$ and k is the smallest possible having that property. Hence, what quite naturally arises is the following:

Problem 3.4 *Is it true that nil clean rings of finite nil-clean index are regularly nil clean? Does there exist a nil clean ring which is not regularly nil clean?*

Note that the nil clean ring from Example 2.13 has nil-clean index 4.

The following somewhat enlarges Definition 1.1.

Problem 3.5 *Describe those rings R, calling them weakly regularly nil clean, for which, for every $a \in R$, there exists $e \in Id(R) \cap Ra$ such that $a(1-e) = a - ae \in Nil(R)$ or $a(1+e) = a + ae \in Nil(R)$. Are they still exchange rings?*

We now will treat some mechanical variations of π-regularity, which is closely related to our considerations alluded to above. As stated above, let us recall that a ring R is strongly π-regular if, for any element $a \in R$, there exists $n \in \mathbb{N}$ such that $a^n \in a^{n+1}R$. This is tantamount to $a^n \in a^{2n}R^n$, where $R^n = \{r^n \mid r \in R\}$, but what can be said if we require the stronger inclusion $a^n \in a^{n+1}R^n$? As in the classical version, is this left-right symmetric in the sense that $a^n \in R^n a^{n+1}$?

On the same vein, π-regular rings are these for which $a^n \in a^n R a^n$ and, as we already have noted above, it is well known that strongly π-regular rings are always π-regular. The reverse is untrue in general, but in the case of central idempotents, these two notions coincide. However, if we modify the requested condition to the stronger one $a^n \in a^n R^n a^n$, what can be said for R? As in the classical situation, does the first condition imply the second and are they equivalent in the abelian case?

And so, we close with our final query.

Problem 3.6 *Are Artinian rings and/or perfect rings equipped with the shared above new stronger π-regular properties?*

Notice just for completeness that both classes are strongly π-regular.

Acknowledgements

The author would like to express his deep thanks to the both expert referees for their insightful comments and suggestions made. The author is also very thankful to the handling editor of the current article, Professor Adnan Tercan, for his professional management.

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