On a new subclass of bi-univalent functions defined by using Salagean operator

Bilal ŞEKER
Department of Mathematics, Faculty of Science, Dicle University, Diyarbakır, Turkey

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Abstract: In this manuscript, by using the Salagean operator, new subclasses of bi-univalent functions in the open unit disk are defined. Moreover, for functions belonging to these new subclasses, upper bounds for the second and third coefficients are found.

Key words: Univalent functions, bi-univalent functions, coefficient bounds and coefficient estimates, Salagean operator

1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1.1)

which are analytic in the open unit disk $\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Let $S$ denote the subclass of functions in $A$, which are univalent in $\mathbb{U}$ (for details, see [5]).

In 1983, Salagean [10] introduced the following differential operator:

$D^n : A \rightarrow A$

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = Df(z) = zf'(z),$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

For the functions given by (1.1), we can easily find that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0).$$

It is known that every univalent function $f$ has an inverse $f^{-1}$ satisfying

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}).$$

*Correspondence: bilal.seker@dicle.edu.tr
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\[
\frac{f \left( f^{-1}(w) \right)}{f^2(w)} = w \quad (|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}).
\]

In fact, the inverse function \( f^{-1} \) is given by
\[
g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.2}
\]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathbb{U} \). We denote by \( \Sigma \) the class of all bi-univalent functions in \( \mathbb{U} \) given by the Taylor–Maclaurin series expansion (1.1).

For more information about functions in the class \( \Sigma \), see [11] (see also [3, 8, 9, 13]).

In recent years, the aforementioned study of Srivastava et al. [11] essentially revived the investigation of various subclasses of the bi-univalent function class \( \Sigma \); it was followed by such studies as those by Ali et al. [2], Srivastava et al. [12], and Jahangiri and Hamidi [7] (see also [1, 4, 6], and the references cited in each of them).

The aim of the this paper is to introduce two new subclasses of the function class \( \Sigma \) related to the Salagean differential operator and find estimates on the coefficients \( |a_2| \) and \( |a_3| \) for functions in these new subclasses. We have to remember here the following lemma so as to derive our basic results:

**Lemma 1.1** [5] If \( p \in \mathcal{P} \) then \( |c_k| \leq 2 \) for each \( k \), where \( \mathcal{P} \) is the family of functions \( p \) analytic in \( \mathbb{U} \) for which \( \text{Re}\{p(z)\} > 0, p(z) = 1 + c_1z + c_2z^2 + \cdots \) for \( z \in \mathbb{U} \).

2. Coefficient bounds for the function class \( H^{m,n}_\Sigma(\alpha) \)

By introducing the function class \( H^{m,n}_\Sigma(\alpha) \), we start by means of the following definition.

**Definition 2.1** A function \( f(z) \) given by (1.1) is said to be in the class \( H^{m,n}_\Sigma(\alpha) \) (\( 0 < \alpha \leq 1, \ m, n \in \mathbb{N}_0, \ m > n \)) if the following conditions are satisfied:

\[
f \in \Sigma \quad \text{and} \quad \left| \arg \left( \frac{D^m f(z)}{D^n f(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (z \in \mathbb{U}) \tag{2.1}
\]

and

\[
\left| \arg \left( \frac{D^m g(w)}{D^n g(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (w \in \mathbb{U}), \tag{2.2}
\]

where the function \( g(w) \) is given by (1.2).

For functions in the class \( H^{m,n}_\Sigma(\alpha) \), we start by finding the estimates on the coefficients \( |a_2| \) and \( |a_3| \).

**Theorem 2.2** Let the function \( f(z) \) given by (1.1) be in the class \( H^{m,n}_\Sigma(\alpha) \) (\( 0 < \alpha \leq 1, \ m, n \in \mathbb{N}_0, \ m > n \)). Then

\[
|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(3^m - 3^n) + (2^m - 2^n)^2 - \alpha(2^{2m} - 2^{2n})}} \tag{2.3}
\]

and

\[
|a_3| \leq \frac{2\alpha}{3^m - 3^n + \frac{4\alpha^2}{(2^m - 2^n)^2}}. \tag{2.4}
\]
Proof It can be written that the inequalities (2.1) and (2.2) are equivalent to

\[
\frac{D^m f(z)}{D^n f(z)} = [p(z)]^\alpha
\] (2.5)

and

\[
\frac{D^m g(w)}{D^n g(w)} = [q(w)]^\alpha
\] (2.6)

where \(p(z)\) and \(q(w)\) are in \(P\) and have the forms

\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots
\] (2.7)

and

\[
q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots
\] (2.8)

Now, equating the coefficients in (2.5) and (2.6), we obtain

\[
(2^m - 2^n) a_2 = \alpha p_1
\] (2.9)

\[
(3^m - 3^n) a_3 - 2^n (2^m - 2^n) a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2
\] (2.10)

\[-(2^m - 2^n) a_2 = \alpha q_1
\] (2.11)

and

\[
(3^m - 3^n)(2a_2^2 - a_3) - 2^n (2^m - 2^n) a_2^2 = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2.
\] (2.12)

From (2.9) and (2.11), we get

\[
p_1 = -q_1
\] (2.13)

and

\[
2(2^m - 2^n)^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2).
\] (2.14)

Also, from (2.10), (2.12), and (2.14), we find that

\[
[2(3^m - 3^n) - 2^n (2^m - 2^n)] a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2} (p_1^2 + q_1^2) = \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2} \frac{2(2^m - 2^n)^2 a_2^2}{\alpha^2}.
\]

Therefore, we have

\[
a_2^2 = \frac{\alpha(p_2 + q_2)}{2\alpha(3^m - 3^n) + (2^m - 2^n)^2 - \alpha(2^m - 2^n)}. \tag{2.15}
\]

If we apply Lemma 1.1 for the coefficients \(p_2\) and \(q_2\), we have

\[
|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha(3^m - 3^n) + (2^m - 2^n)^2 - \alpha(2^m - 2^n)}}.
\]

This gives the desired estimate for \(|a_2|\) as asserted in (2.3).
Next, in order to find the bound on $|a_3|$, by subtracting (2.12) from (2.10), we get

\[ 2(3^m - 3^n)(a_3 - a_2^2) = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 - q_1^2) \]

\[ a_3 = \frac{\alpha(p_2 - q_2)}{2(3^m - 3^n)} + \frac{\alpha^2(p_1^2 + q_1^2)}{2(2^m - 2^n)^2}. \quad (2.16) \]

We apply Lemma 1.1 one more time for the coefficients $p_2, p_2, q_1,$ and $q_2$, obtaining

\[ |a_3| \leq \frac{2\alpha}{(3^m - 3^n)} + \frac{4\alpha^2}{(2^m - 2^n)^2}. \]

This completes the proof of Theorem 2.1.

\[ \square \]

3. Coefficient bounds for the function class $H^{m,n}_\Sigma(\beta)$

**Definition 3.1** A function $f(z)$ given by (1.1) is said to be in the class $H^{m,n}_\Sigma(\beta)$ ($0 \leq \beta < 1$, $m, n \in \mathbb{N}_0$, $m > n$) if the following conditions are satisfied:

\[ f \in \Sigma \text{ and } \text{Re} \left( \frac{D^m f(z)}{D^n f(z)} \right) > \beta \quad (z \in \mathbb{U}) \quad (3.1) \]

and

\[ \text{Re} \left( \frac{D^m g(w)}{D^n g(w)} \right) > \beta \quad (w \in \mathbb{U}), \quad (3.2) \]

where the function $g(w)$ is given by (1.2).

**Theorem 3.2** Let the function $f(z)$ given by (1.1) be in the class $H^{m,n}_\Sigma(\beta)$ ($0 \leq \beta < 1$, $m, n \in \mathbb{N}_0$, $m > n$). Then

\[ |a_2| \leq \left( \frac{2(1 - \beta)}{(3^m - 3^n) - 2n(2^m - 2^n)} \right)^{\frac{1}{2}} \quad (3.3) \]

and

\[ |a_3| \leq \frac{4(1 - \beta)^2}{(2^m - 2^n)^2} + \frac{2(1 - \beta)}{(3^m - 3^n)}. \quad (3.4) \]

**Proof** It follows from (3.1) and (3.2) that there exists $p(z) \in \mathcal{P}$ and $q(z) \in \mathcal{P}$ such that

\[ \frac{D^m f(z)}{D^n f(z)} = \beta + (1 - \beta)p(z) \quad (3.5) \]

and

\[ \frac{D^m g(w)}{D^n g(w)} = \beta + (1 - \beta)q(w), \quad (3.6) \]
where \( p(z) \) and \( q(w) \) have the forms (2.7) and (2.8), respectively. Equating coefficients in (3.5) and (3.6) yields
\[
(2^m - 2^n)a_2 = (1 - \beta)p_1, \tag{3.7}
\]
\[
(3^m - 3^n)a_3 - 2^n(2^m - 2^n)a_2^2 = (1 - \beta)p_2, \tag{3.8}
\]
\[-(2^m - 2^n)a_2 = (1 - \beta)q_1, \tag{3.9}\]
and
\[
(3^m - 3^n)(2a_2^2 - a_3) - 2^n(2^m - 2^n)a_2^2 = (1 - \beta)q_2. \tag{3.10}
\]
From (3.7) and (3.9), we get
\[
p_1 = -q_1, \tag{3.11}
\]
\[
2(2^m - 2^n)^2a_2^2 = (1 - \beta)^2(p_1^2 + q_1^2). \tag{3.12}
\]
Also, from (3.8) and (3.10), we find that
\[
2((3^m - 3^n) - 2^n(2^m - 2^n))a_2^2 = (1 - \beta)(p_2 + q_2). \tag{3.13}
\]
Thus, we have
\[
|a_2^2| \leq \frac{(1 - \beta)(|p_2| + |q_2|)}{2((3^m - 3^n) - 2^n(2^m - 2^n))},
\]
\[
|a_2^2| \leq \frac{2(1 - \beta)}{(3^m - 3^n) - 2^n(2^m - 2^n)},
\]
which is the bound on \(|a_2|\) as given in (3.3).

Next, in order to find the bound on \(|a_3|\), by subtracting (3.10) from (3.8), we get
\[
2(3^m - 3^n)(a_3 - a_2^2) = (1 - \beta)(p_2 - q_2)
\]
or equivalently
\[
a_3 = a_2^2 + \frac{(1 - \beta)(p_2 - q_2)}{2(3^m - 3^n)}.
\]
Upon substituting the value of \(a_2^2\) from (3.12), we have
\[
a_3 = \frac{(1 - \beta)^2(p_1^2 + q_1^2)}{2(2^m - 2^n)^2} + \frac{(1 - \beta)(p_2 - q_2)}{2(3^m - 3^n)}.
\]
Applying Lemma 1.1, once again for the coefficients \(p_2\), \(p_2\), \(q_1\), and \(q_2\), we obtain
\[
|a_3| \leq \frac{4(1 - \beta)^2}{(2^m - 2^n)^2} + \frac{2(1 - \beta)}{(3^m - 3^n)},
\]
which is the bound on \(|a_3|\) as asserted in (3.4). \(\square\)
References


